INTRODUCTION TO POISSON GEOMETRY
LECTURE NOTES, WINTER 2017

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Abstract. These notes are very much under construction. In particular, the references are very incomplete. Apologies!

Contents

1. Poisson manifolds

1.1. **Basic definitions.** Poisson structures on manifolds can be described in several equivalent ways. The quickest definition is in terms of a bracket operation on smooth functions.

Definition 1.1. [27] A *Poisson structure* on a manifold M is a skew-symmetric bilinear map

$$\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

with the *derivation property*

$$\{f,gh\}=\{f,g\}h+g\{f,h\} \tag{1}$$

and the *Jacobi identity*

$$\{f,\{g,h\}\}=\{\{f,g\},h\}+\{g,\{f,h\}\}, \tag{2}$$

for all $f,g,h\in C^\infty(M)$. The manifold M together with a Poisson structure is called a *Poisson manifold*. A map $\Phi\colon N\to M$ between Poisson manifolds is a *Poisson map* if the pull-back map $\Phi^*\colon C^\infty(M)\to C^\infty(N)$ preserves brackets.

Condition (2) means that $\{\cdot,\cdot\}$ is a *Lie bracket* on $C^\infty(M)$, making the space of smooth functions into a Lie algebra. Condition (1) means that for all $f\in C^\infty(M)$, the operator $\{f,\cdot\}$ is a derivation of the algebra of smooth functions $C^\infty(M)$, that is, a vector field. One calls

$$X_f=\{f,\cdot\}$$

the *Hamiltonian vector field* determined by the *Hamiltonian* f. In various physics interpretations, the flow of X_f describes the dynamics of a system with Hamiltonian f.

Example 1.2. The standard Poisson bracket on 'phase space' $\mathbb{R}^{2n}$, with coordinates $q^1,\ldots,q^n$ and $p_1,\ldots,p_n$, is given by

$$\{f,g\}=\sum_{i=1}^n\Big(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\Big).$$

The Jacobi identity may be verified by direct computation, using the formula for the bracket. (Of course, one can do much better than 'direct computation' – see below.) The differential equations defined by the Hamiltonian vector field X_f are *Hamilton's equations*

$$\dot q^i=\frac{\partial f}{\partial p_i},\quad \dot p_i=-\frac{\partial f}{\partial q^i}$$

from classical mechanics. Here our sign conventions (cf. Appendix **??**) are such that a vector field

$$X=\sum_j a^j(x)\frac{\partial}{\partial x^i}$$

on $\mathbb{R}^N$ corresponds to the ODE

$$\frac{dx^j}{dt}=-a^j\big(x(t)\big).$$

A function $g\in C^\infty(M)$ with $X_f(g)=0$ is a *conserved quantity*, that is, $t\mapsto g(x(t))$ is constant for any solution curve $x(t)$ of X_f. One of Poisson's motivation for introducing his bracket was the realization that if g and h are two conserved quantities then $\{g,h\}$ is again a conserved quantity. This was explained more clearly by Jacobi, by introducing the Jacobi identity (1).

Example 1.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, with basis ϵ and corresponding structure constants c_{ij}^k defined by $[\epsilon_i, \epsilon_j] = \sum_k c_{ij}^k \epsilon_k$. On the $C^\infty(\mathfrak{g}^*)$, we have the bracket

$$\{f, g\}(\mu) = \sum_{ijk} c_{ij}^k \mu_k \frac{\partial f}{\partial \mu_i} \frac{\partial g}{\partial \mu_j}. \tag{3}$$

One checks that this does not depend on the choice of basis, and that $\{\cdot, \cdot\}$ is a Poisson bracket. (The Jacobi identity for $\mathfrak{g}$ becomes the Jacobi identity for $\mathfrak{g}^*$.) It is called the *Lie-Poisson structure* on $\mathfrak{g}^*$, since it was discovered by Lie in his foundational work in the late 19th century, and is also known as the *Kirillov-Poisson structure*, since it plays a key role in Kirillov's orbit method in representation theory. The Poisson structure is such that $\phi_\xi \in C^\infty(\mathfrak{g}^*)$ is the linear map defined by a Lie algebra element $\xi \in \mathfrak{g}$, then

$$\{\phi_\xi, \phi_\zeta\} = \phi_{[\xi,\zeta]} \tag{4}$$

The Hamiltonian vector field X_{ϕ_ξ} is the generating vector field corresponding to ξ, for the coadjoint G-action on $\mathfrak{g}^*$. Writing $\xi = \sum_i \xi^i \epsilon_i$, we have $\phi_\xi(\mu) = \sum_i \xi^i \mu_i$, hence

$$X_{\phi_\xi} = \sum_{ijk} c_{ij}^k \, \mu_k \, \xi^i \, \frac{\partial}{\partial \mu_j}.$$

1.2. **Deformation of algebras.** Classical mechanics and Lie theory are thus two of the major inspirations for Poisson geometry. A more recent motivation comes from *deformation theory.* Consider the problem of deforming the product on the algebra of smooth functions $C^\infty(M)$, to a possibly non-commutative product. Thus, we are interested in a family of products $f \cdot_\hbar g$ depending smoothly on a parameter $\hbar$, always with the constant function 1 as a unit, and with $f \cdot_0 g$ the usual (pointwise) product. The commutator $f \cdot_\hbar g - g \cdot_\hbar f$ vanishes to first order in $\hbar$, let $\{f, g\}$ be its linear term:

$$\{f, g\} = \frac{d}{d\hbar}\Big|_{\hbar=0} (f \cdot_\hbar g - g \cdot_\hbar f)$$

so that $f \cdot_\hbar g - g \cdot_\hbar f = \hbar\{f, g\} + O(\hbar^2)$. Then $\{\cdot, \cdot\}$ is a Poisson bracket. This follows since for *any* associative algebra $\mathcal{A}$, the commutation $[a, b] = ab - ba$ satisfies

$$[a, bc] = [a, b]c + b[a, c] \tag{5}$$

and

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]], \tag{6}$$

hence the properties (1) and (2) of the bracket follow by applying these formulas for $\mathcal{A} = C^\infty(M)$ with product $\cdot_\hbar$, and taking the appropriate term of the Taylor expansion in $\hbar$ of both sides. Conversely, $C^\infty(M)$ with the deformed product $\cdot_\hbar$ could then be called a *quantization* of the Poisson bracket on $C^\infty(M)$.

Unfortunately, there are few concrete examples of 'strict' quantizations in this sense. More is known for the so-called *formal deformations* of the algebra $C^\infty(M)$.

Definition 1.4. Let $\mathcal{A}$ be an associative algebra over $\mathbb{R}$. A *formal deformation* of $\mathcal{A}$ is an algebra structure on $\mathcal{A}[[\hbar]]$ (formal power series in $\hbar$ with coefficients in $\mathcal{A}$), such that

(a) The product is $\mathbb{R}[[\hbar]]$-linear in both arguments.

(b) The isomorphism

$$\frac{\mathcal{A}[[\hbar]]}{\hbar\mathcal{A}[[\hbar]]} \cong \mathcal{A}$$

is an isomorphism of algebras.

(Note that by (a), the subspace $\hbar\mathcal{A}[[\hbar]]$ is a two-sided ideal in $\mathcal{A}[[\hbar]]$, hence the quotient space inherits an algebra structure.)

The product of $\mathcal{A}[[\hbar]]$ is usually denoted $*$. We have $\mathcal{A} \subseteq \mathcal{A}[[\hbar]]$ as a subspace, but not as a subalgebra. The product $*$ is uniquely determined by what it is on $\mathcal{A}$. For $a, b \in \mathcal{A}$ we have

$$a * b = ab + \hbar F_1(a,b) + \hbar^2 F_2(a,b) + \cdots$$

As before, any formal deformation of $\mathcal{A} = C^\infty(M)$ gives a Poisson bracket $\{f, g\} = F_1(a,b) - F_2(b,a)$.

Definition 1.5. A *deformation quantization* of a Poisson manifold $(M, \{\cdot,\cdot\})$ is given by a *star product* on $C^\infty(M)[[\hbar]]$, with the following properties:

(i) $(C^\infty(M)[[\hbar]], *)$ is a deformation of the algebra structure on $C^\infty(M)$.
(ii) The terms $F_i(f,g)$ are given by bi-differential operators in f and g.
(iii) $F_1(f,g) - F_2(g,f) = \{f,g\}$.

Conversely, we think of $(C^\infty(M)[[\hbar]], *)$ as a *deformation quantization* of $(C^\infty(M), \cdot, \{\cdot,\cdot\})$. One often imposes the additional condition that $1 * f = f * 1 = f$ for all f.

Example 1.6. An example of a deformation quantization is the *Moyal quantization* of $C^\infty(\mathbb{R}^{2n})$, with the standard Poisson bracket. Let $\mu\colon C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ be the standard pointwise product. Then

$$f * g = \mu\big(D(f \otimes g)\big)$$

where D is the infinite-order 'formal' differential operator on $M \times M$

$$D = \exp\Big(\frac{\hbar}{2}\sum_i \big(\frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i}\big)\Big).$$

It is an exercise to check that this does indeed define an associative multiplication.

Example 1.7. Let $\mathfrak{g}$ be a finite-dimensional algebra. The *universal enveloping algebra* $U\mathfrak{g}$ is the algebra linearly generated by $\mathfrak{g}$, with relations

$$XY - YX = [X, Y]$$

(where the right hand side is the Lie bracket). Note that if the bracket is zero, then this is the symmetric algebra. In fact, as a vector space, $U\mathfrak{g}$ is isomorphic to $S\mathfrak{g}$, the *symmetrization map*

$$S\mathfrak{g} \to U\mathfrak{g}, \quad X_1 \cdots X_r \mapsto \frac{1}{r!}\sum_{s \in \mathfrak{S}_r} X_{s(1)} \cdots X_{s(r)}$$

where the right hand side uses the product in $U\mathfrak{g}$. The fact that this map is an isomorphism is a version of the Poincaré-Birkhoff-Witt theorem. Using this map, we may transfer the product of $U(\mathfrak{g})$ to a product on $S(\mathfrak{g})$. In fact, we putting a parameter $\hbar$ in front of the Lie bracket, we obtain a family of algebra structures on $S(\mathfrak{g})$, which we may also regard as a product on $S(\mathfrak{g})[[\hbar]]$. On low degree polynomials, this product can be calculated by hand: In particular,

$$X * Y = XY + \frac{\hbar}{2}[X,Y]$$

for $X, Y \in \mathfrak{g} \subseteq S(\mathfrak{g})[[\hbar]]$.

The resulting Poisson structure on $S(\mathfrak{g})$ is just the Lie-Poisson structure, if we regard $S(\mathfrak{g})$ as the polynomial functions on $\mathfrak{g}^*$. Hence, we obtain a canonical quantization of the Lie-Poisson structure, given essentially by the universal enveloping algebra.

The question of whether every Poisson structure admits a deformation quantization was settled (in the affirmative) by Kontsevich, in his famous 1997 paper, "*Deformation quantization of Poisson manifolds*".

1.3. **Basic properties of Poisson manifolds.** A skew-symmetric bilinear $\{\cdot,\cdot\}$ satisfying (1) is a derivation in both arguments. In particular, the value of $\{f,g\}$ at any given point depnds only on the differential $\mathrm{d}f$, $\mathrm{d}g$ at that point. This defines a bi-vector field $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ such that

$$\pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\}$$

for all functions f, g. Conversely, given a bivector field π, one obtains a skew-symmetric bracket $\{\cdot,\cdot\}$ on functions satisfying the derivation property. Given bivector fields π_1 on M_1 and π_2 on M_2, with corresponding brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, then a smooth map $\Phi\colon M_1 \to M_2$ is bracket preserving if and only if the bivector fields are Φ-*related*

$$\pi_1 \sim_\Phi \pi_2,$$

that is, $T_m\Phi((\pi_1)_m) = (\pi_2)_{\Phi(m)}$ for all $m \in M_1$ where $T_m\Phi$ is the tangent map (extended to multi-tangent vectors).

We will call π a *Poisson bivector field* (or *Poisson structure*) on M if the associated bracket $\{\cdot,\cdot\}$ is Poisson, that is, if it satisfies the Jacobi identity. Consider the *Jacobiator* $\mathrm{Jac}(\cdot,\cdot,\cdot)$ defined as

$$\mathrm{Jac}(f,g,h) = \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} \tag{7}$$

for $f, g, h \in C^\infty(M)$. Clearly, $\mathrm{Jac}(f,g,h)$ is skew-symmetric in its three arguments. Here $\pi^\sharp\colon T^*M \to TM$ be the bundle map defined by π, i.e. $\pi^\sharp(\alpha) = \pi(\alpha,\cdot)$. Then the Hamiltonian vector field associated to a function f is $X_f = \{f,\cdot\} = \pi^\sharp(\mathrm{d}f)$. We have the following alternative formulas for the Jacobiator:

$$\mathrm{Jac}(f,g,h) = \mathcal{L}_{[X_f,X_g]}h - \mathcal{L}_{X_{\{f,g\}}}h = (\mathcal{L}_{X_f}\pi)(\mathrm{d}g, \mathrm{d}h). \tag{8}$$

The first equality shows that Jac is a derivation in the last argument h, hence (by skew-symmetry) in all three arguments. It follows the values of $\mathrm{Jac}(f,g,h)$ at any given point depend only on the exterior differentials of f, g, h at that point, and we obtain a 3-vector field

$$\Upsilon_\pi \in \mathfrak{X}^3(M), \qquad \Upsilon_\pi(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h) = \mathrm{Jac}(f,g,h).$$

We hence see:

Proposition 1.8. *We have the equivalences,*

$$\begin{aligned} \{\cdot,\cdot\} \text{ is a Poisson bracket } &\Leftrightarrow\ [X_f, X_g] = X_{\{f,g\}} \quad \text{for all } f,g \\ &\Leftrightarrow\ \mathcal{L}_{X_f}\pi = 0 \quad \text{for all } f \\ &\Leftrightarrow\ \mathcal{L}_{X_f}\circ\pi^\sharp = \pi^\sharp\circ\mathcal{L}_{X_f} \quad \text{for all } f \\ &\Leftrightarrow\ \Upsilon_\pi = 0. \end{aligned}$$

Let us point out the following useful consequence:

Remark 1.9. To check if $\{\cdot,\cdot\}$ satisfies the Jacobi identity, it is enough to check on functions whose differentials span T^*M everywhere. (Indeed, to verify $\Upsilon_\pi = 0$ at any given $m \in M$, we only have to check on covectors spanning T^*_mM.)

Remark 1.10. In terms of the *Schouten bracket* of multi-vector fields, the 3-vector field Υ_π associated to a bivector field π is given by $\Upsilon_\pi = -\frac{1}{2}[\pi,\pi]$. Thus, π defines a Poisson structure if and only if $[\pi,\pi] = 0$.

1.4. **Examples of Poisson structures.**

Example 1.11. Every constant bivector field on a vector space is a Poisson structure. Choosing a basis, this means that $\pi = \frac{1}{2}\sum A_{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$ for any skew-symmetric matrix A is a Poisson structure. This follows from Remark 1.9, since we only need to check the Jacobi identity on the coordinate functions. But since the bracket of two linear functions is constant, and the bracket with a constant function is zero, all three terms in the Jacobiator are zero in that case. As a special case, the bivector field on $\mathbb{R}^{2n}$ given as

$$\pi = \sum_{i=1}^n \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i}. \tag{9}$$

is Poisson.

Example 1.12. Similarly, if $\mathfrak{g}$ is a Lie algebra, the bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{g}^*)$ given by (3) corresponds to the bivector field

$$\pi = \frac{1}{2}\sum_{ijk} c_{ij}^k\ \mu^k\,\frac{\partial}{\partial\mu_i}\wedge\frac{\partial}{\partial\mu_j}$$

on $\mathfrak{g}^*$. By Remark 1.9, to verify the Jacobi identity, we only need to check on linear functions $\phi_\xi,\ \xi\in\mathfrak{g}$. But on linear functions, the Jacobi identity for the bracket reduces to the Jacobi identity for the Lie algebra $\mathfrak{g}$.

Example 1.13. Any symplectic manifold (M,ω) becomes a Poisson manifold, in such a way that the Hamiltonian vector fields $X_f = \{f,\cdot\}$ satisfy $\omega(X_f,\cdot) = -\mathrm{d}f$. In local symplectic coordinates $q^1,\ldots,q^n,p_1,\ldots,p_n$, with $\omega = \sum_i \mathrm{d}q^i\wedge\mathrm{d}p_i$, the Poisson structure is given by the formula (9) above. Note that with our sign conventions, the two maps

$$\pi^\sharp\colon T^*M\to TM,\quad \mu\mapsto\pi(\mu,\cdot),$$

and

$$\omega^\flat\colon TM\to T^*M,\quad v\mapsto\omega(v,\cdot)$$

are related by

$$\pi^\sharp = -(\omega^\flat)^{-1}.$$

Example 1.14. If $\dim M = 2$, then *any* bivector field $\pi \in \mathfrak{X}^2(M)$ is Poisson: The vanishing of Υ_π follows because on a 2-dimensional manifold, every 3-vector field is zero.

Example 1.15. If (M_1, π_1) and (M_2, π_2) are Poisson manifolds, then their direct product $M_1 \times M_2$ is again a Poisson manifold, with the Poisson tensor $\pi = \pi_1 + \pi_2$. To check that this is indeed a Poisson tensor field, using Remark 1.9 it suffices to check the Jacobi identity for functions that are pullbacks under one of the projections $\mathrm{pr}_i\colon M_1 \times M_2 \to M_i$, but this is immediate. Put differently, the bracket is such that both projections $\mathrm{pr}_i\colon M_1 \to M_2$ are Poisson maps, and the two subalgebras $\mathrm{pr}_i^* C^\infty(M_i) \subseteq C^\infty(M_1 \times M_2)$ Poisson commute among each other.

Warning: While we usually refer to this operation as a *direct product of Poisson manifolds*, it is not a direct product in the categorical sense. For the latter, it would be required that whenever N is a Poisson manifold with two Poisson maps $f_i\colon N \to M_i$, the diagonal map $N \to M_1 \times M_2$ is Poisson. But this is rarely the case.

