18.965 Differential Geometry, MIT Fall term, 2007

Differentialgeometrieexerzitien III, due: November 5, during class.

I encourage you to work together on the ideas but the solutions must be *individual*. Also, please be *concise* – if your proof is convoluted and takes up several pages, you are probably missing the point. Late assignments will not be accepted without *prior* arrangement.

Exercise 1 (25 points)**.** A *graded derivation* of degree d of the graded algebra $\Omega^\bullet(M)$ is a $\mathbb{R}$-linear map

$$D : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

such that $D(\Omega^k(M)) \subset \Omega^{k+d}(M)$ and $D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^d \alpha \wedge D\beta$. We saw in class that if

$$Q \in C^\infty(M, T \otimes \wedge^k T^*) = \Omega^k(T),$$

then Q determines a graded derivation of degree $k-1$ via the expression, for Q of the form $Q = X \otimes \alpha$ for X a vector field,

$$i_Q \rho = \alpha \wedge i_X \rho.$$

We also defined the de Rham derivative, d, a graded derivation of degree $+1$. We therefore define

$$L_Q = [d, i_Q],$$

where $[\cdot, \cdot]$ denotes the graded commutator $[D_1, D_2] = D_1 D_2 - (-1)^{d_1 d_2} D_2 D_1$.

- Let D be a gr. derivation of $\Omega^\bullet(M)$ such that $D(\Omega^0(M)) = 0$. Show that D must be of the form i_Q for uniquely determined Q. [Hint: you may use the fact that a map $\varphi : C^\infty(M, E) \to C^\infty(M, F)$ is induced by a bundle map $E \to F$ (over the identity) when $\varphi(fs) = f\varphi(s)$ for all $s \in C^\infty(M, E)$ and all $f \in C^\infty(M, \mathbb{R})$, a fact easy to prove]

- Show that $[i_Q, i_R] = i_S$ for S uniquely defined. Compute S in the case that $Q, R \in \Omega^1(T)$.

- There is a canonical element $\mathrm{Id} \in \Omega^1(T)$ given by the identity bundle map $TM \longrightarrow TM$. Compute $L_{\mathrm{Id}}\rho$ for a differential form $\rho \in \Omega^k(M)$.

- Show that the map $Q \mapsto L_Q$ is an injection.

- Show that any graded derivation of $\Omega^\bullet(M)$ may be represented as

$$D = L_Q + i_R,$$

and that $Q = 0$ if and only if $D(\Omega^0(M)) = 0$. Also, show $R = 0$ if and only if $[D, d] = 0$.

- Show that $[L_Q, L_R] = L_S$ for S uniquely defined. S is called the "Frolicher-Nijenhuis bracket" of Q, R, and is sometimes denoted $[Q, R]$.

- Compute $[Q, R]$ when $Q = X \otimes \sigma$ and $R = Y \otimes \tau$, for $X, Y \in C^\infty(M, T)$ and $\sigma, \tau \in \Omega^k(M)$.

- Consider the special case that $Q \in \Omega^1(T) = C^\infty(M, \mathrm{End}(T))$ and suppose that Q is diagonalizable with two distinct eigenvalues $a, b \in \mathbb{R}$ with constant multiplicity throughout the manifold. Show that both the a-eigenspaces and b-eigenspaces form integrable distributions if and only if $[Q, Q] = 0$. The tensor $[Q, Q]$ is often called the Nijenhuis tensor of the endomorphism Q.

Exercise 2. Let $f_0 : M \longrightarrow N$, $f_1 : M \longrightarrow N$ be smoothly homotopic maps, i.e. there exists a smooth map

$$h : M \times \mathbb{R} \longrightarrow N$$

such that $h(x, i) = f_i(x)$ for $i = 0, 1$. Then show that $(f_1^* - f_0^*)\alpha$ is exact when α is closed.

Exercise 3. Let $\{U_i\}$, $i = 1, \ldots, N$ be a finite cover of a compact, oriented n-manifold M, and let $\alpha \in \Omega^n(M)$. Express $\int_M \alpha$ in terms of the integrals

$$\int_{U_{i_1} \cap \cdots \cap U_{i_k}} \alpha$$

for k ranging from 1 to N.

Exercise 4. Warner Ch. 4, Exercise 2,12

Exercise 5. Warner Ch. 4, Exercise 11, 15

Exercise 6. Warner Ch 4, Exercise 16,17

Exercise 7. Warner Ch. 2 Exercise 18, 20.

Exercise 8 (15 points)**.** Compute all these de Rham cohomology groups for all degrees.

- What is the de Rham cohomology of $\mathbb{R}^3 - \{p_1 \cup \cdots \cup p_k\}$ where p_i are a collection of k distinct points?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup \cdots \cup l_m\}$ where l_i are a collection of m non-intersecting lines?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{p_1 \cup \cdots \cup p_k \cup l_1 \cup \cdots \cup l_m\}$, assuming no p_i lies on a l_j?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup l_2\}$, assuming that l_1 intersects l_2 in a point?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup \cdots \cup l_m\}$, assuming that all the lines intersect the origin but are distinct?
- What is the de Rham cohomology of $\mathbb{R}^3 - \{l_1 \cup l_2 \cup l_3\}$, assuming the lines intersect in three distinct points?
- What is the de Rham cohomology of $\mathbb{R}^n - \{X_i\}$, where X_i is a i-dimensional linear subspace?