18.965 Differential Geometry, MIT Fall term, 2007

Differentialgeometrieexerzitien II

Due: October 12, during class.

I encourage you to work together on the ideas but the solutions must be *individual*. Also, please be *concise* – if your proof is convoluted and takes up several pages, you are probably missing the point. Late assignments will not be accepted without *prior* arrangement.

Exercise 1. Let φ_t, ψ_t be automorphisms of an algebra A varying smoothly in time, with $\varphi_0 = \psi_0 = \mathrm{Id}$, so that $\dot{\varphi}_0, \dot{\psi}_0$ are derivations of A. Show that the family of automorphisms $\lambda_t = \varphi_t\psi_t\varphi_t^{-1}\psi_t^{-1}$ has vanishing first derivative at $t = 0$ and has second derivative at $t = 0$ which is a derivation of A, and is given by $[\dot{\varphi}_0, \dot{\psi}_0]$.

Exercise 2. A manifold is said to be "parallelizable" when its tangent bundle TM is isomorphic to the trivial bundle $M \times \mathbb{R}^n$. Show that any connected Lie group G is parallelizable by exhibiting an explicit isomorphism between TG and $G \times \mathfrak{g}$, where $\mathfrak{g} = T_eG$, the tangent space to the identity element.

A manifold is said to be "stably parallelizable" when $TM \oplus V$ is isomorphic to the trivial bundle, for some trivial bundle V. Show that S^n is stably parallelizable. We may be able to show later on that the only spheres which are actually parallelizable are S^1, S^3, and S^7. Why are S^1 and S^3 parallelizable?

Exercise 3. Let $X \in C^\infty(M, TM)$ be a vector field, so that it may be viewed as a smooth section $X : M \longrightarrow TM$. The derivative of this map is a map

$$X_* : TM \longrightarrow T(TM).$$

Is this a vector field on $N = TM$? Prove your result.

Exercise 4. Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

where (r, ϕ, θ) are standard Cartesian coordinates on $\mathbb{R}^3$.

- Show that φ is a diffeomorphism onto its image when restricted to $U = \{(r, \phi, \theta) \;:\; 0 < r, 0 < \phi < \pi, 0 < \theta < 2\pi\}$.
- Compute $\varphi^*dx, \varphi^*dy, \varphi^*dz$ where (x, y, z) are Cartesian coordinates for $\mathbb{R}^3$. (Note: the pullback operation φ^* is denoted $\delta\varphi$ in the textbook).
- Compute $\varphi^*(dx \wedge dy \wedge dz)$.

Exercise 5. Warner Ch. 2 exercise 2, 5.

Exercise 6. Warner Ch. 2 Exercise 8, 9.

Exercise 7. Warner Ch. 2 Exercise 10, 12.

Exercise 8. Warner Ch. 2 Exercise 15, 16.

Exercise 9 (20 pts)**.** Let $N = T^*M$ be the total space of the cotangent bundle of a smooth manifold M, and let $\pi : N \longrightarrow M$ be the usual bundle projection. We now describe a natural 1-form $\theta \in \Omega^1(N)$. At each point $p = (x, \xi) \in N$ (here $x \in M$ is a point and $\xi \in T_x^*M$ is a covector at x), the 1-form takes the following value on a vector $V \in T_pN$:

$$\theta(V) = \xi(\pi_* V).$$

i) Choosing coordinates $(x^1, ..., x^n)$ for an open set U containing x and using coordinates $(x^1, \ldots, x^n, \xi_1, \ldots \xi_n)$ to represent the points $(x \in U, \xi = \sum \xi_i dx^i) \in T^*U$, write the coordinate expression of θ and verify that it is smooth.

ii) Compute $\omega = d\theta \in \Omega^2(N)$. View the result as a smooth family of skew-symmetric 2-forms on N. Compute the rank of this 2-form at the point $p = (x, \xi)$ (i.e. if we write $\omega = \sum \omega_{ij} dx^i \wedge dx^j$, the rank is the rank of the matrix ω_{ij}).

iii) Let $\mu \in \Omega^1(M)$ be a 1-form on M, and view it as a smooth section $\mu : M \longrightarrow T^*M$ of the cotangent bundle. Therefore it defines a smooth map $\mu : M \longrightarrow N$. Compute the pullbacks $\mu^*(\theta) \in \Omega^1(M)$ and $\mu^*(\omega) \in \Omega^2(M)$ as a function of μ.

iv) Let $\varphi_t(x, \xi) = (x, e^t \xi)$ define a 1-parameter group of diffeomorphisms of N (These are called "homotheties" of the vector bundle). Write its generating vector field X in the coordinates from i). Compute $\varphi_t^* \omega$ and $L_X \omega$.

v) Just as a natural 1-form θ was defined on T^*M, define a natural k-form $\theta \in \Omega^k(N_k)$ on the total space of the bundle $N_k = \wedge^k T^*M$. If $\mu \in \Omega^k(M)$, view it as a smooth map $\mu : M \longrightarrow N_k$ and compute $\mu^*(d\theta)$. Does all this work for $k = 0$?