Example 1.16. If A is a skew-symmetric $n \times n$-matrix, then

$$\pi = \frac{1}{2} \sum_{ij} A_{ij} x^i x^j \, \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

is a Poisson structure on $\mathbb{R}^n$. Here are two simple ways of seeing this: (i) On the open, dense subset where all $x_i \neq 0$, the differentials of the functions $f_i(x) = \log(|x^i|)$ span the cotangent space. But the Poisson bracket of two such functions is constant. (ii) Using a linear change of coordinates, we can make A block-diagonal wth 2×2-blocks, and possibly one 1×1-block with entry 0. This reduces the question to the case $n = 2$; but $\pi = x^1 x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ is a Poisson structure by the preceding example.

Example 1.17. $\mathbb{R}^3$ has a Poisson structure π_0 given as

$$\{x, y\}_0 = z, \ \{y, z\}_0 = x, \ \{z, x\}_0 = y.$$

The corresponding Poisson tensor field is

$$\pi_0 = z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

Actually, we know this example already: It is the Poisson structure on $\mathfrak{g}^*$ for $\mathfrak{g} = \mathfrak{so}(3)$ (in a standard basis). Another Poisson structure on $\mathbb{R}^3$ is

$$\pi_1 = xy \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + yz \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + zx \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x},$$

as a special case of the quadratic Poisson structures from Example 1.16. In fact, all

$$\pi_t = (1 - t)\pi_0 + t\pi_1$$

with $t \in \mathbb{R}$ are again Poisson structures. (It suffice to verify the Jacobi identity for $f = x, \ g = y, \ h = z$). This is an example of a *Poisson pencil.*

Exercise: Show that if π_0, π_1 are Poisson structures on a manifold M such that $\pi_t = (1 - t)\pi_0 + t\pi_1$ is a Poisson structure for some $t \neq 0, 1$, then it is a Poisson structure for all $t \in \mathbb{R}$. In other words, given three Poisson structures on an affne line in $\mathfrak{X}^2(M)$, then the entire line consists of Poisson structures.

Example 1.18. Another Poisson structure on $\mathbb{R}^3$:

$$\{x,y\} = xy,\ \{z,x\} = xz,\ \{y,z\} = \phi(x)$$

for any smooth function ϕ. Indeed,

$$\begin{aligned} &\{x,\{y,z\}\} + \{y,\{z,x\}\} + \{z,\{x,y\}\} \\ &= \{x,\phi(x)\} + \{y,xz\} + \{z,xy\} \\ &= -\{x,y\}z + x\{y,z\} + \{z,x\}y - \{y,z\}x \\ &= -(xy)z + (zx)y = 0. \end{aligned}$$

Example 1.19. Let M be a Poisson manifold, and $\Phi\colon M \to M$ a Poisson automorphism. Then the the group $\mathbb{Z}$ acts on $M\times\mathbb{R}$ by Poisson automorphism, generated by $(m,t)\mapsto(\Phi(m),t+1)$, and the *mapping cylinder* $(M\times\mathbb{R})/\mathbb{Z}$ inherits a Poisson structure.

Example 1.20. Given a 2-form $\alpha = \frac{1}{2}\sum_{ij}\alpha_{ij}(q)\,\mathrm{d}q^i\wedge\mathrm{d}q^j$ on $\mathbb{R}^n$, we can change the Poisson tensor on $\mathbb{R}^{2n}$ to the bivector field

$$\pi = \sum_{i=1}^n \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i} + \frac{1}{2}\sum_{i,j=1}^n \alpha_{ij}(q)\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j}. \tag{10}$$

Is π a Poisson structure? When checking the Jacobi identity on linear functions, only the sum over cyclic permutations of $\{p_i,\{p_j,p_k\}\}$ is an issue. One finds

$$\{p_i,\{p_j,p_k\}\} = -\frac{\partial\alpha_{jk}}{\partial q^i},$$

so the sum over cyclic permutations of this expression vanishes if and only if $\mathrm{d}\alpha = 0$. This example generalizes to cotangent bundles T^*Q (with their standard symplectic structure): Given a closed 2-form $\alpha\in\Omega^2(Q)$, we can regard α as a vertical bi-vector field π_α on T^*Q. (The constant bivector fields on T^*_qQ are identified with $\wedge^2 T^*_qQ$, hence a family of such fiberwise constant vertical bivector fields is just a 2-form.)

1.5. **Casimir functions.**

Definition 1.21. Suppose π is a Poisson structure on M. A function $\chi\in C^\infty(M)$ is a *Casimir* function if it Poisson commutes with all functions: $\{\chi,f\}$ for all $f\in C^\infty(M)$.

Note that if π is a Poisson structure, and χ is a Casimir, then $\chi\pi$ is again a Poisson structure. To check whether a given function $\chi\in C^\infty(M)$ is a Casimir function, it suffices to prove $\{f,\chi\}=0$ for functions f whose differentials span T^*M everywhere.

Example 1.22. If $M=\mathbb{R}^3$ with the Poisson structure from example 1.17, the Casimir functions are the smooth functions of $x^2+y^2+z^2$. Indeed, it is immediate that this Poisson commutes with x,y,z. More generally, if $M=\mathfrak{g}^*$, it is enough to consider differentials of linear functions ϕ_ξ with $\xi\in\mathfrak{g}$. The Hamiltonian vector fields X_{ϕ_ξ} are the generating vector field for the co-adjoint action of G on $\mathfrak{g}^*$. Hence, the Casimir functions for $\mathfrak{g}^*$ are exactly the $\mathfrak{g}$-invariant functions on $\mathfrak{g}^*$.

1.6. **Tangent lifts of Poisson structures.** Given a Poisson structure π on M, there is a canonical way of obtaining a Poisson structure π_{TM} on the tangent bundle TM. For every smooth function $f \in C^\infty(M)$, let

$$f_T \in C^\infty(TM)$$

be its *tangent lift*, defined by $f_T(v) = v(f)$ for $v \in TM$. Put differently, f_T is the exterior differential $\mathrm{d}f \in \Gamma(T^*M)$, regarded as a function on TM via the pairing. In local coordinates x^i on M, with corresponding tangent coordinates x^i, y^i on TM (i.e. $y^i = \dot{x}_i$ are the 'velocities') we have

$$f_T = \sum_{i=1}^n \frac{\partial f}{\partial x^i} y^i.$$

Theorem 1.23. *Given a bi-vector field π on M, with associated bracket $\{\cdot,\cdot\}$, there is a unique bi-vector field π_{TM} on the tangent bundle such that the associated bracket satisfies*

$$\{f_T, g_T\}_{TM} = \{f, g\}_T, \tag{11}$$

for all $f, g \in C^\infty(M)$. The bivector field π_{TM} is Poisson if and only if π is Poisson.

Proof. From the description in local coordinates, we see that the differentials $\mathrm{d}f_T$ span the cotangent space $T^*(TM)$ everywhere, except along the zero section $M \subseteq TM$. Hence, there is at most one bracket with the desired property. To show existence, it is enough to write the Poisson bivector in local coordinates: If $\pi = \frac{1}{2}\sum_{ij} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$,

$$\pi_{TM} = \sum_{ij} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2}\sum_{ijk} \frac{\partial \pi^{ij}}{\partial x^k}\, y^k \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$$

It is straightforward to check that this has the desired property (11). Equation (**??**) also implies

$$\Upsilon_{\pi_{TM}}(\mathrm{d}f_T,\ \mathrm{d}g_T,\ \mathrm{d}h_T) = (\Upsilon_\pi(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h))_T.$$

In particular, π_{TM} is Poisson if and only if π is Poisson. □

Remark 1.24. Let $f_V \in C^\infty(TM)$ denote the vertical lift, given simply by pullback. Then

$$\{f_T, g_V\}_{TM} = \{f, g\}_V, \quad \{f_V, g_V\}_{TM} = 0.$$

Example 1.25. If $\mathfrak{g}$ is a Lie algebra, with corresponding Lie-Poisson structure on $\mathfrak{g}^*$, then $T(\mathfrak{g}^*)$ inherits a Poisson structure. Under the identification $T(\mathfrak{g}^*) \cong (T\mathfrak{g})^*$, this is the Lie-Poisson structure for the tangent Lie algebra $T\mathfrak{g} = \mathfrak{g} \ltimes \mathfrak{g}$.

Example 1.26. If (M, ω) is a symplectic manifold, and π the associated Poisson structure, then π_{TM} is again non-degenerate. That is, we obtain a symplectic structure ω_{TM} on TM.

2. Lie algebroids as Poisson manifolds

The Lie-Poisson structure on the dual of a finite-dimensional Lie algebra $\mathfrak{g}$, has the important property of being *linear*, in the sense that the coefficients of the Poisson tensor are linear functions, or equivalently the bracket of two linear functions is again linear. Conversely any linear Poisson structure on a finite-dimensional vector space V defines a Lie algebra structure on its dual space $\mathfrak{g} := V^*$, with $\{\cdot,\cdot\}$ as the corresponding Lie-Poisson structure: One simply identifies $\mathfrak{g}$ with the linear functions on V. This gives a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Vector spaces with} \\ \text{linear Poisson structures} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \text{Lie algebras} \right\}. \tag{12}$$

The correspondence (12) extends to vector bundles, with Lie algebras replaced by Lie *algebroids*.

2.1. Lie algebroids.

Definition 2.1. A *Lie algebroid* $(E, \mathsf{a}, [\cdot,\cdot])$ over M is a vector bundle $E \to M$, together with a bundle map $\mathsf{a}\colon E \to TM$ called the *anchor*, and with a Lie bracket $[\cdot,\cdot]$ on its space $\Gamma(E)$ of sections, such that for all $\sigma, \tau \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$[\sigma, f\tau] = f[\sigma,\tau] + \big(\mathsf{a}(\sigma)(f)\big)\tau. \tag{13}$$

Remarks 2.2. (a) One sometimes sees an additional condition that the map $\mathsf{a}\colon \Gamma(E) \to \mathfrak{X}(M)$ should preserve Lie brackets. But this is actually automatic. (Exercise.)

(b) It is not necessary to include the anchor map as part of the structure. An equivalent formulation is that $[\sigma, f\tau] - f[\sigma,\tau]$ is multiplication of τ by *some function*. (In other words, $\mathrm{ad}_\sigma := [\sigma,\cdot]$ is a first order differential operator on $\Gamma(E)$ with scalar principal symbol.) Denoting this function by $X(f)$, one observes $f \mapsto X(f)$ is a derivation of $C^\infty(M)$, hence X is a vector field depending linearly on σ. Denoting this vector field by $X = \mathsf{a}(\sigma)$, one next observes that $\mathsf{a}(f\sigma) = f\mathsf{a}(\sigma)$ for al functions f, so that a actually comes from a bundle map $E \to TM$.

Some examples:

- $E = TM$ is a Lie algebroid, with anchor the identity map.
- More generally, the tangent bundle to a regular foliation of M is a Lie algebroid, with anchor the inclusion.
- A Lie algebroid over $M = \mathrm{pt}$ is the same as a finite-dimensional Lie algebra $\mathfrak{g}$.
- A Lie algebroid with zero anchor is the same as a family of Lie algebras E_m parametrized by M. Note that the Lie algebra structure can vary with $m \in M$; hence it is more general than what is known as a 'Lie algebra bundle'. (For the latter, one requires the existence of local trivializations in which the Lie algebra structure becomes constant.)
- Given a $\mathfrak{g}$-action on M, the trivial bundle $E = M \times \mathfrak{g}$ has a Lie algebroid structure, with anchor given by the action map, and with the Lie bracket on sections extending the Lie bracket of $\mathfrak{g}$ (regarded as constant sections of $M \times \mathfrak{g}$). Concretely, if $\phi, \psi\colon M \to \mathfrak{g}$ are $\mathfrak{g}$-valued functions,
$$[\phi,\psi](m) = [\phi(m),\psi(m)] + (\mathcal{L}_{\mathsf{a}(\phi)}\psi)(m) - (\mathcal{L}_{\mathsf{a}(\psi)}\phi)(m).$$
- For a principal G-bundle $P \to M$, the bundle $E = TP/G$ is a Lie algebroid, with anchor the obvious projection to $T(P/G) = TM$. This is known as the *Atiyah algebroid*. Its sections are identified with the G-invariant vector fields on M. It fits into an exact sequence
$$0 \to P \times_G \mathfrak{g} \to TP/G \xrightarrow{\mathsf{a}} TM \to 0;$$
a splitting of this sequence is the same as a principal connection on P.

- A closely related example: Let $V \to M$ be a vector bundle. A derivation of V is a first order differential operator $D\colon \Gamma(V) \to \Gamma(V)$, such that there exists a vector field X on M with
$$D(f\sigma) = fD(\sigma) + X(f)\sigma$$
for all sections $\sigma \in \Gamma(V)$ and functions $f \in C^\infty(M)$. These are the sections of a certain Lie algebroid E, with anchor given on sections by $\mathsf{a}(D) = X$. In fact, it is just the Atiyah algebroid of the frame bundle of V.
- Let $N \subseteq M$ be a codimension 1 submanifold. Then there is a Lie algebroid E of rank $\dim M$, whose space of sections are the vector fields on M tangent to N. [31]. In local coordinates $x^1, \ldots, x^k$, with N defined by an equation $x^k = 0$, it is spanned by the vector fields
$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{k-1}},\ x^k \frac{\partial}{\partial x^k}.$$
Note that it is important here that N has codimension 1; in higher codimension the space of vector fields vanishing along N would not be a free $C^\infty(M)$-module, so it cannot be the sections of a vector bundle.
- Let $\omega \in \Omega^2(M)$ be a closed 2-form. Then $E = TM \oplus (M \times \mathbb{R})$ acquires the structure of a Lie algebroid, with anchor the projection to the first summand, and with the Lie bracket on sections,
$$[X + f, Y + g] = [X, Y] + \mathcal{L}_X g - \mathcal{L}_Y f + \omega(X, Y).$$
(A similar construction works for any Lie algebroid E, and closed 2-form in $\Gamma(\wedge^2 E^*)$.

2.2. **Linear Poisson structures on vector bundles.** Given a vector bundle $V \to M$, let $\kappa_t\colon V \to V$ be scalar multiplication by $t \in \mathbb{R}$. For $t \neq 0$ this is a diffeomorphism. A function $f \in C^\infty(V)$ is called linear if it is homogeneous of degree 1, that is, $\kappa_t^* f = f$ for all $t \neq 0$. A multi-vector field $u \in \mathfrak{X}^k(V)$ on the total space of V will be called (fiberwise) *linear* if it is homogeneous of degree $1-k$, that is,
$$\kappa_t^* u = t^{1-k}\ u$$
for $t \neq 0$. An equivalent condition is that $u(\mathrm{d}f_1, \ldots, \mathrm{d}f_k)$ is linear whenever the f_i are all linear. In terms of a local vector bundle coordinates, with x^i the coordinates on the base and y^j the coordinates on the fiber, such a fiberwise linear multi-vector field is of the form
$$u = \sum_{i_1<\cdots<i_k} \sum_j c^j_{i_1\cdots i_k}\ y_j \frac{\partial}{\partial y_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y_{i_k}} + \sum_{i_1<\cdots<i_{k-1}} \sum_r a_{i_1\cdots i_{k-1},r}\ \frac{\partial}{\partial x^r} \wedge \frac{\partial}{\partial y_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y_{i_{k-1}}}$$
where the coefficients are smooth functions on U.

An example of a linear vector field on V is the *Euler vector field*, given in local coordinates as
$$\mathcal{E} = \sum_i y^i \frac{\partial}{\partial y^i}.$$
It is the unique vector field on V with the property that $\mathcal{L}_\mathcal{E} f = f$ for all linear functions f. In turn, the linearity of a multi-vector field $u \in \mathfrak{X}^k(V)$ can be expressed in terms of the Euler vector field as the condition
$$\mathcal{L}_\mathcal{E} u = (1-k)u.$$

As a special case, a bivector field π is linear if it is homogeneous of degree -1, or equivalently $\mathcal{L}_{\mathcal{E}}\pi = -\pi$. The following theorem gives a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Vector bundles with} \\ \text{linear Poisson structures} \end{array} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \text{Lie algebroids} \right\} \tag{14}$$

For any section $\sigma \in \Gamma(E)$, let $\phi_\sigma \in C^\infty(E^*)$ be the corresponding linear function on the dual bundle E^*.

Theorem 2.3 (Courant [11, Theorem 2.1.4]). *For any Lie algebroid $E \to M$, the total space of the dual bundle $p\colon E^* \to M$ has a unique Poisson bracket such that for all sections $\sigma, \tau \in \Gamma(E)$,*

$$\{\phi_\sigma, \phi_\tau\} = \phi_{[\sigma,\tau]}. \tag{15}$$

The anchor map is described in terms of the Poisson bracket as

$$p^*(\mathsf{a}(\sigma)f) = \{\phi_\sigma,\ p^*f\}, \tag{16}$$

for $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, while $\{p^*f, p^*g\} = 0$ for all functions f, g. The Poisson structure on E^* is linear; conversely, every fiberwise linear Poisson structure on a vector bundle $V \to M$ arises in this way from a unique Lie algebroid structure on the dual bundle V^*.

Proof. Let $E \to M$ be a Lie algebroid. Pick local bundle trivializations $E|_U = U \times \mathbb{R}^n$ over open subsets $U \subseteq M$, and let $\epsilon_1, \ldots, \epsilon_n$ be the corresponding basis of sections of E. Let x^j be local coordinates on U. The differentials of functions $y_i = \phi_{\epsilon_i}$ and functions $x^j y_i = \phi_{x^j\epsilon_i}$ span $T^*(E^*)$ everywhere, except where all $y^j = 0$. This shows that he differentials of the linear functions ϕ_σ span the cotangent spaces to E^* everywhere, except along the zero section $M \subseteq E^*$. Hence, there can be at most one bivector field $\pi \in \mathfrak{X}^2(E^*)$ such that

$$\pi(\mathrm{d}\phi_\sigma, \mathrm{d}\phi_\tau) = \phi_{[\sigma,\tau]} \tag{17}$$

(so that the corresponding bracket $\{\cdot,\cdot\}$ satisfies (15)). To show its existence, define 'structure functions' $c^k_{ij} \in C^\infty(U)$ by $[\epsilon_i, \epsilon_j] = \sum_k c^k_{ij}\epsilon_k$, and let $\mathsf{a}_i = \mathsf{a}(\epsilon_i) \in \mathfrak{X}(U)$. Letting y_i be the coordinates on $(\mathbb{R}^n)^*$ corresponding to the basis, one finds that

$$\pi = \frac{1}{2}\sum_{ijk} c^k_{ij}\, y_k \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} + \sum_i \frac{\partial}{\partial y_i} \wedge \mathsf{a}_i \tag{18}$$

is the unique bivector field on $E^*|_U = U \times (\mathbb{R}^n)^*$ satisfying (17). (Evaluate the two sides on $\mathrm{d}\phi_\sigma, \mathrm{d}\phi_\tau$ for $\sigma = f\epsilon_i$ and $\tau = g\epsilon_j$.) This proves the existence of $\pi \in \mathfrak{X}^2(E^*)$. The Jacobi identity for $\{\cdot,\cdot\}$ holds true since it is satisfied on linear functions, by the Jacobi identity for $\Gamma(E)$.

