IGA Lecture II: Dirac Geometry

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Dirac geometry was introduced by T. Courant and A. Weinstein as a common geometric framework for

- Poisson structures
  \[ \pi \in \Gamma(\wedge^2 TM), \ [\pi, \pi] = 0, \]
- closed 2-forms
  \[ \omega \in \Gamma(\wedge^2 T^* M), \ d\omega = 0. \]

The name comes from relation with Dirac theory of constraints.
Linear Dirac geometry

- $\mathbb{V}$ vector space, $\mathbb{V} = V \oplus V^*$,
- $\langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle$.

**Definition**

$E \subset \mathbb{V}$ is Lagrangian if $E = E^\perp$. The pair $(\mathbb{V}, E)$ is called a linear Dirac structure.
Examples of Lagrangian subspaces:

1. $\omega \in \bigwedge^2(V^*) \Rightarrow \text{Gr}(\omega) = \{v + \iota_v \omega \mid v \in V\} \in \text{Lag}(\mathbb{V})$.
2. $\pi \in \bigwedge^2(V) \Rightarrow \text{Gr}(\pi) = \{\iota_\alpha \pi + \alpha \mid \alpha \in V^*\} \in \text{Lag}(\mathbb{V})$.
3. $S \subseteq V \Rightarrow S + \text{ann}(S) \in \text{Lag}(\mathbb{V})$.

Most general $E \subset \mathbb{V}$ given by $S \subset V$, $\omega \in \bigwedge^2(S^*)$:

$$E = \{v + \alpha \mid v \in V, \ i_v \omega = \alpha|_S\}.$$
Let $V, V'$ be vector spaces.

**Definition**

A **morphism** $R: V \rightarrow V'$ is a Lagrangian subspaces $R \subset V' \times \overline{V}$ whose projection to $V' \times V$ is the graph of a map $A: V \rightarrow V'$.

A morphism defines a relation $x \sim_R x'$; composition of morphisms is composition of relations.

**Definition**

A **morphism of Dirac structures** $R: (V, E) \rightarrow (V', E')$ is a morphism $R: V \rightarrow V'$ such that

$$E' = R \circ E, \quad E \cap \ker(R) = 0.$$ 

Here $\ker(R) = \{x \in V | x \sim_R 0\}$. 
Equivalently, a morphism $R: \mathbb{V} \rightarrow \mathbb{V}'$ is given by a linear map $A: V \rightarrow V'$ together with a 2-form $\omega \in \wedge^2(V^*)$, where

$$v + \alpha \sim_R v' + \alpha' \iff \begin{cases} v' = A(v) \\ \alpha = A^*(\alpha') + \iota_v \omega \end{cases}$$

Hence, we will also refer to such pairs $(A, \omega)$ as morphisms. Composition of morphisms reads:

$$(A', \omega') \circ (A, \omega) = (A' \circ A, \omega + A^* \omega').$$
The conditions $E' = R \circ E$, $\ker(R) \cap E = 0$ for Dirac morphisms mean that
\[ \forall x' \in E' \quad \exists! x \in E : x \sim_R x'. \]
This defines a map $E' \to E$, $x' \mapsto x$. 
Let $M$ be a manifold, $\mathbb{T}M = TM \oplus T^*M$.

**Definition**

The **Courant bracket** on $\Gamma(\mathbb{T}M)$ is

$$\left[ [v_1 + \alpha_1, v_2 + \alpha_2] = [v_1, v_2] + \mathcal{L}_{v_1} \alpha_2 - \iota_{v_2} d\alpha_1. $$

**Definition**

A **Dirac structure on $M$** is a sub-bundle $E \subseteq \mathbb{T}M$ such that

- $E = E^\perp$,
- $\Gamma(E)$ is closed under $[\cdot, \cdot]$. 
Examples of Dirac structures:

1. For $\omega \in \Gamma(\Lambda^2 T^* M)$, $\text{Gr}(\omega)$ is a Dirac structure $\iff d\omega = 0$.
2. For $\pi \in \Gamma(\Lambda^2 TM)$, $\text{Gr}(\pi)$ is a Dirac structure $\iff [\pi, \pi] = 0$.
3. For $S \subset TM$ a distribution, $S + \text{ann}(S)$ is a Dirac structure $\iff S$ is integrable.
More generally, one can twist by a closed 3-form $\eta \in \Omega^3(M)$. Put $T^M \eta = TM \oplus T^*M$.

