18.965, MIT Fall term, 2007

Differentialgeometrieexerzitien I

Due: Sept. 21 during class.

I encourage you to work together on the ideas but the solutions must be *individual*. Also, please be *concise* – if your proof is convoluted and takes up several pages, you are probably missing the point. Late assignments will not be accepted without *prior* arrangement.

Exercise 1. Warner Ch. 1 Ex. 2, 12

Exercise 2. Warner Ch. 1 Ex. 13, 15

Exercise 3. Warner Ch. 1 Ex. 21, 22

Exercise 4. Warner Ch. 1 Ex. 23

Exercise 5. Show that $U(n)$ is diffeomorphic to $SU(n) \times S^1$. Determine if they are isomorphic as groups.

Exercise 6. Let Γ be a discrete group (a group with a countable number of elements, each one of which is an open set). Show (easy) that Γ is a zero-dimensional Lie group. Suppose that Γ acts smoothly on a manifold $\tilde{M}$, meaning that the action map

$$\begin{aligned} \theta : &\Gamma \times \tilde{M} \longrightarrow \tilde{M} \\ &(h, x) \mapsto h \cdot x \end{aligned}$$

is C^∞. Suppose also that the action is *free* , i.e. that the only group element with a fixed point is the identity. Finally suppose the action is *properly discontinuous*, meaning that the following conditions hold:

i) Each $x \in \tilde{M}$ has a nbhd U s.t. $\{h \in \Gamma \ : \ (h \cdot U) \cap U \neq \emptyset\}$ is finite.

ii) If $x, y \in \tilde{M}$ are not in the same orbit, then there are nbhds U, V containing x, y respectively, such that $U \cap (\Gamma \cdot V) = \emptyset$.

Then show the following:

a) Show that the quotient topological space $M = \tilde{M}/\Gamma$ is Hausdorff and has a countable basis of open sets.

b) Show that M naturally inherits the structure of a smooth manifold.

c) Let $\tilde{M} = S^n$ and Let $\Gamma = \mathbb{Z}/2\mathbb{Z}$ act via $h \cdot x = -x$, where h is the generator of Γ. Show that the hypotheses above are satisfied, and identify the resulting quotient manifold.

d) Let $\tilde{M} = \mathbb{C}^n - \{0\}$ and let the generator of $\Gamma = \mathbb{Z}$ act via $x \mapsto 2x$, for $x \in \tilde{M}$. Verify the hypotheses above and show the quotient manifold is diffeomorphic to $S^{2n-1} \times S^1$.

d) Show that any discrete subgroup of a Lie group G acts freely and properly discontinuously on G by left multiplication.

e) Show that the unipotent 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \ : \ x, y, z \in \mathbb{R} \right\}$$

form a Lie group G, and that the unipotent matrices with integer entries forms a discrete subgroup Γ. Show that G/Γ is a compact smooth 3-dimensional manifold, where Γ acts by left multiplication.

Exercise 7. Show the following:

a) Consider the vector field

$$x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

on $\mathbb{R}^2$. Is it complete? Prove your answer.

b) Prove that any compactly supported smooth vector field is complete; conclude that any smooth vector field on a compact manifold is complete.

c) Let X be a smooth vector field on a manifold M. Show there is a positive function $f \in C^\infty(M)$ such that fX is complete.

Exercise 8. Show that there is no embedded surface in $\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$ whose tangent planes are spanned by the two vector fields $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$.

Exercise 9. Let M be the manifold of $n \times n$ real matrices, which is of course a vector space. For a fixed matrix $a \in M$, we define the vector field a^L whose value at the point $x \in M$ is xa (matrix multiplication). Similarly we define $a^R(x) = ax$. Note that we are using the vector space structure to identify $T_xM = M$ naturally.

Show the following:

a) Let $y \in M$ and let $L_y : M \longrightarrow M$ be left matrix multiplication by y. Also let $R_y : M \longrightarrow M$ be right matrix multiplication by y. Show that L_y and R_y are smooth maps.

   Show that a^L is L_y-related to itself, and that a^R is R_y-related to itself, for any y. For this reason, a^L and a^R are called left- and right-invariant vector fields, respectively.

b) Compute the Lie bracket $[a^L, b^L]$ for $a, b \in M$ and show it is left-invariant. Similarly compute $[a^R, b^R]$ and show it is right-invariant.

b) Compute the flow of the vector fields a^L, a^R explicitly. Hint: use the exponential function of a matrix

$$\exp(a) = \sum_{i=0}^{\infty} \frac{a^k}{k!}.$$

c) Compute the flow of the vector field $a^L - a^R$ explicitly.

d) Show that the functions

$$\begin{aligned} \mathrm{Tr}_k : & M \longrightarrow \mathbb{R}, && (1) \\ & x \mapsto \mathrm{Tr}(x^k), \quad k = 1, \ldots, n && (2) \end{aligned}$$

satisfy $(a^L - a^R)(\mathrm{Tr}_k) = 0$ for all a, i.e. their derivatives in the $a^L - a^R$ direction vanish.