Conversely, suppose $p\colon V \to M$ is a vector bundle with a linear Poisson structure π. Let $E = V^*$ be the dual bundle. We define the Lie bracket on sections and the anchor $\mathsf{a}\colon E \to TM$ by (15) and (16). This is well-defined: for instance, since ϕ_σ and p^*f have homogeneity 1 and 0 respectively, their Poisson bracket is homogeneous of degree $1 + 0 - 1 = 0$. Also, it is straightforward to check that $\mathsf{a}(\sigma)$ is a vector field, and that the map $\sigma \mapsto \mathsf{a}(\sigma)$ is $C^\infty(M)$-linear. The Jacobi identity for the bracket $[\cdot,\cdot]$ follows from that of the Poisson bracket, while the Leibnitz rule (13) for the anchor a follows from the derivation property of the Poisson bracket, as follows:

$$\begin{aligned}\phi_{[\sigma,f\tau]} &= \{\phi_\sigma, \phi_{f\tau}\}\\ &= \{\phi_\sigma, (p^*f)\,\phi_\tau\}\\ &= p^*(\mathsf{a}(\sigma)f)\,\phi_\tau + (p^*f)\,\phi_{[\sigma,\tau]}.\end{aligned}$$ □

As a simple (if unsurprising) consequence of this result, we see that if $E_1 \to M_1$ and $E_2 \to M_2$ are two Lie algebroids, then the exterior direct sum $E_1 \times E_2 \to M_1 \times M_2$ is again a Lie algebroid. The corresponding Poisson manifold is the product of Poisson manifolds:

$$(E_1 \times E_2)^* = E_1^* \times E_2^*.$$

Note also that if E^- is the Lie algebroid with the opposite $\mathcal{LA}$-structure (that is, E^- is E as a vector space, but the Lie bracket on sections given by *minus* the bracket on E, and with minus the anchor of E), then

$$(E^-)^* = (E^*)^-$$

as vector bundles with linear Poisson structure, where the superscript $-$ on the right hand side signifies the opposite Poisson structure.

Example 2.4. Consider $E = TM$ as a Lie algebroid over M. In local coordinates, the sections of TM are of the form

$$\sigma = \sum_i a^i \frac{\partial}{\partial q^i},$$

with corresponding linear function $\phi_\sigma(q,p) = \sum_i a_i(q)p^i$. The Lie bracket with another such section $\tau = \sum_j b^j(q)\frac{\partial}{\partial q^j}$ is (as the usual Lie bracket of vector fields)

$$[\sigma,\tau] = \sum_k \Big(\sum_i a^i \frac{\partial b^k}{\partial q^i} - \sum_i b^i \frac{\partial a^k}{\partial q^i}\Big)\frac{\partial}{\partial q^k}$$

It corresponds to

$$\phi_{[\sigma,\tau]} = \sum_k \Big(\sum_i a^i \frac{\partial b^k}{\partial q^i} - \sum_i b^i \frac{\partial a^k}{\partial q^i}\Big)p_k = \sum_{ik}\Big(\frac{\partial \phi_\sigma}{\partial p_i}\frac{\partial \phi_\tau}{\partial q^i} - \frac{\partial \phi_\sigma}{\partial q^i}\frac{\partial \phi_\tau}{\partial p_i}\Big).$$

The resulting Poisson structure on T^*M is the *opposite* of the standard Poisson structure.

Example 2.5. Given a Lie algebra action of $\mathfrak{g}$ on M, let $E = M \times \mathfrak{g}$ with dual bundle $E^* = M \times \mathfrak{g}^*$. The Poisson tensor on E^* is given by (18), with a_i the generating vector fields for the action.

Example 2.6. For a principal G-bundle $P \to M$, we obtain a linear Poisson structure on $(TP/G)^*$. This is called by Sternberg [**?**] and Weinstein [**?**] the 'phase space of a classical particle in a Yang-Mills field'. It may be identified with T^*P/G, with the Poisson structure induced from the opposite of the standard Poisson structure on T^*P.

Example 2.7. For the Lie algebroid E associated to a hypersurface $N \subseteq M$, with local coordinates $x^1,\ldots,x^k$ so that N is given by the vanishing of the k-th coordinate, we have $\epsilon_i = \frac{\partial}{\partial x^i}$ for

$i < k$ and $\epsilon_k = x^k \frac{\partial}{\partial x^k}$ as a basis for the sections of E. Denote by $y_1, \ldots, y_n$ the corresponding linear functions. The Poisson bracket reads

$$\{x^i, y_j\} = \delta^i_j \text{ for } i < k, \quad \{x^k, y_j\} = x^k \delta^k_j.$$

2.3. **The cotangent Lie algebroid of a Poisson manifold.** As a particularly important example, let (M, π) be a Poisson manifold. As we saw, the tangent bundle $V = TM$ inherits a Poisson structure π_{TM} such that $\{f_T, g_T\}_{TM} = \{f, g\}_T$ for all f, g. The functions f_T are homogeneous of degree 1, hence π_{TM} is homogeneous of degree -1. That is, π_{TM} is a linear Poisson structure, and hence determines a Lie algebroid structure on the dual bundle T^*M. It is common to use the notation $T^*_\pi M$ for the cotangent bundle with this Lie algebroid structure. From $\{f_T, g_V\} = \{f, g\}_V$ we see that the anchor map satisfies $\mathsf{a}(df) = X_f = \pi^\sharp(\mathrm{d}f)$. That is,

$$\mathsf{a} = \pi^\sharp \colon T^*_\pi M \to TM.$$

Since $\phi_{\mathrm{d}f} = f_T$ for $f \in C^\infty(M)$, the bracket on sections is such that

$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\}$$

for all $f, g \in C^\infty(M)$. The extension to 1-forms is uniquely determined by the Leibnitz rule, and is given by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}\pi(\alpha, \beta).$$

This Lie bracket on 1-forms of a Poisson manifold was first discovered by Fuchssteiner [19].

2.4. **Lie algebroid comorphisms.** As we saw, linear Poisson structures on vector bundles $V \to M$ correspond to Lie algebroid structures on $E = V^*$. One therefore expects that the category of vector bundles with linear Poisson structures should be the same as the category of Lie algebroids. This turns out to be true, but we have to specify what kind of morphisms we are using.

The problem is that a vector bundle map $V_1 \to V_2$ does not dualize to a vector bundle map $E_1 \to E_2$ for $E_i = V_i^*$ (unless the map on the base is a diffeomorphism). We are thus forced to allow more general kinds of vector bundle morphisms, either for $V_1 \to V_2$ (if we insist that $E_1 \to E_2$ is an actual vector bundle map), or for $E_1 \to E_2$ (if we insist that $V_1 \to V_2$ is an actual vector bundle map. Both options are interesting and important, and lead to the notions of Lie algebroid morphisms and Lie algebroid comorphisms, respectively.

Definition 2.8. A *vector bundle comorphism*, depicted by a diagram

$$\begin{array}{ccc} E_1 & \overset{\Phi_E}{\dashrightarrow} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \underset{\Phi_M}{\longrightarrow} & M_2 \end{array}$$

is given by a base map $\Phi_M \colon M_1 \to M_2$ together with a family of linear maps (going in the 'opposite' direction)

$$\Phi_E \colon (E_2)_{\Phi_M(m)} \to (E_1)_m$$

depending smoothly on m, in the sense that the resulting map $\Phi_M^* E_2 \to E_1$ is smooth.

Given such a vector bundle comorphism, one obtains a pullback map on sections,

$$\Phi_E^*\colon \Gamma(E_2) \to \Gamma(E_1) \tag{19}$$

which is compatible with the pullback of functions on M. Comorphisms can be composed in the obvious way, hence one obtains a category $\mathcal{VB}^\vee$ the category of vector bundles and vector bundle morphisms.

Remark 2.9. Letting $\mathcal{VS}$ be the category of vector spaces, and $\mathcal{VS}^{\mathrm{op}}$ the opposite category, one has the isomorphism $\mathcal{VS} \xrightarrow{\cong} \mathcal{VS}^{\mathrm{op}}$ taking a vector space to its dual space. Taking the opposite category ('reversing arrows') ensures that this is a covariant functor. Similarly, taking a vector bundle to its dual is an isomorphism of categories $\mathcal{VB} \xrightarrow{\cong} \mathcal{VB}^\vee$. In this sense, the introduction of $\mathcal{VB}^\vee$ may appear pointless. It becomes more relevant if the vector bundles have additional structure, which is not so easy to dualize.

Definition 2.10. Let $E_1 \to M_1$ and $E_2 \to M_2$ be Lie algebroids. A *Lie algebroid comorphism* $\Phi_E\colon E_1 \dashrightarrow E_2$ is a vector bundle comorphism such that

(i) the pullback map (19) preserves brackets,

(ii) The anchor maps satisfy
$$\mathsf{a}_1(\Phi_E^*\sigma) \sim_{\Phi_M} \mathsf{a}_2(\sigma)$$
(Φ_M-related vector fields).

We denote by $\mathcal{LA}^\vee$ the category of Lie algebroids and Lie algebroid comorphisms.

The second condition means that we have a commutative diagram

$$\begin{array}{ccc} E_1 & \overset{\Phi_E}{\dashrightarrow} & E_2 \\ {\scriptstyle \mathsf{a}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{a}_2} \\ TM_1 & \underset{T\Phi_M}{\longrightarrow} & TM_2 \end{array}$$

Note that this condition (ii) is not automatic. For instance, take $M_1 = M_2 = M$, with Φ_M the identity map, let $E_2 = TM$ the tangent bundle and let $E_1 = 0$ the zero Lie algebroid. Take $X \in \Gamma(TM)$ be a non-zero vector field. There is a unique comorphism $\Phi_E\colon 0 \dashrightarrow E$ covering $\Phi_M = \mathrm{id}_M$; the pull-back map on sections is the zero map, and in particular preserves brackets. But the condition (ii) would tell us $0 \sim_{\mathrm{id}_M} X$, i.e. $X = 0$.

Example 2.11. Let M be a manifold, and $\mathfrak{g}$ a Lie algebra. A comorphism of Lie algebroids $TM \dashrightarrow \mathfrak{g}$ is the same as a Lie algebra action of $\mathfrak{g}$ on M. In this spirit, a comorphism from TM to a general Lie algebroid E may be thought of as a Lie algebroid action of E on M.

Remark 2.12. On the open set of all $m \in M_1$ where the pullback map $\Phi_E^*\colon (E_2)_{\Phi(m)} \to (E_1)_m$ is non-zero, condition (ii) is automatic. To see this let σ, τ be sections of E_2, and $f \in C^\infty(M_2)$. Then $\Phi^*[\sigma, f\tau] = [\Phi^*\sigma, (\Phi^*f)\Phi^*\tau]$. Expanding using the Leibnitz rule, nd cancelling like terms, one arrives at the formula

$$\big(\Phi^*(\mathsf{a}_2(\sigma)f) - \mathsf{a}(\sigma_1)(\Phi^*f)\big)\ \Phi^*\tau = 0.$$

This shows that $\Phi^*(\mathsf{a}_2(\sigma)f) = \mathsf{a}(\sigma_1)(\Phi^*f)$ at all those points $m \in M_1$ where $\Phi^*\tau|_m \neq 0$ for some $\tau \in \Gamma(E_2)$.

Now, let $\mathcal{VB}_{Poi}$ be the category of vector bundles with linear Poisson structures; morphisms in this category are vector bundle maps that are also Poisson maps. (It is tempting to call these 'Poisson vector bundles', but unfortunately that terminology is already taken.) The following result shows that there is an isomorphism of categories

$$\mathcal{VB}_{Poi} \xrightarrow{\cong} \mathcal{LA}^\vee.$$

Proposition 2.13. *Let $E_1 \to M_1$ and $E_2 \to M_2$ be two Lie algebroids. A vector bundle comorphism $\Phi_E \colon E_1 \dashrightarrow E_2$ is a Lie algebroid comorphism if and only if the dual map $\Phi_{E^*} \colon E_1^* \to E_2^*$ is a Poisson map.*

Proof. To simplify notation, we denote all the pull-back maps $\Phi_M^*, \Phi_E^*, \Phi_{E^*}^*$ by Φ^*. For any $\mathcal{VB}$-comorphism $\Phi_E \colon E_1 \dashrightarrow E_2$, and any $\sigma \in \Gamma(E_2)$, we have that

$$\phi_{\Phi^*\sigma} = \Phi^* \phi_\sigma. \tag{20}$$

Given sections $\sigma, \tau \in \Gamma(E_2)$ and a function $f \in C^\infty(M_2)$, we have

$$\phi_{\Phi^*[\sigma,\tau]} = \Phi^* \phi_{[\sigma,\tau]} = \Phi^*\{\phi_\sigma, \phi_\tau\}, \tag{21}$$

$$\phi_{[\Phi^*\sigma,\Phi^*\tau]} = \{\phi_{\Phi^*\sigma}, \phi_{\Phi^*\tau}\} = \{\Phi^*\phi_\sigma, \Phi^*\phi_\tau\}, \tag{22}$$

and

$$p_1^*\Phi^*(\mathsf{a}_2(\sigma)f) = \Phi^* p_2^*(\mathsf{a}_2(\sigma)f) = \Phi^*\{\phi_\sigma, p_2^* f\}, \tag{23}$$

$$p_1^*\big(\mathsf{a}_1(\Phi^*\sigma)(\Phi^* f)\big) = \{\phi_{\Phi^*\sigma},\, p_1^*\Phi^* f\} = \{\Phi^*\phi_\sigma, \Phi^* p_2^* f\} \tag{24}$$

Here we have only used (20), and the description of the Lie algebroid structures of E_1, E_2 in terms of the Poisson structures on E_1^*, E_2^*, see (15) and (16).

Φ_E being an $\mathcal{LA}$-morphism is equivalent to the equality of the left hand sides of equations (21), (22) and equality of the left hand sides of equations (23), (24), while Φ_{E^*} being a Poisson map is equivalent to the equality of the corresponding right hand sides. □

2.5. **Lie subalgebroids and $\mathcal{LA}$-morphisms.** To define Lie algebroid morphisms $F_E \colon E_1 \to E_2$, we begin with the case of injective morphisms, i.e. subbundles.

Definition 2.14. Let $E \to M$ be a Lie algebroid, and $F \subseteq E$ a vector subbundle along $N \subseteq M$. Then F is called a *Lie subalgebroid* if it has the following properties:

- If $\sigma, \tau \in \Gamma(E)$ restrict over N to sections of F, then so does their bracket $[\cdot,\cdot]$,
- $\mathsf{a}(F) \subseteq TN$.

As the name suggests, a Lie subalgebroid is itself a Lie algebroid:

Proposition 2.15. *if $F \subseteq E$ is a sub-Lie algebroid along $N \subseteq M$, then F acquires a Lie algebroid structure, with anchor the restriction of $\mathsf{a} \colon E \to TN$, and with the unique bracket such that*

$$[\sigma|_N, \tau|_N] = [\sigma, \tau]|_N$$

whenever $\sigma|_N, \tau|_N \in \Gamma(F)$.

Proof. To show that this bracket is well-defined, we have to show that $[\sigma,\tau]|_N = 0$ whenever $\tau|_N = 0$. (In other words, the sections vanishing along N are an ideal in the space of sections of E that restrict to sections of N.) Write $\tau = \sum_i f_i\tau_i$ where $f_i \in C^\infty(M)$ vanish on N. Then

$$[\sigma,\tau]\Big|_N = \sum_i f_i|_N\,[\sigma,\tau_i]|_N + (\mathsf{a}(\sigma)f_i)|_N\ \tau_i|_N = 0$$

where we used that $\mathsf{a}(\sigma)f_i = 0$, since $\mathsf{a}(\sigma)$ is tangent to N and the f_i vanish on N. □

Here is one typical example of how Lie subalgebroids arise:

Proposition 2.16. *Let $E \to M$ be a Lie algebroid, on which a compact Lie group G acts by Lie algebroid automorphisms. Then the fixed point set $E^G \subseteq E$ is a Lie subalgebroid along $M^G \subseteq M$.*

Proof. Recall first that since G is compact, the fixed point set M^G is a submanifold, and $E^G \to M^G$ is a vector subbundle. By equivariance, $\mathsf{a}(E^G) \subseteq (TM)^G = T(M^G)$. Let $\Gamma(E)^G$ be the G-invariant sections. The restriction of such a section to M^G is a section of E^G, and the resulting map

$$\Gamma(E)^G \to \Gamma(E^G)$$

is surjective. (Given a section of E^G, we can extend extend to a section of E, and than achieve G-invariance by averaging.) But the bracket of G-invariant sections of E is again G-invariant, and hence restricts to a section of E^G. □

Proposition 2.17. *Let $E \to M$ be a Lie algebroid, and $N \subseteq M$ a submanifold. Suppose that $\mathsf{a}^{-1}(TN)$ is a smooth subbundle of E. Then $\mathsf{a}^{-1}(TN) \subseteq E$ is a Lie subalgebroid along $N \subseteq M$.*

Proof. This follows from the fact that $\mathsf{a}\colon \Gamma(E) \to \mathfrak{X}(M)$ is a Lie algebra morphism, and the Lie bracket of vector fields tangent to N is again tangent to N. □

Let $\iota\colon N \to M$ be the inclusion map. We think of

$$\iota^! E := \mathsf{a}^{-1}(TN)$$

as the proper notion of 'restriction' of a Lie algebroid. Two special cases:

(a) If a is tangent to N (i.e. $\mathsf{a}(E|_N) \subseteq TN$), then $\iota^! E = E|_N$ coincides with the vector bundle restriction.
(b) If a is transverse to N, then the restriction $\iota^! E$ is well-defined, with

$$\operatorname{rank}(\iota^! E) = \operatorname{rank}(E) - \dim(M) + \dim(N).$$

Note that $\iota^! TM = TN$.