**Definition**

The **Courant bracket** on $\Gamma(T^M \eta)$ is

$$\left[ [v_1 + \alpha_1, v_2 + \alpha_2] = [v_1, v_2] + \mathcal{L}_{v_1} \alpha_2 - \iota_{v_2} d\alpha_1 + \iota_{v_1} \iota_{v_2} \eta. $$

**Definition**

A **Dirac structure on** $M$ is a sub-bundle $E \subseteq T^M \eta$ such that

- $E = E^\perp$,
- $\Gamma(E)$ is closed under $\left[ \cdot, \cdot \right]$. 
Examples of Dirac structures in $\mathbb{T}M_\eta$:

1. For $\omega \in \Gamma(\bigwedge^2 T^*M)$, $\text{Gr}(\omega)$ is a Dirac structure $\iff d\omega = \eta$.

2. For $\pi \in \Gamma(\bigwedge^2 TM)$, $\text{Gr}(\pi)$ is a Dirac structure
   $\iff \frac{1}{2} [\pi, \pi] = -\pi^\#(\eta)$.

3. For $S \subset TM$ a distribution, $S + \text{ann}(S)$ is a Dirac structure
   $\iff S$ is integrable and $\eta|\bigwedge^3 S = 0$. 
Definition

A map $\Phi: M \rightarrow M'$ together with $\omega \in \Omega^2(M)$ is called a Courant morphism $(\Phi, \omega): \mathbb{T}M_\eta \rightarrow \mathbb{T}M'_{\eta'}$ if

$$\eta = \Phi^*\eta' + d\omega.$$

Definition

A Dirac morphism $(\Phi, \omega): (\mathbb{T}M, E) \rightarrow (\mathbb{T}M', E')$ is a Courant morphism such that $(d\Phi, \omega)$ defines linear Dirac morphisms fiberwise.
Application to Hamiltonian geometry

- $G \circ \mathfrak{g}^*$ coadjoint action
- $d\mu \in \Omega^1(\mathfrak{g}^*, \mathfrak{g}^*)$ tautological 1-form

For $\xi \in \mathfrak{g}$ put $e(\xi) = \xi_{\mathfrak{g}^*} + \langle d\mu, \xi \rangle \in \Gamma(T\mathfrak{g}^*)$. These satisfy

$$[e(\xi_1), e(\xi_2)] = e([\xi_1, \xi_2]),$$

hence span a Dirac structure $E_{\mathfrak{g}^*} \subset T\mathfrak{g}^*$.

Remark

*Equivalently, $E_{\mathfrak{g}^*}$ is the graph of the Kirillov-Poisson bivector on $\mathfrak{g}^*$.*
A Dirac morphism

$$(\Phi, \omega): (\mathbb{T}M, TM) \rightarrow (\mathbb{T}g^*, E_g^*)$$

is a Hamiltonian $g$-space. That is, $g$ acts on $M$, $\omega, \Phi$ are invariant, and

$$\omega(\xi_{g^*}, \cdot) + \langle d\Phi, \xi \rangle = 0, \quad d\omega = 0, \quad \ker(\omega) = 0.$$
Application to q-Hamiltonian geometry

- $G \circlearrowleft G$ conjugation action,
- · invariant metric on $g = \text{Lie}(G)$,
- $\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L]$ Cartan 3-form,

For $\xi \in g$ put $e(\xi) = \xi_G + \frac{1}{2} (\theta^L + \theta^R) \cdot \xi \in \Gamma(\mathbb{T}G_\eta)$. These satisfy

$$\llbracket e(\xi_1), e(\xi_2) \rrbracket = e([\xi_1, \xi_2]),$$

hence span a Dirac structure $E_G \subset \mathbb{T}G_\eta$. 
Application to q-Hamiltonian geometry

<table>
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<tr>
<th>Theorem (Bursztyn-Crainic)</th>
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A q-Hamiltonian $g$-space is a Dirac morphism

$$(\Phi, \omega): (TM, TM) \rightarrow (T\mathbb{G}_\eta, E_G).$$

This new viewpoint is extremely useful.
Lemma

Let $\varsigma = \frac{1}{2} \operatorname{pr}_1^* \theta^L \cdot \operatorname{pr}_2^* \theta^R \in \Omega^2(G \times G)$. Then $(\operatorname{Mult}_G, \varsigma)$ defines a Dirac morphism

$$(\operatorname{Mult}_G, \varsigma) : (\mathbb{T} G_\eta, E_G) \times (\mathbb{T} G_\eta, E_G) \rightarrow (\mathbb{T} G_\eta, E_G).$$

Hence, given two q-Hamiltonian $G$-spaces $(M_i, \omega_i, \Phi_i)$, one can define their fusion product by composition

$$(\operatorname{Mult}_G, \varsigma) \circ ((\Phi_1, \omega_1) \times (\Phi_2, \omega_2)).$$
Use \( \cdot \) to identify \( g \cong g^* \).