More generally, we can sometimes define 'pull-backs' of Lie algebroids $E \to M$ under smooth maps $\Phi\colon N \to M$. Here, we assume that Φ is transverse to $\mathsf{a}\colon E \to TM$. Then the fiber product $E \times_{TN} TM \subseteq E \times TN$ is a well-defined subbundle along the graph of Φ, and is exactly the pre-image of $T\,\mathrm{Gr}(\Phi)$. It hence acquires a Lie algebroid structure. We let

$$\Phi^! E = E \times_{TN} TM \tag{25}$$

under the identification $\mathrm{Gr}(\Phi) \cong N$.

Remarks 2.18. (a) As a special case, $\Phi^!(TM) = TN$.
(b) If $\Phi = \iota$ is an embedding as a submanifold, then $\Phi^! E$ coincides with the 'restriction'.

(c) Under composition of maps, $(\Phi\circ\Psi)^! E=\Psi^!\Phi^! E$ (whenever the two sides are defined).

We can use Lie subalgebroids also to define *morphisms of Lie algebroids.*

Definition 2.19. Given Lie algebroids $E_1\to M_1$, $E_2\to M_2$, a vector bundle map

$$\Phi_E\colon E_1\to E_2$$

is a *Lie algebroid morphism* if its graph $\mathrm{Gr}(\Phi_E)\subseteq E_2\times E_1^-$ is a Lie subalgebroid along $\mathrm{Gr}(\Phi_M)$. The category of Lie algebroids and Lie algebroid morphisms will be denote $\mathcal{LA}$.

It will take some time and space (which we don't have right now) to get acquainted with this definition. At this point, we just note some simple examples:

(a) For any smooth map $\Phi\colon M_1\to M_2$, the tangent map $T\Phi\colon TM_1\to TM_2$ is an $\mathcal{LA}$-morphism.
(b) For any Lie algebroid E, the anchor map $\mathsf{a}\colon E\to TM$ is an $\mathcal{LA}$-morphism.
(c) Let E be a Lie algebroid over M, and $\Phi\colon N\to M$ a smooth map for which the pull-back $\Phi^! E$ is defined. Then the natural map $\Phi^! E\to E$ is a Lie algebroid morphism.
(d) Given $\mathfrak{g}$-actions on M_1, M_2, and an equivariant map $M_1\to M_2$, the bundle map

$$M_1\times\mathfrak{g}\to M_2\times\mathfrak{g}$$

is an $\mathcal{LA}$ morphism.
(e) If $\mathfrak{g}$ is a Lie algebra, then a Lie algebroid morphism $TM\to\mathfrak{g}$ is the same as a Maurer-Cartan form $\theta\in\Omega^1(M,\mathfrak{g})$, that is,

$$\mathrm{d}\theta+\frac{1}{2}[\theta,\theta]=0.$$

(See e.g. [???])

Having defined the category $\mathcal{LA}$, it is natural to ask what corresponds to it on the dual side, in terms of the linear Poisson structures on vector bundles. The answer will have to wait until we have the notion of a *Poisson morphism.*

3. Submanifolds of Poisson manifolds

Given a Poisson manifold (M,π), there are various important types of submanifolds.

3.1. **Poisson submanifolds.** A submanifold $N\subseteq M$ is called a *Poisson submanifold* if the Poisson tensor π is everywhere tangent to N, in the sense that $\pi_n\in\wedge^2 T_nN\subseteq\wedge^2T_nM$. Taking the restrictions pointwise defines a bivector field $\pi_N\in\mathfrak{X}^2(N)$, with the property that

$$\pi_N\sim_j\pi$$

where $j\colon N\to M$ is the inclusion. The corresponding Poisson bracket $\{\cdot,\cdot\}_N$ is given by

$$\{j^*f,j^*g\}_N=j^*\{f,g\}.$$

The Jacobi identity for π_N follows from that for π. The Poisson submanifold condition can be expressed in various alternate ways.

Proposition 3.1. *The following are equivalent:*

(a) *N is a Poisson submanifold.*
(b) $\pi^\sharp(T^*M|_N)\subseteq TN$.

(c) $\pi^\sharp(\mathrm{ann}(TN)) = 0$.
(d) *All Hamiltonian vector fields* X_f, $f \in C^\infty(M)$ *are tangent to* N.
(e) *The space of functions* f *with* $f|_N = 0$ *are a Lie algebra ideal in* $C^\infty(M)$*, under the Poisson bracket.*

Proof. It is clear that (a),(b),(c), are equivalent. The equivalence of (b) and (d) follows since for all $m \in M$, the range $\mathrm{ran}(\pi_m^\sharp)$ is spanned by the Hamiltonian vector fields X_f. Furthermore, if (d) holds, then the functions vanishing on N are an ideal since $g|_N = 0$ implies $\{f, g\}|_N = X_f(g)|_N = 0$ since X_f is tangent to N. This gives (e). Conversely, if (e) holds, so that $\{f, g\}|_N = 0$ whenever $g|_N = 0$, it follows that $\langle \mathrm{d}g, X_f\rangle|_N = X_f(g)|_N = 0$ whenever $g|_N = 0$. The differentials $\mathrm{d}g|_N$ for $g|_N = 0$ span $\mathrm{ann}(TN)$, hence this means that $X_f|_N \in \Gamma(TN)$, which gives (d). □

Examples 3.2. (a) If $\chi \in C^\infty(M)$ is a Casimir function, then all the smooth level sets of χ are Poisson submanifolds. Indeed, since $X_f\chi = \{f, \chi\} = 0$ shows that the Hamiltonian vector fields are tangent to the level sets of χ.
(b) As a special case, if $\mathfrak{g}$ is a Lie algebra with an invariant metric, defining a metric on the dual space, then the set of all $\mu \in \mathfrak{g}^*$ such that $||\mu|| = R$ (a given constant) is a Poisson submanifold.
(c) For any Poisson manifold M, and any $k \in \mathbb{N} \cup \{0\}$ one can consider the subset $M_{(2k)}$ of elements where the Poisson structure has given rank $2k$. If this subset is a submanifold, then it is a Poisson submanifold. For example, if $M = \mathfrak{g}^*$ the components of the set of elements with given dimension of the stabilizer group G_μ are Poisson submanifolds.

3.2. **Symplectic leaves.** As mentioned above, the subspaces

$$\mathrm{ran}(\pi_m^\sharp) \subseteq T_mM$$

are spanned by the Hamiltonian vector fields. The subset $\mathrm{ran}(\pi^\sharp) \subseteq TM$ is usually a *singular distribution*, since the dimensions of the subspaces $\mathrm{ran}(\pi_m^\sharp)$ need not be constant. It doesn't prevent us from considering leaves:

Definition 3.3. A maximal connected injectively immersed submanifold $S \subseteq M$ of a connected manifold S is called a *symplectic leaf* of the Poisson manifold (M, π) if

$$TS = \pi^\sharp(T^*M|_S).$$

By definition, the symplectic leaves are Poisson submanifolds. Since $\pi_S^\sharp$ is onto TS everywhere, this Poisson structure is non-degenerate, that is, it corresponds to a symplectic 2-form ω_S with $\omega_S^\flat = -(\pi_S^\sharp)^{-1}$. The Hamiltonian vector fields X_f are a Lie subalgebra of $\mathfrak{X}(M)$, since $[X_{f_1}, X_{f_2}] = X_{\{f_1,f_2\}}$. If the distribution spanned by these vector fields has constant rank, then we can use Frobenius' theorem to conclude that the distribution is integrable: Through every point there passes a unique symplectic leaf. However, in general Frobenius's theorem is not applicable since the rank may jump. Nevertheless, we have the following fundamental result:

Theorem 3.4. [39] *Every point* m *of a Poisson manifold* M *is contained in a unique symplectic leaf* S.

Thus, M has a decomposition into symplectic leaves. One can prove this result by obtaining the leaf through a given point m as the 'flow-out' of m under all Hamiltonian vector fields, and this is Weinstein's argument in [39]. We will not present this proof, since we will later obtain this result as a corollary to the *Weinstein splitting theorem* for Poisson structures.

Example 3.5. For $M = \mathfrak{g}^*$ the dual of a Lie algebra $\mathfrak{g}$, the symplectic leaves are the orbits of coadjoint action G on $\mathfrak{g}^*$. Here G is any connected Lie group integrating $\mathfrak{g}$.

Example 3.6. For a Poisson structure π on a 2-dimensional manifold M, let $Z \subseteq M$ be its set of zeros, i.e. points $m \in M$ where $\pi_m = 0$. Then the 2-dimensional symplectic leaves of π are the connected components of $M - Z$, while the 0-dimensional leaves are the points of Z.

Remark 3.7. The Poisson structure is uniquely determined by its symplectic leaves, and can sometimes be described in these terms. Suppose for instance M is a manifold with a (regular) foliation, and with a 2-form ω whose pull-back to every leaf of the foliation is closed and non-degenerate, i.e., symplectic. Then M becomes a Poisson structure with the given foliation as its symplectic foliation. The Poisson bracket of two functions on M may be computed leafwise; it is clear that the result is again a smooth function on M. (See Vaisman [**?**, Proposition 3.6].)

Remark 3.8. Since the dimension of the symplectic leaf S through $m \in M$ equals the rank of the bundle map $\pi_m^\sharp\colon T_m^*M \to T_mM$, we see that this dimension is a lower semi-continuous function of m. That is, the nearby leaves will have dimension greater than or equal to the dimension of S. In particular, if π has maximal rank $2k$, then the union of $2k$-dimensional symplectic leaves is an open subset of M.

3.3. **Coisotropic submanifolds.**

Lemma 3.9. *The following are equivalent:*

(a) $\pi^\sharp(\mathrm{ann}(TN)) \subseteq TN$
(b) *For all f such that $f|_N = 0$, the vector field X_f is tangent to N.*
(c) *The space of functions f with $f|_N = 0$ are a Lie subalgebra under the Poisson bracket.*
(d) *The annihilator* $\mathrm{ann}(TN)$ *is a Lie subalgebroid of the cotangent Lie algebroid.*

Proof. Equivalence of (a) and (b) is clear, since $\mathrm{ann}(TN)$ is spanned by $\mathrm{d}f|_N$ such that $f|_N = 0$. If (b) holds, then $f|_N = 0,\ g|_N = 0$ implies $\{f, g\}|_N = X_f(g)|_N = 0$. Conversely, if (c) holds, and $f|_N = 0$, then X_f is tangent to N since for all g with $g|_N = 0$, $X_f(g)|_N = \{f, g\}|_N = 0$. If $\mathrm{ann}(TN)$ is a Lie subalgebroid of $T^*_\pi M$, then in particular its image under the anchor is tangent to N, which is (a). Conversely, if the equivalent conditions (a),(c), hold, then $\mathrm{ann}(TN)$ is a Lie subalgebroid because its space of sections is generated by $\mathrm{d}f$ with $f|_N = 0$, and $[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\}$. □

A submanifold $N \subseteq M$ is called a *coisotropic submanifold* if it satisfies any of these equivalent conditions. Clearly,

$$\{\text{open subsets of symplectic leaves }\} \subseteq \{\text{ Poisson submanifolds }\} \subseteq \{\text{ coisotropic submanifolds }\}.$$

Remark 3.10. By (d), we see in particular that for any coisotropic submanifold N, the normal bundle

$$\nu(M, N) = TM|_N/TN = \mathrm{ann}(TN)^*$$

inherits a linear Poisson structure $\pi_{\nu(M,N)}$. By the tubular neighborhood theorem, there is an diffeomorphism of open neighborhoods of N inside $\nu(M, N)$ and inside M. Hence, $\nu(M, N)$ wit this linear Poisson structure is thought of as the *linear approximation* of the Poisson structure π_M along N. As special cases, we obtain linear Poisson structures on the normal bundles of Poisson submanifolds, and in particular on normal bundles of symplectic leaves.

Remark 3.11. There are also notions of *Lagrangian submanifold* and *isotropic submanifold* of a Poisson manifold, defined by the conditions that $\pi^\sharp(\mathrm{ann}(TN)) = TN$ and $\pi^\sharp(\mathrm{ann}(TN)) \supseteq TN$. However, it seems that these notions rarely appears in practice.

Example 3.12. Let $E \to M$ be a Lie algebroid, so that $E^* \to M$ has a linear Poisson structure. For any submanifold $N \subseteq M$, the restriction $E^*|_N$ is a coisotropic submanifold. Indeed, the conormal bundle to $E^*|_N$ is spanned by $\mathrm{d}(p^*f)$ such that $f|_N = 0$, but $\{p^*f, p^*g\}$ for *all* functions on M.

Example 3.13. If (M, ω) is a symplectic manifold, regarded as a Poisson manifold, then the notions of coisotropic in the Poisson sense coincides with that in the symplectic sense. Indeed, in this case $\pi^\sharp(\mathrm{ann}(TN))$ equals the *ω-orthogonal space* TN^ω, consisting of $v \in TM$ such that $\omega^\flat(v) \in \mathrm{ann}(TN)$. But $TN^\omega \subseteq TN$ is the coisotropic condition in symplectic geometry. For a Poisson manifold, it follows that the intersection of coisotropic submanifolds with symplectic leaves are coisotropic.

Theorem 3.14 (Weinstein)**.** *A smooth map* $\Phi\colon M_1 \to M_2$ *of Poisson manifolds* (M_1, π_1) *and* (M_2, π_2) *is a Poisson map if and only if its graph* $\mathrm{Gr}(\Phi) \subseteq M_2 \times M_1^-$ *is a coisotropic submanifold. (Here* M_1^- *is* M_1 *with the Poisson structure* $-\pi_1$*.*

Proof. The condition that $\pi_1 \sim_\Phi \pi_2$ means that for covectors $\alpha_1 \in T^*_m M_1$, $\alpha_2 \in T^*_{\Phi(m)} M_2$,

$$\alpha_1 = \Phi^*\alpha_2 \ \Rightarrow\ \pi_1^\sharp(\alpha_1) \sim_\Phi \pi_2^\sharp(\alpha_2).$$

But $\alpha_1 = \Phi^*\alpha_2$ is equivalent to $(\alpha_2, -\alpha_1) \in \mathrm{ann}(T\,\mathrm{Gr}\,\Phi)$, while $\pi_1^\sharp(\alpha_1) \sim_\Phi \pi_2^\sharp(\alpha_2)$ is equivalent to $(\pi_2^\sharp(\alpha_2), \pi_1^\sharp(\alpha_1)) \in \mathrm{Gr}(T\Phi) = T\,\mathrm{Gr}\,\Phi$. ☐

Theorem 3.14 is the Poisson counterpart to a well-known result from symplectic geometry: If M_1, M_2 are symplectic manifolds, then a diffeomorphism $\Phi\colon M_1 \to M_2$ is symplectomorphism if and only if its graph $\mathrm{Gr}(\Phi) \subseteq M_2 \times M_1^-$ is a Lagrangian submanifold. This leads to the idea of viewing Lagrangian submanifolds of $M_2 \times M_1^-$ as 'generalized morphisms' from M_1 to M_2, and idea advocated by Weinstein's notion of a symplectic category [40]. In a similar fashion, Weinstein defined:

Definition 3.15. Let M_1, M_2 be Poisson manifolds. A *Poisson relation* from M_1 to M_2 is a coisotropic submanifold $N \subseteq M_2 \times M_1^-$, where M_1^- is M_1 equipped with the opposite Poisson structure.

Poisson relations are regarded as generalized 'morphisms'. We will thus write

$$N\colon M_1 \dashrightarrow M_2$$

for a submanifold $N \subseteq M_2 \times M_1$ thought of as such a 'morphism'. However, 'morphism' is in quotes since relations between manifolds cannot always be composed: Given submanifolds $N \subseteq M_2 \times M_1$ and $N' \subseteq M_3 \times M_2$, the composition $N' \circ N$ need not be a submanifold.

Definition 3.16. We say that two relations $N\colon M_1 \dashrightarrow M_2$ and $N'\colon M_2 \dashrightarrow M_3$ (given by submanifolds $N \subseteq M_2 \times M_1$ and $N' \subseteq M_3 \times M_2$) *have clean composition* if

(i) $N' \circ N$ is a submanifold, and
(ii) $T(N' \circ N) = TN' \circ TN$ *fiberwise.*

By (ii), we mean that for all $m_i \in M_i$ with $(m_3, m_2) \in N'$ and $(m_2, m_1) \in N$, we have that

$$T_{(m_3,m_1)}(N' \circ N) = T_{(m_3,m_2)}N' \circ T_{(m_2,m_1)}N.$$

We stress that there are various versions of 'clean composition' in the literature, and the condition here is weaker (but also simpler) than the one found in [**?**] or [41, Definition (1.3.7)]. Our goal is to show that a clean composition of Poisson relations is again a Poisson relation.

We will need some facts concerning the composition of linear relations. For any linear relation $R\colon V_1 \dashrightarrow V_2$, given by a subspace $R \subseteq V_2 \times V_1$, define a relation $R^\diamond\colon V_1^* \dashrightarrow V_2^*$ of the dual spaces, by

$$R^\diamond = \{(\alpha_2, \alpha_1) \in V_2^* \times V_1^* |\ (\alpha_2, -\alpha_1) \in \operatorname{ann}(R)\}. \tag{26}$$

For example, if $\Delta_V \subseteq V \times V$ is the diagonal (corresponding to the identity morphism), then $\Delta_V^\diamond = \Delta_{V^*}$. The main reason for including a sign change is the following property under composition of relations.

Lemma 3.17. *(Cf.* [25, Lemma A.2]*) For linear relations $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$, with composition $R' \circ R\colon V_1 \dashrightarrow V_3$, we have that*

$$(R' \circ R')^\diamond = (R')^\diamond \circ R^\diamond\colon V_1^* \dashrightarrow V_3^*.$$

Proof. It is a well-known fact in linear symplectic geometry that the composition of linear Lagrangian relations in symplectic vector spaces is again a Lagrangian relation. (No transversality assumptions are needed.) We will apply this fact, as follows. If V is a vector space, let $W = T^*V = V \oplus V^*$ with its standard symplectic structure, and let W^- be the same space with the opposite symplectic structure. If $S \subseteq V$ is any subspace, then $S \oplus \operatorname{ann}(S)$ is Lagrangian in W. In our situation, let $W_i = V_i \oplus V_i^*$. Then

$$R \oplus R^\diamond \subseteq W_2 \oplus W_1^-, \quad R' \oplus (R')^\diamond \subseteq W_3 \oplus W_2^-$$

are Lagrangian relations, hence so is their composition $(R' \circ R) \oplus ((R')^\diamond \circ R^\diamond)$. This means that $(R' \circ R)^\diamond = (R')^\diamond \circ R^\diamond$. ☐

Put differently, the Lemma says that

$$\operatorname{ann}(R' \circ R) = \big\{(\alpha_3, -\alpha_1) \big| \exists \alpha_2\colon (\alpha_3, -\alpha_2) \in \operatorname{ann}(R'),\ (\alpha_2, -\alpha_1) \in \operatorname{ann}(R)\big\}. \tag{27}$$

The following result was proved by Weinstein [41] under slightly stronger assumptions.

Proposition 3.18 (Weinstein)**.** *Let $N\colon M_1 \dashrightarrow M_2$ and $N'\colon M_2 \dashrightarrow M_3$ be Poisson relations with clean composition $N' \circ N\colon M_1 \dashrightarrow M_3$. Then $N' \circ N$ is again a Poisson relation.*

Proof. We have to show that $N' \circ N$ is a coisotropic submanifold. Let

$$(\alpha_3, -\alpha_1) \in \operatorname{ann}(T(N' \circ N))$$

be given, with base point $(m_3, m_1) \in N' \circ N$. Choose $m_2 \in M_2$ with $(m_3, m_2) \in N'$ and $(m_2, m_1) \in N$. Since

$$T_{(m_3,m_1)}(N' \circ N) = T_{(m_3,m_2)}N' \circ T_{(m_2,m_1)}N,$$

Equation (27) gives the existence of $\alpha_2 \in T^*_{m_2}M_2$ such that $(\alpha_3, -\alpha_2) \in \mathrm{ann}(TN')$ and $(\alpha_2, -\alpha_1) \in \mathrm{ann}(TN)$. Letting $v_i = \pi_i^\sharp(\alpha_i)$ we obtain $(v_3, v_2) \in TN'$ and $(v_2, v_1) \in TN$, since N', N are coisotropic. Hence $(v_3, v_1) \in TN' \circ TN = T(N' \circ N)$, proving that $N' \circ N$ is coisotropic. □

Example 3.19. [**?**, Corollary (2.2.5)] Suppose $\Phi\colon M_1 \to M_2$ is a Poisson map, and $N \subseteq M_1$ is a coisotropic submanifold. Suppose $\Phi(N) \subseteq M_2$ is a submanifold, with

$$(T_m\Phi)(T_mN) = T_{\Phi(m)}(\Phi(N)) \tag{28}$$

for all $m \in M_1$. Then $\Phi(N)$ is a coisotropic submanifold. Indeed, this sat my be regarded as a composition of relations $\Phi(N) = \mathrm{Gr}(\Phi) \circ N$, and the assumptions given are equivalent to the clean composition assumption. Similarly, if $Q \subseteq M_2$ is a coisotropic submanifold, such that $\Phi^{-1}(Q)$ is a submanifold with $T_m(\Phi^{-1}(Q)) = (T_m\Phi)^{-1}(T_{\Phi(m)}Q)$, then $\Phi^{-1}(Q)$ is a submanifold.

If $R \subseteq M_2 \times M_1^-$ is a Poisson relation, we can consider the *transpose* (or inverse) Poisson relation

$$R^\top \subseteq M_1 \times M_2^-$$

consisting of all (m_1, m_2) such that $(m_2, m_1) \in R$. We may then define new relations $R^\top \circ R$ and $R \circ R^\top$, provided that clean composition assumptions are satisfied. As a special case, suppose $R = \mathrm{Gr}(\Phi)$ is the graph of a Poisson map $\Phi\colon M_1 \to M_2$. Then

$$R^\top \circ R = \{(m, m') \in M_1 \times M_1^- \mid \Phi(m) = \Phi(m')\} = M_1 \times_{M_2} M_1$$

(the fiber product of M_1 with itself over M_2). is coisotropic, provided that the composition is clean. The cleanness assumption is automatic if Φ is a submersion. In this case, one has a partial converse, which may be regarded as a criterion for *reducibility* of a Poisson structure.

Proposition 3.20 (Weinstein)**.** *Let $\Phi\colon M_1 \to M_2$ be a surjective submersion, where M_1 is a Poisson manifold. Then the following are equivalent:*

(a) *The Poisson structure on M_1* descends *to M_2. That is, M_2 has a Poisson structure such that Φ is a Poisson map.*
(b) *The fiber product $M_1 \times_{M_2} M_1^- \subseteq M_1 \times M_1^-$ is a coisotropic submanifold of $M_1 \times M_1^-$.*

Proof. One direction was discussed above. For the converse, suppose $S := M_1 \times_{M_2} M_1$ is coisotropic. To show that the Poisson structure descends, we have to show that functions of the form Φ^*f with $f \in C^\infty(M_2)$ form a Poisson subalgebra. For any such function f, note that $F = \mathrm{pr}_1^*\Phi^*f - \mathrm{pr}_2^*\Phi^*f \in C^\infty(M_1 \times M_1^-)$ vanishes on S. Given another function $f' \in C^\infty(M_2)$, with corresponding function F', we have that $\{F, F'\}$ vanishes on S. But the vanishing of

$$\{F, F'\} = \mathrm{pr}_1^*\{\Phi^*f, \Phi^*f'\} - \mathrm{pr}_2^*\{\Phi^*f, \Phi^*f'\}$$

on S means precisely that $\{\Phi^*f, \Phi^*f'\}$ is constant along the fibers of Φ. In other words, it lies in $\Phi^*(C^\infty(M_2))$. □

Remark 3.21. In [41], Weinstein also discussed the more general Marsden-Ratiu reduction procedure along similar lines.

3.4. **Applications to Lie algebroids.** Recall that $F \subseteq E$ is a Lie subalgebroid if and only if $\{\sigma \in \Gamma(E) |\ \sigma|_N \in \Gamma(F)\}$ is a Lie subalgebra, with $\{\sigma \in \Gamma(E) |\ \sigma|_N = 0\}$ as an ideal (the latter condition being equivalent to $\mathsf{a}(E) \subseteq TN$). In the dual picture,

$$\sigma|_N \in \Gamma(F) \ \Leftrightarrow\ \phi_\sigma \text{ vanishes on } \operatorname{ann}(F) \subseteq T^*M|_N$$

$$\sigma|_N = 0 \ \Leftrightarrow\ \phi_\sigma \text{ vanishes on } T^*M|_N.$$

Proposition 3.22. *Let E be a Lie algebroid, and $F \subseteq E$ a vector subbundle along $N \subseteq M$. Then F is a Lie subalgebroid if and only if $\operatorname{ann}(F) \subseteq E^*$ is a coisotropic submanifold.*

Proof. "$\Leftarrow$". Suppose that $\operatorname{ann}(F) \subseteq E^*$ is coisotropic. If $\sigma|_N \in \Gamma(F)$ and $f|_N = 0$, then ϕ_σ and p^*f vanish on $\operatorname{ann}(F)$, hence so does their Poisson bracket

$$\{\phi_\sigma, p^*f\} = p^*(\mathsf{a}(\sigma)(f)).$$

Hence $\mathsf{a}(\sigma)(f)|_N = 0$, which proves that $\mathsf{a}(\sigma)$ is tangent to N. Since σ was any section restricting to a section of N, this shows $\mathsf{a}(F) \subseteq TN$. Similarly, if σ, τ restrict to sections of F, then ϕ_σ, ϕ_τ vanish on $\operatorname{ann}(F)$, hence so does

$$\{\phi_\sigma, \phi_\tau\} = \phi_{[\sigma,\tau]}$$

which means that $[\sigma, \tau]$ restricts to a section of F. This shows that F is a Lie subalgebroid.

"$\Rightarrow$". Suppose F is a Lie subalgebroid. Then, for all σ, τ that restrict to sections of F, and all $f, g \in C^\infty(M)$ that restrict to zero on N, the Poisson brackets

$$\{\phi_\sigma, \phi_\tau\} = \phi_{[\sigma,\tau]},\ \{\phi_\sigma, p^*f\} = p^*(\mathsf{a}(\sigma)(f)),\quad \{p^*f, p^*g\} = 0$$

all restrict to 0 on $\operatorname{ann}(F)$. Since these functions generate the vanishing ideal of $\operatorname{ann}(F)$ inside $C^\infty(E^*)$, this shows that this vanishing ideal is a Lie subalgebra; that is, $\operatorname{ann}(N)$ is coisotropic. □

Remark 3.23. Note the nice symmetry:

- For a Poisson manifold (M, π), we have that $N \subseteq M$ is a coisotropic submanifold if and only if $\operatorname{ann}(TN) \subseteq T^*M$ is a Lie subalgebroid.
- For a Lie algebroid E, a a vector subbundle $F \subseteq E$ is a Lie subalgebroid if and only if $\operatorname{ann}(F) \subseteq E^*$ is a coisotropic submanifold.

Definition 3.24. We denote by $\mathcal{VB}^\vee_{Poi}$ the category of vector bundles with linear Poisson structures, with morphisms the vector bundle comorphisms that are also Poisson relations.

Proposition 3.25. *Let $E_1 \to M_1,\ E_2 \to M_2$ be Lie algebroids. Then $\Phi_E \colon E_1 \to E_2$ is a Lie algebroid morphism if and only if the dual comorphism $\Phi_{E^*} \colon E_1^* \dashrightarrow E_2^*$ is a Poisson relation. We conclude that there is an isomorphism of categories,*

$$\mathcal{VB}^\vee_{Poi} \xrightarrow{\cong} \mathcal{LA}.$$

Proof. By definition, Φ_E is an $\mathcal{LA}$-morphism if and only if its graph is a Lie subalgebroid. By Proposition 3.22, this is the case if and only if the dual comorphism Φ_{E^*} is a Poisson relation □

3.5. **Poisson-Dirac submanifolds.** Aside from the Poisson submanifolds, there are other classes of submanifolds of Poisson manifolds M, with naturally induced Poisson structures. For example, suppose a submanifold $N \subseteq M$ has the property that its intersection with every symplectic leaf of M is a symplectic submanifold of that leaf. Then one can ask if the resulting decomposition of N into symplectic submanifolds defines a Poisson structure on N. This is not automatic, as the following example shows.

Example 3.26. Let $M = \mathbb{R}^2 \times \mathbb{R}^3$ as a product of Poisson manifolds, where the first factor has the standard Poisson structure $\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$, and the second factor has the zero Poisson structure. Let $N \subseteq M$ be the image of the embedding

$$\mathbb{R}^3 \to M,\ (q,p,t) \mapsto (q,p,tq,tp,t).$$

Then N contains the symplectic leaf $\mathbb{R}^2 \times \{0\} \subseteq M$, but intersects all other leaves transversally. The resulting decomposition of N into a single 2-dimensional submanifold together with isolated points cannot correspond to a symplectic foliation. (Cf. Remark 3.8.) See Crainic-Fernandes [**?**, Section 8.2] for a similar example.

Definition 3.27. Let M be a Poisson manifold. A submanifold $N \subseteq M$ is called a *Poisson-Dirac submanifold* if every $f \in C^\infty(N)$ admits an extension $\widetilde{f} \in C^\infty(M)$ (i.e., $\widetilde{f}|_N = f$) for which $X_{\widetilde{f}}$ is tangent to N.

Note that in particular, every Poisson submanifold is a Poisson-Dirac submanifold.

Remark 3.28. Definition 3.27 follows Laurent-Gengoux, Pichereau and Vanhaecke, see [24, Section 5.3.2]. Cranic-Fernandes [**?**] use the term for any submanifold N with a Poisson structure π_N such that $\mathrm{ran}(\pi_N^\sharp) = \mathrm{ran}(\pi^\sharp) \cap TN$ everywhere.

An equivalent condition is the following:

Lemma 3.29. *$N \subseteq M$ is a Poisson-Dirac submanifold if and only if every 1-form $\alpha \in \Omega^1(N)$ is the pull-back of a 1-form $\widetilde{\alpha} \in \Omega^1(M)$ such that $\pi^\sharp(\widetilde{\alpha})$ is tangent to N.*

Proof. The direction "$\Rightarrow$" is obvious. For the other direction, we have to show that every $f \in C^\infty(M)$ admits an extension $\widetilde{f}$ whose hamiltonian vector field is tangent to N. By using a partition of unity, we may assume that f is contained in a submanifold chart of N. Thus suppose x^i, y^j are local coordinates so that N is given by $y^j = 0$. Let $\alpha = \mathrm{d}f$, and choose an extension $\widetilde{\alpha}$ as in the statement of the lemma. Then $\widetilde{\alpha}|_N$ has the form

$$\widetilde{\alpha}|_N = \mathrm{d}f + \sum_j c_j(x)\mathrm{d}y^j.$$

The formula

$$\widetilde{f}(x,y) = f(x) + \sum_j c_j(x) y^j.$$

defines an extension of f, and since $\mathrm{d}\widetilde{f}|_N = \widetilde{\alpha}|_N$ we have that $X_{\widetilde{f}} = \pi^\sharp(\widetilde{\alpha})$ is tangent to N. $\square$

Proposition 3.30. *If N is a Poisson-Dirac submanifold, then N inherits a Poisson structure via*

$$\{f,g\}_N = \{\widetilde{f},\widetilde{g}\}|_N$$

where $\widetilde{f}, \widetilde{g}$ are extensions of f, g whose Hamiltonian vector fields are tangent to N. In terms of bivector fields,

$$\pi_N(\alpha, \beta) = \pi(\widetilde{\alpha}, \widetilde{\beta})|_N \tag{29}$$

whenever $\widetilde{\alpha} \in \Omega^1(M)$ pulls back to $\alpha \in \Omega^1(N)$ and $\pi^\sharp(\widetilde{\alpha})$ is tangent to N, and similarly for $\widetilde{\beta}$. The symplectic leaves of N with respect to π_N are the components of the intersections of N with the symplectic leaves of M.

Proof. To show that the bracket is well-defined, we have to show that the right hand side vanishes if $\widetilde{g}|_N = 0$. But this follows from $\{\widetilde{f}, \widetilde{g}\}|_N = X_{\widetilde{f}}(\widetilde{g})|_N$ since $X_{\widetilde{f}}$ is tangent to N. The Jacobi identity for $\{\cdot,\cdot\}_N$ follows from that for $\{\cdot,\cdot\}$. The formula in terms of bivector fields reduces to the one in terms of brackets if the 1-forms are all exact. To show that it is well-defined in the general case, we have to show that the right hand vanishes if the pullback of $\widetilde{\beta}$ to N is zero, or equivalently if $\beta|_N$ takes values in $\mathrm{ann}(TN)$. But this is clear since

$$\pi(\widetilde{\alpha}, \widetilde{\beta})|_N = \langle \widetilde{\beta}|_N,\, \pi^\sharp(\widetilde{\alpha})|_N \rangle = 0$$

using that $\pi^\sharp(\widetilde{\alpha})|_N$ takes values in TN. From the formula in terms of 1-forms, we see that $\pi_N^\sharp(\alpha) = \pi^\sharp(\widetilde{\alpha})$, whenever the right hand side takes values in TN and $\widetilde{\alpha}$ pulls back to α. This shows that the range of $\pi_N^\sharp$ is exactly the intersection of TN with the range of $\pi^\sharp$. □

What are conditions to guarantee that a given submanifold is Poisson-Dirac? The vector field $\pi^\sharp(\widetilde{\alpha})$ is tangent to N if and only if $\widetilde{\alpha}|_N$ takes values in $(\pi^\sharp)^{-1}(TN)$. Hence, a necessary condition is that the pullback map $T^*M|_N \to T^*N$ restricts to a surjection $(\pi^\sharp)^{-1}(TN) \to T^*N$. The kernel of this map is $\mathrm{ann}(TN)$, hence the necessary condition reads as

$$T^*M|_N = \mathrm{ann}(TN) + (\pi^\sharp)^{-1}(TN). \tag{30}$$

Taking annihilators on both sides, this is equivalent to

$$TN \cap \pi^\sharp(\mathrm{ann}(TN)) = 0. \tag{31}$$

If this condition holds, then one obtains a pointwise bivector $\Pi_N|_m$ for all $m \in N$, defined by the pointwise version of (29). However, the collection of these pointwise bivector fields do not define a smooth bivector field, in general. For instance, in Example 3.26 the condition (31) is satisfied, but N is not Poisson-Dirac. A *sufficient* condition for N to be Poisson-Dirac is the following.

Proposition 3.31. *The submanifold $N \subseteq M$ is Poisson-Dirac if and only if the exact sequence*

$$0 \to \mathrm{ann}(TN) \to T^*M|_N \to T^*N \to 0, \tag{32}$$

admits a splitting $j\colon T^*N \to T^*M|_N$ whose image is contained in $(\pi^\sharp)^{-1}(TN)$. That is, N is Poisson-Dirac if and only if

$$T^*M|_N = \mathrm{ann}(TN) \oplus K$$

where K is a subbundle contained in $(\pi^\sharp)^{-1}(TN)$.

Proof. Suppose such a splitting $j\colon T^*N \to T^*M_N$ is given. Given $\alpha \in \Omega^1(N)$, let $\widetilde{\alpha} \in \Omega^1(M)$ be any extension of $j(\alpha) \in \Gamma(T^*M|_N)$. Then $\widetilde{\alpha}$ pulls back to α, and $\pi^\sharp(\widetilde{\alpha})$ is tangent to N. This shows that N is Poisson-Dirac. Conversely, suppose that N is Poisson Dirac. Given a local frame $\alpha_1, \ldots, \alpha_k$ for T^*N, we may choose lifts $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k$ as in Lemma 3.29. These lifts span a complement to $\mathrm{ann}(TN)$ in $T^*M|_N$, giving the desired splitting $j\colon T^*N \to T^*M|_N$ *locally.* But convex linear combinations of splittings are again splittings; and if these splittings take values in $(\pi^\sharp)^{-1}(TN)$, then so does their linear combination. Hence, we may patch the local splittings with a partition of unity to obtain a global splitting with the desired property. ☐

Remark 3.32. If $\pi^\sharp(\mathrm{ann}(TN))$ has constant rank, and zero intersection with TN, then N is a Poisson-Dirac submanifold.

Here is a typical example of a Poisson-Dirac submanifold.