**Lemma**

Let \( \varpi \in \Omega^2(g) \) be the standard primitive of \( \exp^* \eta \). Then \( (\exp, \varpi) \) defines a Dirac morphism

\[
(\exp, \varpi): (T_g, E_g) \to (T G_\eta, E G)
\]

over the subset of \( g \) where \( \exp \) is regular.

Hence, if \( (M, \omega_0, \Phi_0) \) is a Hamiltonian \( G \)-space, such that \( \exp \) regular over \( \Phi_0(M) \), then

\[
(\Phi, \omega) := (\exp, \varpi) \circ (\Phi_0, \omega_0)
\]

defines a q-Hamiltonian \( G \)-space.
We will use the Dirac geometry viewpoint to explain the following fact. Suppose $G$ is compact and simply connected.

**Fact:** q-Hamiltonian $G$-spaces $(M, \omega, \Phi)$ carry distinguished invariant volume forms.

These are the analogues of the ‘Liouville forms’ of symplectic manifolds.

We will need the concept of ‘pure spinors’.
Pure spinors

Return to the linear algebra set-up: \( \mathbb{V} = V \oplus V^*, \langle \cdot, \cdot \rangle. \)

**Definition**

- The **Clifford algebra** \( \mathbb{C}l(V) \) is the unital algebra with generators \( x \in \mathbb{V} \) and relations

  \[ x_1 x_2 + x_2 x_1 = \langle x_1, x_2 \rangle. \]

- The **spinor module** over \( \mathbb{C}l(V) \) is given by

  \[ \varrho: \mathbb{C}l(V) \to \text{End}(\wedge V^*), \quad \varrho(v + \alpha)\phi = \iota_v \phi + \alpha \wedge \phi. \]
For $\phi \in \wedge V^*$ let

$$N(\phi) = \{ x \in V | \varrho(x)\phi = 0 \}.$$ 

**Lemma**

For $\phi \neq 0$, the space $N(\phi) \subseteq V$ is isotropic.

*(Exercise!)*

**Definition (E. Cartan)**

$\phi \in \wedge V^*$ is a **pure spinor** if $N(\phi)$ is Lagrangian.

**Fact:** Every $E \in \text{Lag}(V)$ is given by a pure spinor, unique up to scalar.
**Example**

- \( \text{Gr}(\omega) = N(\phi) \) for \( \phi = e^{-\omega} \).
- \( \text{Gr}(\pi) = N(\phi) \) for \( \phi = e^{-\iota(\pi)} \Lambda \), where \( \Lambda \in \wedge^{\text{top}} V^* - \{0\} \).
- \( S + \text{ann}(S) = N(\phi) \) for \( \phi \in \wedge^{\text{top}}(\text{ann}(S)) - \{0\} \).

**Lemma**

*Suppose \( \phi \in \wedge (V^*) \) is a pure spinor. Then*

\[
\phi^{[\text{top}]} \neq 0 \iff N(\phi) \cap V = 0.
\]

*(Exercise!)*

**Example**

Let \( \phi = e^{-\omega} \). Then \( N(\phi) \cap V = \text{Gr}(\omega) \cap V = \text{ker}(\omega) \) is trivial if and only if \( (e^{-\omega})^{[\text{top}]} \neq 0 \).
Lemma

Suppose \((A, \omega): (\mathbb{V}, E) \rightarrow (\mathbb{V}', E')\) is a Dirac morphism. If \(\phi' \in \wedge(\mathbb{V}')^*\) is a pure spinor with \(E' \cap N(\phi') = 0\), then

\[ \phi = e^{-\omega} A^* \phi' \]

is a pure spinor with \(E \cap N(\phi) = 0\).

Exercise!

In particular if \(E = V\) then \((e^{-\omega} A^* \phi')[^{\text{top}}]\) is a volume form.
Back to q-Hamiltonian $G$-spaces, viewed as Morita morphisms

$$(\Phi, \omega): (\mathbb{T}M, TM) \to (\mathbb{T}G_\eta, E_G)$$

If we can find $\psi \in \Gamma(G, \wedge T^*G)$ with $E \cap N(\psi) = 0$, then 
$(e^{-\omega \Phi^* \psi})^{[\text{top}]}$ is a volume form on $M$. 
Recall: \(E_G\) is spanned by sections

\[
e(\xi) = (\xi^L - \xi^R) + \frac{1}{2}(\theta^L + \theta^R) \cdot \xi.
\]

Let \(F_G\) be spanned by sections

\[
f(\xi) = \frac{1}{2}(\xi^L + \xi^R) + \frac{1}{4}(\theta^L - \theta^R) \cdot \xi.
\]

Then \(T G_\eta = E_G \oplus F_G\) is a Lagrangian splitting.
Suppose $G$ is 1-connected. (Actually, it suffices that $\text{Ad}: G \to \text{SO}(g)$ lifts to $\text{Spin}(g)$.)