Proposition 3.33 (Damianou-Fernandes)**.** *Suppose a compact Lie group G acts on a Poisson manifold M by Poisson diffeomorphisms. Then M^G is a Poisson-Dirac submanifold.*

Proof. We have a G-equivariant direct sum decomposition

$$T^*M|_{M^G} = \mathrm{ann}(TM^G) \oplus (T^*M)^G.$$

By equivariance of the anchor map, $\pi^\sharp((T^*M)^G) \subseteq (TM)^G = T(M^G)$ as required. ☐

Remark 3.34. In [42], Xu introduces a special type of Poisson-Dirac submanifolds which he called *Dirac submanifolds*, but were later renamed as *Lie-Dirac submanifolds*. We will return to this later. In the case of a compact group action, Fernandes-Ortega-Ratiu [16] prove that M^G is in fact a *Lie-Dirac submanifold* in the sense of Xu [42].

Remark 3.35. Given splitting of the exact sequence (32), with image $K \subseteq T^*M|_N$ such that $\pi^\sharp(K) \subseteq TN$, the restriction of the Poisson tensor decomposes as $\pi|_N = \pi_N + \pi_K$ where $\pi_N \in \Gamma(\wedge^2 TN)$ and $\pi_K \in \Gamma(\wedge^2 K)$. As shown in [42, Lemma 2.5], having such a decomposition already implies that π_N is a Poisson tensor.

Remark 3.36. Crainic-Fernandes [**?**] give an example showing that it is possible for a submanifold of a Poisson manifold M to admit a Poisson structure π_N with $\mathrm{ran}(\pi_N^\sharp) = \mathrm{ran}(\pi^\sharp) \cap TN$, without admitting a splitting of (32).

3.6. **Cosymplectic submanifolds.** An important special case of Poisson-Dirac submanifold is the following.

Definition 3.37. A submanifold $N \subseteq M$ is called *cosymplectic* if

$$TM|_N = TN + \pi^\sharp(\mathrm{ann}(TN)).$$

Remark 3.38. Compare with the definition of a coisotropic submanifold, where $\pi^\sharp(\mathrm{ann}(TN)) \subseteq TN$.

Remark 3.39. If M is symplectic, then the cosymplectic submanifolds are the same as the symplectic submanifolds.

Proposition 3.40. *Let N be a submanifold of a Poisson manifold M. The following are equivalent:*

(a) *N is cosymplectic*
(b) $TM|_N = TN \oplus \pi^\sharp(\mathrm{ann}(TN))$.
(c) $T^*M|_N = \mathrm{ann}(TN) \oplus (\pi^\sharp)^{-1}(TN)$.
(d) $\mathrm{ann}(TN) \cap (\pi^\sharp)^{-1}(TN) = 0$.
(e) *The restriction of π to* $\mathrm{ann}(TN) \subseteq T^*M|_N$ *is nondegenerate.*

(f) *N intersects every symplectic leaf of M* transversally*, with intersection a symplectic submanifold of that leaf.*

Proof. if N is cosymplectic, then the pointwise rank of $\pi^\sharp(\mathrm{ann}(TN))$ must be at least equal to the codimension of N. Hence, it is automatic that the sum in Definition 3.37 is a direct sum, which gives the equivalence with (b). Condition (c) is equivalent to (b) by dualization, and (d) is equivalent to (a) by taking annihilators on both sides.

Next, condition (e) means that if $\alpha \in \mathrm{ann}(TN)$ is π-orthogonal to all of $\mathrm{ann}(TN)$, then $\alpha = 0$. The space of elements that are π-orthogonal to $\mathrm{ann}(TN)$ is

$$\mathrm{ann}(\pi^\sharp(\mathrm{ann}(TN)) = (\pi^\sharp)^{-1}(TN),$$

so we see that (e) is equivalent to (d).

Condition (b) means in particular that $TM|_N = TN \oplus \mathrm{ran}(\pi^\sharp)$, so that N intersects the symplectic leaves transversally. Let ω_m be the symplectic form on $\mathrm{ran}(\pi^\sharp_m)$. If $\alpha \in T^*_m M$ is such that $v = \pi^\sharp_m(\alpha) \in T_m N$ is non-zero, then by (c) we can find $\beta \in T^*_m M$ with $w = \pi^\sharp_m(\beta) \in T_m N$ and $\langle \beta, v\rangle \neq 0$. But this means $\omega_m(v, w) \neq 0$, thus $T_m N \cap \mathrm{ran}(\pi^\sharp_m)$ is symplectic. This proves (f); the converse is similar. □

The main example of a cosymplectic submanifold is the following:

Example 3.41*.* Let M be a Poisson manifold. Suppose $m \in M$, and N is a submanifold passing through m with

$$T_m M = T_m N \oplus \mathrm{ran}(\pi^\sharp_m).$$

In other words, N intersects the symplectic leaf transversally and is of complementary dimension. Dualizing the condition means

$$T^*_m M = \mathrm{ann}(T_m N) \oplus \ker(\pi^\sharp_m),$$

which shows that π_m is non-degenerate on $\mathrm{ann}(TN)$ at the point m. But then π remains non-degenerate on an open neighborhood of m in N. This neighborhood is then a cosymplectic submanifold, with an induced Poisson structure. One refers to this Poisson structure on N near m as the 'transverse Poisson structure' at m. [**?**]

Remark 3.42*.* Cosymplectic submanifolds are already discussed in Weinstein's article [**?**], although the terminology appears later [42, 9]. They are also known as *Poisson transversals* [17], presumably to avoid confusion with the so-called cosymplectic structures.

4. Dirac structures

Dirac structures were introduced by Courant and Weinstein [**?**, 11] as a differential geometric framework for Dirac brackets in classical mechanics. The basic idea is to represent Poisson structures in terms of their graphs

$$\mathrm{Gr}(\pi) = \{\pi^\sharp(\alpha) + \alpha |\ \alpha \in T^*M\} \subseteq \mathbb{T}M = TM \oplus T^*M.$$

The maximal isotropic subbundles $E \subseteq \mathbb{T}M$ arising as graphs of Poisson bivector fields are characterized by a certain integrability condition; dropping the assumption that E is the graph of a map from T^*M to TM one arrives at the notion of a Dirac structure. Dirac geometry is extremely interesting in its own right; here we will use it mainly to prove facts about Poisson manifolds. Specifically, we will use Dirac geometry to discuss, among other things,

(a) the Lie algebroid structure of the cotangent bundle of a Poisson manifold
(b) the Weinstein splitting theorem
(c) symplectic realizations and symplectic groupoids for Poisson manifolds
(d) Poisson Lie groups and Drinfeld's classification

We begin with a discussion of the *Courant algebroid structure* of $\mathbb{T}M$.

4.1. **The Courant bracket.** Let M be a manifold, and

$$\mathbb{T}M = TM \oplus T^*M \tag{33}$$

the direct sum of the tangent and cotangent bundles. Elements of $\mathbb{T}M$ will be written $x = v+\mu$, with $v \in T_mM$ and $\mu \in T_m^*M$, and similarly sections will be written as $\sigma = X + \alpha$, where X is a vector field and α a 1-form. The projection to the summand TM will be called the *anchor map*

$$\mathsf{a}\colon \mathbb{T}M \to TM \tag{34}$$

thus $\mathsf{a}(v + \mu) = v$. Let $\langle\cdot,\cdot\rangle$ denote the bundle metric, i.e. non-degenerate symmetric bilinear form,

$$\langle v_1 + \mu_1,\ v_2 + \mu_2\rangle = \langle \mu_1, v_2\rangle + \langle \mu_2, v_1\rangle; \tag{35}$$

here $v_1, v_2 \in TM$ and $\mu_1, \mu_2 \in T^*M$ (all with the same base point in M).[1] We will use this metric to identify $\mathbb{T}M$ with its dual; for example, the anchor map dualizes to the map $\mathsf{a}^*\colon T^*M \to \mathbb{T}M^* \cong \mathbb{T}M$ given by the inclusion. The *Courant bracket* [11] (also know as the *Dorfman bracket* [15]) is the following bilinear operation on sections $\sigma_i = X_i + \alpha_i \in \Gamma(\mathbb{T}M)$,

$$[\![\sigma_1, \sigma_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \iota_{X_2}\mathrm{d}\alpha_1 \tag{36}$$

Remark 4.1. Note that this bracket is *not* skew-symmetric, and indeed Courant in [11] used the skew-symmetric version $[X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1$. However, the non-skew symmetric version (36), introduced by Dorfman [15], turned out to be much easier to deal with; in particular it satisfies a simple Jacobi identity (see (38) below). For this reason the skew-symmetric version is rarely used nowadays.

Remark 4.2. One motivation for the bracket (36) is as follows. Using the metric on $\mathbb{T}M$, one can form the bundle of Clifford algebras $\mathrm{Cl}(\mathbb{T}M)$. Thus, $\mathrm{Cl}(\mathbb{T}_mM)$ is the algebra generated by the elements of $\mathbb{T}_mM$, subject to relations $[x_1, x_2] \equiv x_1x_2 + x_2x_1 = \langle x_1, x_2\rangle$ for $x_i \in \mathbb{T}_mM$ (using *graded* commutators). The Clifford bundle has a spinor module $\wedge T^*M$, with the Clifford action given on generators by $\varrho(x) = \iota(v) + \epsilon(\mu)$ for $x = v + \mu$; here $\iota(v)$ is contraction by v and $\epsilon(\mu)$ is wedge product with μ. Hence, the algebra $\Gamma(\mathrm{Cl}(\mathbb{T}M))$ acts on the space $\Gamma(\wedge T^*M) = \Omega(M)$ of differential forms. But on the latter space, we also have the exterior differential d. The Courant bracket is given in terms of this action by

$$[[\mathrm{d}, \varrho(\sigma_1)], \varrho(\sigma_2)] = \varrho([\![\sigma_1, \sigma_2]\!]).$$

It exhibits the Courant bracket as a *derived bracket*. For more on this viewpoint see [**?**, **?**, **?**, 7].

[1] Note that $\mathbb{T}M$ also has a natural fiberwise symplectic form, but it will not be used here.

Proposition 4.3. *The Courant bracket* (36) *has the following properties, for all sections* σ_i, σ, τ *and all* $f \in C^\infty(M)$*:*

$$\mathsf{a}(\sigma)\langle\tau_1, \tau_2\rangle = \langle[\![\sigma, \tau_1]\!], \tau_2\rangle + \langle\tau_1, \ [\![\sigma, \tau_2]\!]\rangle, \tag{37}$$

$$[\![\sigma, [\![\tau_1, \tau_2]\!]]\!] = [\![[\![\sigma, \tau_1]\!], \tau_2]\!] + [\![\tau_1, [\![\sigma, \tau_2]\!]]\!], \tag{38}$$

$$[\![\sigma, \tau]\!] + [\![\tau, \sigma]\!] = \mathsf{a}^* \,\mathrm{d}\, \langle\sigma, \tau\rangle. \tag{39}$$

Furthermore, it satisfies the Leibniz rule

$$[\![\sigma, f\tau]\!] = f[\![\sigma, \tau]\!] + (\mathsf{a}(\sigma)f)\ \tau, \tag{40}$$

and the anchor map is bracket preserving:

$$\mathsf{a}([\![\sigma, \tau]\!]) = [\mathsf{a}(\sigma), \mathsf{a}(\tau)]. \tag{41}$$

All of these properties are checked by direct calculation.

Generalizing these properties, one defines a *Courant algebroid* over M [28, **?**] to be a vector bundle $\mathbb{A} \to M$, together with a bundle metric $\langle\cdot, \cdot\rangle$, a bundle map $\mathsf{a}\colon \mathbb{A} \to TM$ called the *anchor*, and a bilinear *Courant bracket* on $\Gamma(A)$ satisfying properties (37), (38), and (39) above. One can show [37] that the properties (40) and (41) are consequences. The bundle $\mathbb{T}M$ is called the *standard Courant algebroid* over M. We will encounter more general Courant algebroids later on.

Remark 4.4. For a vector bundle $V \to M$, denote by $\mathrm{Aut}(V)$ the group of vector bundle automorphisms of V. Its elements are diffeomorphism A of the total space of V respecting the linear structure; any automorphism restricts to a diffeomorphism Φ of the base. It defines an action $A\colon \Gamma(V) \to \Gamma(V)$ on sections; here $A.\tau = A \circ \tau \circ \Phi^{-1}$ where on the right hand side, the section is regarded as a map $\tau\colon M \to V$. This has the property

$$A(f\tau) = (\Phi_* f)\, A(\tau)$$

for all $f \in C^\infty(M)$ and $\tau \in \Gamma(V)$, conversely, any such operator on $\Gamma(V)$ describes an automorphism of V. Taking derivatives, we see that the *infinitesimal automorphism* of a vector bundle $V \to M$ may be described by operators $D\colon \Gamma(V) \to \Gamma(V)$ such that there exists a vector field X satisfying the Leibniz rule,

$$D(f\tau) = fD(\tau) + X(f)\tau.$$

For a Lie algebroid, the operator given by the Lie algebroid bracket with a fixed section is such a vector bundle automorphisms; the property $\mathsf{a}([\sigma, \tau]) = [\mathsf{a}(\sigma), \mathsf{a}(\tau)]$ says that this automorphism preserves the anchor, and the Jacobi identity for the bracket signifies that this infinitesimal automorphism preserves the bracket. In a similar fashion, for a Courant algebroid $\mathbb{A}$ be a Courant algebroid (e.g., the standard Courant algebroid $\mathbb{T}M$), the operator $[\![\sigma, \cdot]\!]$ on sections defines an infinitesimal vector bundle automorphism. The property (37) says that this infinitesimal automorphism preserves the metric, (41) says that it preserves the anchor, and (38) says that it preserves the bracket $[\![\cdot, \cdot]\!]$ itself.

4.2. **Dirac structures.** For any subbundle $E \subseteq \mathbb{T}M$, we denote by $E^\perp$ its orthogonal with respect to the metric $\langle\cdot,\cdot\rangle$. The subbundle E is called *isotropic* if $E \subseteq E^\perp$, *co-isotropic* if $E \supset E^\perp$, and *maximal isotropic*, or *Lagrangian* if $E = E^\perp$. The terminology is borrowed from symplectic geometry, where it is used for subspaces of a vector space with a non-degenerate *skew-symmetric* bilinear form. Immediate examples of Lagrangian subbundles are TM and T^*M. Given a bivector field $\pi \in \mathfrak{X}^2(M)$, its *graph*

$$\mathrm{Gr}(\pi) = \{\pi^\sharp(\mu) + \mu |\ \mu \in T^*M\} \subseteq \mathbb{T}M,$$

is Lagrangian; in fact, the Lagrangian subbundles $E \subseteq \mathbb{T}M$ with $E \cap TM = 0$ are exactly the graphs of bivector fields. Similarly, given a 2-form ω its graph

$$\mathrm{Gr}(\omega) = \{v + \omega^\flat(v) |\ v \in TM\} \subseteq \mathbb{T}M$$

is Lagrangian; the Lagrangian subbundles $E \subseteq \mathbb{T}M$ with $E \cap TM = 0$ are exactly the graphs of 2-forms.

Note that although the Courant bracket is not skew-symmetric, it restricts to a skew-symmetric bracket on sections of Lagrangian subbundles, because the right hand side of (39) is zero on such sections.

Definition 4.5. A *Dirac structure* on M is a Lagrangian subbundle $E \subseteq \mathbb{T}M$ whose space of sections is closed under the Courant bracket.

Proposition 4.6. *Any Dirac structure $E \subseteq \mathbb{T}M$ acquires the structure of a Lie algebroid, with the Lie bracket on sections given by the Courant bracket on $\Gamma(E) \subseteq \Gamma(\mathbb{T}M)$, and with the anchor obtained by restriction of the anchor $\mathsf{a}\colon \mathbb{T}M \to TM$.*

Proof. By (39), the Courant bracket is skew-symmetric on sections of E, and (38) gives the Jacobi identity. The Leibnitz identity follows from that for the Courant bracket, Equation (40). ☐

The integrability of a Lagrangian subbundle $E \subseteq \mathbb{T}M$ is equivalent to the vanishing of the expression

$$\Upsilon_E(\sigma_1, \sigma_2, \sigma_3) = \langle \sigma_1,\ [\![\sigma_2, \sigma_3]\!]\rangle \tag{42}$$

for all $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(E)$. Indeed, given $\sigma_2, \sigma_3 \in \Gamma(E)$, the vanishing for all $\sigma_1 \in \Gamma(E)$ means precisely that $[\![\sigma_2, \sigma_3]\!]$ takes values in $E^\perp = E$. Using the properties (37) and (39) of the Courant bracket, one sees that Υ_E is skew-symmetric in its entries. Since Υ_E is clearly tensorial in its first entry, it follows that it is tensorial in all three entries: that is

$$\Upsilon_E \in \Gamma(\wedge^3 E^*).$$

In particular, to calculate Υ_E it suffices to determine its values on any collection of sections that span E everywhere.

Proposition 4.7. *For a 2-form ω, the graph $\mathrm{Gr}(\omega)$ is a Dirac structure if and only if $\mathrm{d}\omega = 0$. In this case, the projection $\mathrm{Gr}(\omega) \to TM$ along T^*M is an isomorphism of Lie algebroids.*

Proof. We calculate,

$$\begin{aligned} [\![X + \omega^\flat(X), Y + \omega^\flat(Y)]\!] &= [X,Y] + \mathcal{L}_X \iota_Y \omega - \iota_Y \mathrm{d}\iota_X \omega \\ &= [X,Y] + \iota_{[X,Y]}\omega + \iota_Y \iota_X \mathrm{d}\omega. \end{aligned}$$

This takes values in $\mathrm{Gr}(\omega)$ if and only if the last term is zero, that is, $\mathrm{d}\omega = 0$. In fact, the calculation shows that $\Upsilon_{\mathrm{Gr}(\omega)}$ coincides with $\mathrm{d}\omega$ under the isomorphism $\mathrm{Gr}(\pi) \cong TM$. □

Proposition 4.8. *A bivector field $\pi \in \mathfrak{X}^2(M)$ is Poisson if and only if its graph $\mathrm{Gr}(\pi)$ is a Dirac structure. In this case, the projection $\mathrm{Gr}(\pi) \to T^*M$ (along TM) is an isomorphism of Lie algebroids, where T^*M has the cotangent Lie algebroid structure determined by π.*

Proof. We want to show that $\Upsilon_{\mathrm{Gr}(\pi)}$ vanishes if and only if π is a Poisson structure. It suffices to evaluate $\Upsilon_{\mathrm{Gr}(\pi)}$ on sections of the form $X_f + \mathrm{d}f$ for $f \in C^\infty(M)$, where $X_f = \pi^\sharp(\mathrm{d}f)$. Thus let $f_1, f_2, f_3 \in C^\infty(M)$ and put $\sigma_i = X_{f_i} + \mathrm{d}f_i$. We have

$$[\![\sigma_2, \sigma_3]\!] = [X_{f_2}, X_{f_3}] + \mathrm{d}\,\mathcal{L}_{X_{f_2}}(f_3),$$

hence

$$\langle \sigma_1, [\![\sigma_2, \sigma_3]\!]\rangle = \mathcal{L}_{[X_{f_2}, X_{f_3}]}(f_1) + \mathcal{L}_{X_{f_1}}\mathcal{L}_{X_{f_2}}(f_3) = \mathrm{Jac}(f_1, f_2, f_3).$$

The result follows. In fact, we have shown that $\Upsilon_{\mathrm{Gr}(\pi)}$ coincides with Υ_π under the isomorphism $\mathrm{Gr}(\pi) \cong T^*M$.

Finally, it is immediate from the formulas for the Courant bracket and the cotangent Lie algebroid that the isomorphism $\mathrm{Gr}(\pi) \cong T^*M$ intertwines the anchor with the map $\pi^\sharp$, and takes the bracket of two sections of $\mathrm{Gr}(\pi)$ to the Lie bracket of the corresponding 1-forms,

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \iota_{\pi^\sharp(\beta)}\mathrm{d}\alpha \tag{43}$$

□

4.3. **Tangent lifts of Dirac structures.** As we had explained earlier, the cotangent Lie alegbroid structure on T^*M for a Poisson manifold (M, π) corresponds to the tangent lift to a Poisson structure on TM. What about tangent lifts of more general Dirac structures? Let $p\colon TM \to M$ be the bundle projection.

We had defined tangent lifts and vertical lifts of functions. The tangent lift of a vector field X is characterized by $X_T(f_T) = X(f)_T$; the vertical lift by $X_V(f_T) = X(f)_V$. In local tangent coordinates, if $X = \sum_i a^i(x)\frac{\partial}{\partial x^i}$,

$$X_T = \sum_i a^i(x)\frac{\partial}{\partial x^i} + \sum_{ij}\frac{\partial a^i}{\partial x^j}y^j\frac{\partial}{\partial y^i}, \qquad X_V = \sum_i a^i(x)\frac{\partial}{\partial y^i}.$$

We have

$$[X_T, Y_T] = [X, Y]_T, \quad [X_V, Y_T] = [X, Y]_V, \quad [X_V, Y_V] = 0.$$

Similar formulas define the tangent and vertical lifts of multi-vector fields, e.g. for a bivector field $\pi_T(\mathrm{d}f_T, \mathrm{d}g_T) = (\pi(\mathrm{d}f, \mathrm{d}g))_T$, $\pi_V(\mathrm{d}f_T, \mathrm{d}g_T) = (\pi(\mathrm{d}f, \mathrm{d}g))_V$. With this notation, the tangent lift of a Poisson structure π_{TM} is indeed just π_T. For differential forms, we define the vertical lift α_V to be simply the pull-back. The tangent lift of functions extends uniquely to a tangent lift of differential forms, in such a way that $(\mathrm{d}f)_T = \mathrm{d}(f_T)$ and

$$(\alpha \wedge \beta)_T = \alpha_V \wedge \beta_T + \alpha_T \wedge \beta_V.$$

For 1-forms $\alpha = \sum_i \alpha_i \mathrm{d}x^i$, one finds,

$$\alpha_T = \sum_i \alpha_i\,\mathrm{d}y^i + \sum_{ij}\frac{\partial \alpha_i}{\partial x^j}\,y^j\,\mathrm{d}x^i.$$

Here are some basic formulas for tangent and vertical lifts:

$$\iota(X_T)\alpha_T = (\iota(X)\alpha)_T, \quad \iota(X_T)\alpha_V = (\iota(X)\alpha)_V = \iota(X_V)\alpha_T, \quad \iota(X_V)\alpha_V = 0;$$

$$\mathcal{L}(X_T)\alpha_T = (\mathcal{L}(X)\alpha)_T, \quad \mathcal{L}(X_T)\alpha_V = (\mathcal{L}(X)\alpha)_V = \mathcal{L}(X_V)\alpha_T, \quad \mathcal{L}(X_V)\alpha_V = 0.$$

For $\sigma = X + \alpha \in \Gamma(\mathbb{T}M)$, consider $\sigma_T = X_T + \alpha_T \in \Gamma(\mathbb{T}(TM))$ and $\sigma_V = X_V + \alpha_V$. From the properties of tangent and vertical lifts of 1-forms and vector fields, we obtain,

$$\langle \sigma_T, \tau_T \rangle = \langle \sigma, \tau \rangle_T, \quad \langle \sigma_V, \tau_V \rangle = 0, \ \langle \sigma_V, \tau_T \rangle = \langle \sigma, \tau \rangle_T$$

$$[\![\sigma_T, \tau_T]\!] = [\![\sigma, \tau]\!]_T, \quad [\![\sigma_V, \tau_V]\!] = 0, \ [\![\sigma_V, \tau_T]\!] = [\![\sigma, \tau]\!]_V = [\![\sigma_T, \tau_V]\!]$$

and finally,

$$\mathsf{a}(\sigma_T) = (\mathsf{a}(\sigma))_T, \quad \mathsf{a}(\sigma_V) = \mathsf{a}(\sigma)_V.$$

As an application, we can prove:

Theorem 4.9. *For any Dirac structure $E \subseteq \mathbb{T}M$ there is a unique Dirac structure $E_T \subseteq \mathbb{T}M_T$ such that $\sigma_T \in \Gamma(E_T)$ for all $\sigma \in \Gamma(E)$.*

Proof. For non-zero $v \in TM$, there is at least one function f such that $f_T(v) \neq 0$. Since $(f\sigma)_T = f_V\sigma_T + f_T\sigma_V$, we conclude that the subspace $(E_T)_v \subseteq \mathbb{T}_v(TM)$ spanned by the tangent lifts of sections of E is the same as the subspace spanned by the tangent and cotangent lifts of sections of E.

Inside $\mathbb{T}(TM)$, we have a subbundle $(\mathbb{T}M)_V$, spanned by all vertical lifts of sections of $\mathbb{T}M$. It is canonically isomorphic to the vector bundle pull-back of $p^*(\mathbb{T}M)$. The quotient space $(\mathbb{T}M)_H := \mathbb{T}(TM)/(\mathbb{T}M)_V$ is isomorphic to $p^*(\mathbb{T}M)$ as well; looking at the explicit formulas we see that it is spanned by image of horizontal lifts. The vertical lifts of sections of E span a subbundle $E_V \cong p^*E$, while the images of tangent lifts in $(\mathbb{T}M)_H$ span a subbundle $E_H \cong p^*E$. It hence follows that at any $v \in TM$, the span $(E_T)_v$ of the vertical and tangent lifts of sections of E has dimension at least $2\,\mathrm{rank}(E) = 2\dim M$. From the properties of tangent and vertical lifts, it is immediate that this subspace is isotropic, hence its dimension is exactly $2\dim M$. We conclude that E_T is a subbundle, and using the Courant bracket relations of tangent and vertical lifts it is clear that E_T defines a Dirac structure. □

5. Gauge transformations of Poisson and Dirac structures

One simple way to produce new Dirac structures from given ones is to apply a bundle automorphism of $\mathbb{T}M$ preserving the Courant algebroid structures.

5.1. **Automorphisms of the Courant algebroid structure.**

Recall from Remark 4.4 that if $V \to M$ is any vector bundle, we denote by $\mathrm{Aut}(V)$ the group of vector bundle automorphisms. Any such automorphisms restricts to a diffeomorphism Φ of the zero section; the kernel of the restriction map $\mathrm{Aut}(V) \to \mathrm{Diff}(M)$ is denoted $\mathrm{Gau}(V)$; its elements are called *gauge transformations* of V. We have an exact sequence,

$$1 \to \mathrm{Gau}(V) \to \mathrm{Aut}(V) \to \mathrm{Diff}(M)$$

where the last map need not be surjective, in general. Similarly, we denote by $\mathfrak{gau}(V)$ the kernel of the restriction map $\mathfrak{aut}(V) \to \mathfrak{X}(M)$, it fits into an exact sequence of Lie algebras

$$0 \to \mathfrak{gau}(V) \to \mathfrak{aut}(V) \to \mathfrak{X}(M) \to 0.$$

Let $\mathrm{Aut}_{CA}(\mathbb{T}M)$ denote the group of Courant algebroid automorphisms of $\mathbb{T}M$, that is, $A \in \mathrm{Aut}(\mathbb{T}M)$, preserves the metric, bracket and anchor. In terms of the resulting action on sections, letting $\Phi\colon M \to M$ be the base map, this means

$$\langle A\sigma, A\tau\rangle = \Phi_*\langle\sigma,\tau\rangle,$$
$$[\![A\sigma, A\tau]\!] = A[\![\sigma,\tau]\!],$$
$$\mathsf{a}\circ A = T\Phi\circ\mathsf{a}.$$

The group homomorphism $\mathrm{Aut}_{CA}(\mathbb{T}M) \to \mathrm{Diff}(M)$ is surjective, since every diffeomorphism $\Phi \in \mathrm{Diff}(M)$ defines a *standard* Courant algebroid automorphism

$$\mathbb{T}\Phi \in \mathrm{Aut}_{CA}(\mathbb{T}M)$$

by taking the sum of the tangent and cotangent maps. This gives a split exact sequence

$$1 \to \mathrm{Gau}_{CA}(\mathbb{T}M) \to \mathrm{Aut}_{CA}(\mathbb{T}M) \to \mathrm{Diff}(M),$$

where the splitting identifies the Courant automorphisms with a semi-direct product,

$$\mathrm{Aut}_{CA}(\mathbb{T}M) = \mathrm{Gau}_{CA}(\mathbb{T}M) \rtimes \mathrm{Diff}(M).$$

For any 2-form ω, define an automorphism

$$\mathcal{R}_\omega \in \mathrm{Aut}(\mathbb{T}M), \quad x \mapsto x + \iota_{\mathsf{a}(x)}\omega.$$

Obviously, $\mathcal{R}_{\omega_1+\omega_2} = \mathcal{R}_{\omega_1}\circ\mathcal{R}_{\omega_2}$.

Proposition 5.1. [21] *The automorphism $\mathcal{R}_\omega$ preserves the metric and anchor; it preserves the Courant bracket if and only if $\mathrm{d}\omega = 0$. The map*

$$\Omega_{cl}(M) \to \mathrm{Gau}_{CA}(\mathbb{T}M), \quad \omega \mapsto \mathcal{R}_{-\omega}$$

is a group isomorphism.

Proof. For $\sigma = X + \alpha$ and $\tau = Y + \beta$, we have that

$$\begin{aligned}[\![\mathcal{R}_\omega\sigma, \mathcal{R}_\omega\tau]\!] &= [X,Y] + \mathcal{L}_X(\beta + \iota_Y\omega) - \iota_Y\mathrm{d}(\alpha + \iota_X\omega)\\ &= [X,Y] + \mathcal{L}_X\beta - \iota_Y\mathrm{d}\alpha + \mathcal{L}_X\iota_Y\omega - \iota_Y\mathrm{d}\iota_X\omega\\ &= [X,Y] + \mathcal{L}_X\beta - \iota_Y\mathrm{d}\alpha + \iota_{[X,Y]}\omega + \iota_Y\iota_X\mathrm{d}\omega\\ &= \mathcal{R}_\omega[\![\sigma,\tau]\!] + \iota_Y\iota_X\mathrm{d}\omega\end{aligned}$$

The fact that $\mathcal{R}_\omega$ preserves the metric and anchor is similar, but easier. Suppose now that $A \in \mathrm{Gau}_{CA}(\mathbb{T}M)$ is given. In particular, $\mathsf{a}\circ A = \mathsf{a}$. Since A preserves the metric, we have $A^* = A^{-1}$ under the identification of $\mathbb{T}M$ with its dual. Hence $A^{-1}\circ\mathsf{a}^* = \mathsf{a}^*$, which means that A^{-1}, hence also A, fixed T^*M pointwise, while on the other hand $A'v - v \in T^*M$ for all $v \in TM$. Since A preserves the metric,

$$0 = \langle v, w\rangle\langle Av, Aw\rangle = \langle Av, w\rangle + \langle v, Aw\rangle$$

for all $v, w \in TM$. Hence there is a well-defined 2-form ω such that

$$\omega(v,w) = \langle v,\ Aw\rangle,$$

and $A = \mathcal{R}_{-\omega}$. But $\mathcal{R}_{-\omega}$ preserves the Courant bracket if and only if ω is closed. ∎

Remark 5.2. The calculation applies more generally to twisted Courant brackets: Given a closed 3-form $\eta \in \Omega^3(M)$, one has the η-twisted Courant bracket

$$[\![\sigma,\tau]\!]_\eta = [X,Y] + \mathcal{L}_X\beta - \iota_Y \mathrm{d}\alpha + \iota_X\iota_Y\eta.$$

For $\Phi \in C^\infty(M)$ one has $[\![\mathbb{T}\Phi.\sigma,\ \mathbb{T}\Phi.\tau]\!]_\eta = \mathbb{T}\Phi.([\![\sigma,\tau]\!]_{\Phi^*\eta})$. Given a 2-form ω, one has that

$$[\![\mathcal{R}_\omega\sigma, \mathcal{R}_\omega\tau]\!]_{\eta+\mathrm{d}\omega} = \mathcal{R}_\omega[\![\sigma,\tau]\!]_\eta.$$

In summary, we have shown that

$$\mathrm{Aut}_{CA}(\mathbb{T}M) = \Omega^2_{cl}(M) \rtimes \mathrm{Diff}(M),$$

where (ω,Φ) acts as $\mathcal{R}_{-\omega}\circ\mathbb{T}\Phi$.

We can similarly discuss the Lie algebra $\mathfrak{aut}_{CA}(\mathbb{T}M)$ of *infinitesimal Courant algebroid automorphisms*. Regarded as operators on sections, these are the linear maps $D\colon \Gamma(\mathbb{T}M)\to\Gamma(\mathbb{T}M)$ such that there exists a vector field X with

$$D(f\sigma) = fD(\sigma) + X(f)\sigma,$$
$$\langle D\sigma,\tau\rangle + \langle\sigma, D\tau\rangle = X\langle\sigma,\tau\rangle,$$
$$[\![D\sigma,\tau]\!] + [\![\sigma, D\tau]\!] = D[\![\sigma,\tau]\!],$$
$$\mathsf{a}(D\tau) = [X, \mathsf{a}(\tau)].$$

As mentioned before, $D = [\![\sigma,\cdot]\!]$ has all these properties.

Proposition 5.3 (Infinitesimal automorphisms of the Courant bracket)**.** [21]

(a) *The Lie algebra of infinitesimal Courant automorphisms is a semi-direct product*

$$\mathfrak{aut}_{CA}(\mathbb{T}M) = \Omega^2_{cl}(M)\rtimes\mathfrak{X}(M),$$

where the action of (γ, X) *on a section* $\tau = Y+\beta\in\Gamma(\mathbb{T}M)$ *is given by*

$$(\gamma,X).\tau = [X,Y] + \mathcal{L}_X\beta - \iota_Y\gamma. \tag{44}$$

(b) *For any section* $\sigma = X+\alpha\in\Gamma(\mathbb{T}M)$*, the Courant bracket* $[\![\sigma,\cdot]\!]$ *is the infinitesimal automorphism* $(\mathrm{d}\alpha,\ X)$*.*

Proof. The proof of a) is similar to the global case. The given formula defines an injective map from $\Omega^2_{cl}(M)\rtimes\mathfrak{X}(M)$ to $\mathfrak{aut}_{CA}(\mathbb{T}M)$. To see that it is surjective, let $D\in\mathfrak{aut}_{CA}(\mathbb{T}M)$ be given, with base vector field X. By subtracting $\mathcal{L}_X\in\mathfrak{aut}_{CA}(\mathbb{T}M)$, we obtain $D' = D-\mathcal{L}_X$ with corresponding vector field equal to 0. Since $D'(f\sigma) = fD'(\sigma)$, it follows that D' is given by an infinitesimal gauge transformation of $\mathbb{T}M$, i.e. by a section of the bundle of endomorphisms of $\mathbb{T}M$. Since $\mathsf{a}\circ D' = 0$, we see that this endomorphism takes values in T^*M. Dually, we obtain $D'\circ\mathsf{a}^* = 0$, hence D' vanishes on T^*M. Hence, it is given by a bundle map $TM\to T^*M$. Since D' preserves metrics,

$$\langle D'X, Y\rangle + \langle X, D'Y\rangle = 0.$$

Hence, there is a well-defined 2-form γ such that $\gamma(X,Y) = \langle X, D'(Y)\rangle$, and the action of D' is

$$D'(X+\alpha) = -\iota_X\gamma.$$

Finally, using that D' preserves brackets one finds that γ must be closed. Property b) is immediate from (44) and the formula for the Courant bracket. ∎

We are interested in the integration of infinitesimal Courant automorphisms, especially those generated by sections of $\mathbb{T}M$. In the discussion below, we will be vague about issues of completeness of vector fields; in the general case one has to work with *local* flows. The following result is an infinite-dimensional instance of a formula for time dependent flows on semi-direct products $V \rtimes G$, where G is a Lie group and V a G-representation.