**Fact:** $F_G = N(\psi)$ is given by a distinguished pure spinor:

$$\psi = \det^{1/2} \left( \frac{1 + \text{Ad}_g}{2} \right) \exp \left( \frac{1}{4} \left( \frac{1 - \text{Ad}_g}{1 + \text{Ad}_g} \right) \theta^L \cdot \theta^L \right) \in \Omega(G).$$

Putting all together:

**Theorem**

For any q-Hamiltonian $G$-space $(M, \omega, \Phi)$, the top degree part of $e^{-\omega} \Phi^* \psi$

defines an invariant volume form on $M$. 
Remark

Assuming only the existence of the invariant metric (=non-degenerate symmetric bilinear form), one still gets an invariant measure on $G$. 
This result applies in particular to conjugacy classes in $G$.

<table>
<thead>
<tr>
<th>Example</th>
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<tbody>
<tr>
<td>If $G$ is a simply connected semi-simple Lie group, then the conjugacy classes $\mathcal{C} \subseteq G$ carry distinguished volume forms. (Take $\cdot$ the Killing form.)</td>
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<tr>
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<tr>
<td>$G = \text{SO}(3)$ has a non-orientable conjugacy class $\mathcal{C} \cong \mathbb{RP}(2)$.</td>
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<tr>
<td>Let $G$ be the 2-dimensional group $\mathbb{R}_{&gt;0} \ltimes \mathbb{R}$ (acting on $\mathbb{R}$ by dilations and translations). Then $G$ has conjugacy classes not admitting invariant measures. Here $\mathfrak{g}$ does not admit an invariant metric $\cdot$.</td>
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A pure spinor defining $F_G$

We should still explain how $\psi$ is obtained.

**Explanation:** $TG$ carries a Riemannian metric $B$ (from inner product on $\mathfrak{g}$). There is an isometric isomorphism

$$TG \oplus \overline{TG} \rightarrow TG.$$  

⇒ get embedding $\kappa: SO(TG) \hookrightarrow SO(TG)$.

$SO(TG) \cong G \times SO(\mathfrak{g})$ has distinguished section $g \mapsto \text{Ad}_g$. We have

$$E_G = \kappa(\text{Ad})(T^*G), \quad F_G = \kappa(\text{Ad})(TG).$$
A pure spinor defining $F_G$

$$F_G = \kappa(\text{Ad})(TG).$$

Suppose $G$ simply connected. Then the section $\kappa(\text{Ad})$ of $SO(TG)$ lifts to a section $\tilde{\kappa}(\text{Ad})$ of Spin($TG$) $\subset$ Cl($TG$).

Since $TG$ is given by the pure spinor $1 \in \Gamma(\wedge T^*G) = \Omega(G)$, the bundle $F_G$ is given by a pure spinor

$$\psi = \kappa(\text{Ad}).1 \in \Gamma(\wedge T^*G).$$

One can calculate this.
Some basic properties of the q-Hamiltonian volume form $\Gamma$:

- Suppose $(M, \omega, \Phi)$ is the ‘exponential’ of a Hamiltonian $G$-space $(M, \omega_0, \Phi_0)$. Then

$$\Gamma = \Phi_0^* J^{1/2} \Gamma_0$$

where $\Gamma_0 = (\exp(-\omega_0))[\text{top}]$ is the Liouville form, and $J$ is the Jacobian determinant of $\exp$.

- The volume form for a fusion product of q-Hamiltonian spaces $(M_i, \omega_i, \Phi_i)$ is the product of the volume forms.

- The volume form for $D(G) = G \times G$ is given by the canonical orientation and Haar measure.
Properties of q-Hamiltonian volume forms

Let $m = \Phi_*|\Gamma| \in D'(G)$ be the q-Hamiltonian Duistermaat-Heckman measure. $m$ is continuous, and

$$m|_e = c \text{Vol}(M//G)$$

where $c$ is the number of elements in a generic stabilizer.

Recall $\mathcal{M}(\Sigma^0_h) = D(G)^h//G$. Hence we get a formula for the symplectic volume $\text{Vol}(\mathcal{M}(\Sigma^0_h))$: Push-forward Haar measure on $G^{2h}$ under the map

$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod a_i b_i a_i^{-1} b_i^{-1}$$

and evaluate at $e$. The result gives Witten’s volume formula for $\mathcal{M}(\Sigma^0_h)$. 