Proposition 5.4. [21, 22] *Let $(\omega_t, \Phi_t) \in \mathrm{Aut}_{CA}(\mathbb{T}M)$ be the family of automorphisms integrating the time-dependent infinitesimal automorphisms $(\gamma_t, X_t) \in \mathfrak{aut}_{CA}(\mathbb{T}M)$. Then Φ_t is the flow of X_t, while*

$$\omega_t = \int_0^t \big((\Phi_s)_*\gamma_s\big)\mathrm{d}s.$$

Proof. Recall (cf. Appendix **??**) that the flow Φ_t of a time dependent vector field X_t is defined in terms of the action on functions by $\frac{d}{dt}(\Phi_t)_* = (\Phi_t)_* \circ \mathcal{L}_{X_t}$. Similarly, the 1-parameter family of Courant automorphisms (ω_t, Φ_t) integrating (γ_t, X_t) is defined in terms of the action on sections $\tau \in \Gamma(\mathbb{T}M)$ by

$$\frac{d}{dt}\Big((\omega_t, \Phi_t).\tau\Big) = (\omega_t, \Phi_t).(\gamma_t, X_t).\tau.$$

Write $\tau = Y + \beta \in \Gamma(\mathbb{T}M)$. Then

$$\begin{aligned}\frac{d}{dt}\Big((\omega_t, \Phi_t).\tau\Big) &= \frac{d}{dt}\Big((\Phi_t)_*\tau - \iota\big((\Phi_t)_*Y\big)\omega_t\Big)\\ &= (\Phi_t)_*\mathcal{L}_{X_t}\tau - \iota\big((\Phi_t)_*\mathcal{L}_{X_t}Y\big)\omega_t - \iota\big((\Phi_t)_*Y\big)\frac{d\omega_t}{dt}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\omega_t, \Phi_t).(\gamma_t, X_t).\tau &= (\omega_t, \Phi_t).\Big(\mathcal{L}_{X_t}\tau - \iota(Y)\gamma_t\Big)\\ &= (\Phi_t)_*\mathcal{L}_{X_t}\tau - \iota\big((\Phi_t)_*Y\big)(\Phi_t)_*\gamma_t - \iota\big((\Phi_t)_*\mathcal{L}_{X_t}Y\big)\omega_t.\end{aligned}$$

Comparing, we see $(\Phi_t)_*\gamma_t = \frac{d}{dt}\omega_t$. □

This calculation applies in particular to the infinitesimal automorphisms (γ_t, X_t) defined by $\sigma_t = X_t + \alpha_t \in \Gamma(\mathbb{T}M)$; here $\gamma_t = \mathrm{d}\alpha_t$. Note that in this case,

$$\omega_t = \mathrm{d}\int_0^t \big((\Phi_s)_*\alpha_s\big)\mathrm{d}s$$

is a family of *exact* 2-forms.

5.2. **Moser method for Poisson manifolds.** For any Dirac structure $E \subseteq \mathbb{T}M$, and closed 2-form $\omega \in \Omega^2_{cl}(M)$, one obtains a new Dirac structure $E^\omega = \mathcal{R}_\omega(E)$ called the *gauge transformation* of E by ω. We are interested in the special case that E is the graph of a Poisson bivector field.

Lemma 5.5. *Let π be a Poisson structure on M, and $\omega \in \Omega^2_{cl}(M)$ a closed 2-form. Then $\mathrm{Gr}(\pi)^\omega$ is transverse to TM if and only if the bundle map*

$$\mathrm{id} + \omega^\flat \circ \pi^\sharp \colon T^*M \to T^*M$$

is invertible. In this case, the Poisson structure π^ω *defined by* $\mathrm{Gr}(\pi)^\omega = \mathrm{Gr}(\pi^\omega)$ *satisfies*

$$(\pi^\omega)^\sharp = \pi^\sharp \circ (\mathrm{id} + \omega^\flat \circ \pi^\sharp)^{-1}. \tag{45}$$

Proof. By definition,

$$\mathrm{Gr}(\pi)^\omega = \{\pi^\sharp(\mu) + \mu + \iota_{\pi^\sharp(\mu)}\omega |\ \mu \in T^*M\}$$

This is transverse to TM if and only if the projection to T^*M is an isomorphism, that is, if and only if for all $\nu \in T^*M$ there is a unique solution of

$$\nu = \mu + \iota_{\pi^\sharp(\mu)}\omega \equiv (\mathrm{id} + \omega^\flat \circ \pi^\sharp)\mu.$$

Furthermore, in this case the resulting π^ω is given by $(\pi^\omega)^\sharp(\nu) = \pi^\sharp(\mu)$, which proves (45). □

One calls π^ω the *gauge transformation* of π by the closed 2-form ω.

Lemma 5.6. [5] *The Poisson structure* π^ω *and* π *define the same symplectic foliation. The 2-forms on leaves are related by pull-back of* ω*.*

Proof. From (45), it is immediate that $\mathrm{ran}((\pi^\omega)^\sharp) = \mathrm{ran}(\pi^\sharp)$. For the second claim, given $m \in M$, let σ be the 2-form on $\mathsf{S}_m = \mathrm{ran}(\pi_m^\sharp)$ defined by π_m. If $v_1, v_2 \in \mathsf{S}_m$ with $v_i = \pi_m^\sharp(\mu_i)$, we have

$$\sigma(v_1, v_2) = -\pi_m(\mu_1, \mu_2) = \langle \mu_1, v_2 \rangle.$$

Similarly, let σ^ω be the 2-form defined by π^ω:

$$\sigma^\omega(v_1, v_2) = \langle \nu_1, v_2 \rangle,$$

where ν_1 is such that $(\pi^\omega)^\sharp \nu_1 = v_1$. As we saw above,

$$\nu_1 = (\mathrm{id} + \omega^\flat \pi^\sharp)\mu_1 = \mu_1 + \iota_{v_1}\omega.$$

Hence, $\sigma^\omega(v_1, v_2) = \langle \nu_1, v_2 \rangle = \sigma(v_1, v_2) + \omega(v_1, v_2)$. □

The standard *Moser argument* for symplectic manifolds shows that for a compact symplectic manifold, any 1-parameter family of deformations of the symplectic forms in a prescribed cohomology class is obtained by the action of a 1-parameter family of diffeomorphisms. The following version for Poisson manifolds can be proved from the symplectic case, arguing 'leaf-wise', or more directly using the Dirac geometric methods described above.

Theorem 5.7. [1, 2] *Suppose* $\pi_t \in \mathfrak{X}^2(M)$ *is a 1-parameter family of Poisson structures related by gauge transformations,*

$$\pi_t = (\pi_0)^{\omega_t},$$

where $\omega_t \in \Omega^2(M)$ *is a family of closed 2-forms with* $\omega_0 = 0$*. Suppose that*

$$\frac{d\omega_t}{dt} = -\mathrm{d}a_t,$$

with a smooth family of 1-forms $a_t \in \Omega^1(M)$*, defining a time dependent vector field* $X_t = \pi_t^\sharp(a_t)$*. Let* Φ_t *be the flow of* X_t*. Then*

$$(\Phi_t)_* \pi_t = \pi_0.$$

Proof. Let

$$b_t = a_t - \iota(X_t)\omega_t,$$

so that $X + t + b + t = \mathcal{R}_{-\omega_t}(X_t + a_t)$. Since $X_t + a_t$ is a section of $\mathrm{Gr}(\pi_t) = \mathcal{R}_{\omega_t}\big(\mathrm{Gr}(\pi_0)\big)$ it follows that $X_t + b_t$ is a section of $\mathrm{Gr}(\pi_0)$. Hence, Courant bracket with $X_t + b_t$ preserves $\Gamma(\mathrm{Gr}(\pi_0))$. Equivalently, the family of infinitesimal automorphisms $(\mathrm{d}b_t,\ X_t) \in \mathfrak{aut}_{CA}(\mathbb{T}M)$ preserves $\mathrm{Gr}(\pi_0)$, hence so does its flow $(u_t, \Phi_t) \in \mathrm{Aut}_{CA}(\mathbb{T}M)$. By Proposition 5.4, the 2-forms u_t are given in terms of their derivative by

$$\begin{aligned}\frac{d}{dt}u_t &= (\Phi_t)_*\mathrm{d}b_t \\ &= (\Phi_t)_*(\mathrm{d}a_t - \mathcal{L}(X_t)\omega_t) \\ &= -(\Phi_t)_*(\frac{\mathrm{d}\omega_t}{\mathrm{d}t} + \mathcal{L}(X_t)\omega_t) \\ &= -\frac{d}{dt}((\Phi_t)_*\omega_t).\end{aligned}$$

Thus, $u_t = -(\Phi_t)_*\omega_t$. It follows that

$$\begin{aligned}\mathrm{Gr}(\pi_0) &= \mathcal{R}_{-u_t} \circ \mathbb{T}\Phi_t(\mathrm{Gr}(\pi_0)) \\ &= \mathbb{T}\Phi_t \circ \mathcal{R}_{\omega_t}(\mathrm{Gr}(\pi_0)) \\ &= \mathbb{T}\Phi_t(\mathrm{Gr}(\pi_t)) \\ &= \mathrm{Gr}\left((\Phi_t)_*\pi_t\right)\end{aligned}$$

which shows that $\pi_0 = (\Phi_t)_*\pi_t$. □

6. Dirac morphisms

We still have to express 'Poisson maps' in terms of Dirac geometry.

6.1. **Morphisms of Dirac structures.** Let $\varphi \in C^\infty(N, M)$ be a smooth map. Recall that the vector bundle morphism $T\varphi\colon TN \to TM$ dualizes to a comorphism $T^*\varphi\colon T^*N \dashrightarrow T^*M$, given fiberwise by maps in the opposite direction, $T^*_{\varphi(n)}M \to T^*_nN$. The comorphism $T^*\varphi$, or rather its graph $\mathrm{Gr}(T^*\varphi) \subseteq T^*M \times T^*N$ defines a relation from T^*N to T^*M. Similarly, we define

$$\mathbb{T}\varphi\colon \mathbb{T}N \dashrightarrow \mathbb{T}M$$

as a relation from $\mathbb{T}N$ to $\mathbb{T}M$; its graph is the sum of the graphs of the tangent and cotangent maps. We write

$$y \sim_\varphi x \quad\Leftrightarrow\quad (x, y) \in \mathrm{Gr}(\mathbb{T}\varphi).$$

Given $x = v + \mu \in \mathbb{T}_mM$, $y = w + \nu \in \mathbb{T}_nN$ this means $m = \varphi(n)$ and $v = (T_n\varphi)\,w,\ \nu = (T_n\varphi)^*\mu$. For sections $\sigma, \tau \in \Gamma(\mathbb{T}M)$ we write

$$\tau \sim_\varphi \sigma \quad\Leftrightarrow\quad (\sigma, \tau) \text{ restricts to a section of } \mathrm{Gr}(\mathbb{T}\varphi).$$

For $\sigma = X + \alpha,\ \ \tau = Y + \beta$, this means $Y \sim_\varphi X$ (related vector fields[2]) and $\beta = \varphi^*\alpha$.

[2]Recall that vector fields $Y \in \mathfrak{X}(N)$ and $X \in \mathfrak{X}(M)$ are *φ-related* (written $Y \sim_\varphi X$) if $(T_n\varphi)(Y_n) = X_{\varphi(n)}$ for all $n \in N$. If $Y_1 \sim_\varphi X_1$ and $Y_2 \sim_\varphi X_2$ then $[Y_1, Y_2] \sim_\varphi [X_1, X_2]$.

Lemma 6.1. *The relation $\mathbb{T}\varphi$ preserves Courant brackets, in the sense that*

$$\tau_1 \sim_\varphi \sigma_1, \quad \tau_2 \sim_\varphi \sigma_2 \quad \Rightarrow \quad \langle \tau_1, \tau_2 \rangle = \varphi^* \langle \sigma_1, \sigma_2 \rangle,$$

$$\tau_1 \sim_\varphi \sigma_1, \quad \tau_2 \sim_\varphi \sigma_2 \quad \Rightarrow \quad [\![\tau_1, \tau_2]\!] \sim_\varphi [\![\sigma_1, \sigma_2]\!]$$

$$\tau \sim_\varphi \sigma \quad \Rightarrow \quad \mathsf{a}(\tau) \sim_\varphi \mathsf{a}(\sigma)$$

Proof. Straightforward computation, using that if two pairs of vector fields are related, then their Lie brackets are related. □

Definition 6.2. Let $F \subseteq \mathbb{T}N$ and $E \subseteq \mathbb{T}M$ be Dirac structures. Then $\varphi \in C^\infty(N, M)$ defines a (forward) *Dirac morphism*

$$\mathbb{T}\varphi \colon (\mathbb{T}N, F) \dashrightarrow (\mathbb{T}M, E)$$

if it has the following property: For every $n \in N$ and $x \in E_{\varphi(n)}$, there exists a *unique* $y \in F_n$ such that $y \sim_\varphi x$.

Remark 6.3. We will use the term *weak Dirac morphism* for a similar definition where we omit the uniqueness condition. For instance, $\mathbb{T}\varphi \colon (\mathbb{T}N, T^*N) \dashrightarrow (\mathbb{T}M, T^*M)$ is a Dirac morphism, but $\mathbb{T}\varphi \colon (\mathbb{T}N, TN) \dashrightarrow (\mathbb{T}M, TM)$ is only a weak Dirac morphism.

Proposition 6.4. *Let (N, π_N) and (M, π_M) be Poisson manifolds. Then $\varphi \colon N \to M$ is a Poisson map if and only if $\mathbb{T}\varphi \colon (\mathbb{T}N, \mathrm{Gr}(\pi_N)) \dashrightarrow (\mathbb{T}M, \mathrm{Gr}(\pi_M))$ is a Dirac morphism.*

Proof. φ is a Poisson map if and only if

$$\pi_N(\varphi^*\mu_1, \varphi^*\mu_2) = \pi_M(\mu_1, \mu_2)$$

for all $\mu_1, \mu_2 \in T^*M$. Equivalently, this means

$$T\varphi(\pi_N^\sharp(\varphi^*\mu)) = \pi_M^\sharp(\mu)$$

for all $\mu \in T^*M$, i.e.

$$\pi_N^\sharp(\varphi^*\mu)) + \varphi^*\mu \sim_\varphi \pi_M^\sharp(\mu) + \mu$$

for all $\mu \in T^*M$. This precisely means that for every $x \in \mathrm{Gr}(\pi_M)_{\varphi(n)}$ there exists $y \in \mathrm{Gr}(\pi_N)_n$, necessarily unique, with $y \sim x$. □

Let $\mathbb{T}\varphi \colon (\mathbb{T}N, F) \dashrightarrow (\mathbb{T}M, E)$ be a Dirac morphism. Using the uniqueness assumption in the definition, one obtains a linear maps $E_{\varphi(n)} \to F_n$, taking $x \in E_{\varphi(n)}$ to the unique $y \in F_n$ such that $y \sim_\varphi x$. It is not hard to see that this depends smoothly on n, and hence gives a comorphism of vector bundles.

Proposition 6.5. *Any Dirac morphism $\mathbb{T}\varphi \colon (\mathbb{T}N, F) \dashrightarrow (\mathbb{T}M, E)$ defines a Lie algebroid comorphism $F \dashrightarrow E$.*

Proof. We have to check that (i) the pull-back map $\varphi^* \colon \Gamma(E) \to \Gamma(F)$ preserves brackets, and (ii) the anchor satisfies $T\varphi(\mathsf{a}(y)) = \mathsf{a}(x)$ for $y \sim_\varphi x$. But both properties are immediate from the previous Lemma. □

6.2. **Pull-backs of Dirac structures.** In general, there is no natural way of pulling back a Poisson structure under a smooth map $\varphi\colon N \to M$. However, such pull-back operations are defined for Dirac structures, under transversality assumptions.

Proposition 6.6. *Suppose $E \subseteq \mathbb{T}M$ is a Dirac structure, and $\varphi\colon N \to M$ is transverse to the anchor of E. Then*

$$\varphi^! E = \{y \in \mathbb{T}N |\ \exists x \in E\colon y \sim_\varphi x\}$$

is again a Dirac structure.

Put differently, $\varphi^! E$ is the pre-image of E under the relation $\mathbb{T}\varphi$.

Proof. Consider first the case that φ is the embedding of a submanifold, $\varphi\colon N \hookrightarrow M$. The transversality condition ensures that $\varphi^! E$ is a subbundle of $\mathbb{T}N$ of the right dimension; since $\mathbb{T}\varphi$ preserves metrics it is isotropic, hence Lagrangian. It also follows from the transversality that for any $y \in \varphi^! E$, the element $x \in E$ such that $y \sim_\varphi x$ is *unique*; this defines an inclusion

$$\varphi^! E \hookrightarrow E|_N$$

with image $\mathsf{a}^{-1}(TN) \cap E$. Hence, every section $\tau \in \Gamma(\varphi^! E)$ admits an extension to a section $\sigma \in \Gamma(E)$; thus $\tau \sim_\varphi \sigma$. Conversely, given $\sigma \in \Gamma(E)$ such that $\mathsf{a}(\sigma)$ is tangent to N we have $\tau \sim_\varphi \sigma$ for a (unique) section τ. Suppose τ_1, τ_2 are sections of $\varphi^! E$, and choose $\sigma_i \in \Gamma(E)$ such that $\tau_i \sim_\varphi \sigma_i$. Then $[\![\tau_1, \tau_2]\!] \sim_\varphi [\![\sigma_1, \sigma_2]\!]$, so that $[\![\tau_1, \tau_2]\!]$ is a section of $\varphi^! E$. This proves the proposition for the case of an embedding.

In the general case, given φ consider the embedding of N as the graph of φ,

$$j\colon N \to M \times N,\ n \mapsto (\varphi(n), n).$$

It is easy to see that

$$\varphi^! E = j^!(E \times TN)$$

as subsets of $\mathbb{T}N$. Since $j^!(E \times TN)$ is a Dirac structure by the above, we are done. □

In particular, if π is a Poisson structure on M, and $\varphi\colon N \to M$ is transverse to the map $\pi^\sharp$, we can define the pull-back $\varphi^! \operatorname{Gr}(\pi) \subseteq \mathbb{T}N$ *as a Dirac structure*. In general, this is not a Poisson structure. We have the following necessary and sufficient condition.

Proposition 6.7. *Suppose (M, π) is a Poisson manifold, and $\varphi\colon N \to M$ is transverse to $\pi^\sharp$. Then $\varphi^! \operatorname{Gr}(\pi) \subseteq \mathbb{T}N$ defines a Poisson structure π_N if and only if φ is an immersion as a cosymplectic submanifold. That is,*

$$TM|_N = TN \oplus \pi^\sharp(\operatorname{ann}(TN)). \tag{46}$$

Proof. $\varphi^! \operatorname{Gr}(\pi)$ defines a Poisson structure if and only if it is transverse to $TN \subseteq \mathbb{T}N$. But $\varphi^! \operatorname{Gr}(\pi) \cap TN$ contains in particular elements $y \in \mathbb{T}N$ with $y \sim_\varphi 0$. Writing $y = w + \nu$ with a tangent vector w and covector ν, this means that $\nu = 0$ and $w \in \ker(T\varphi)$. Hence, it is necessary that $\ker(T\varphi) = 0$.

Let us therefore assume that φ is an immersion. For $w \in TN \subseteq \mathbb{T}N$, we have

$$w \sim_\varphi \pi^\sharp(\mu) + \mu \in \operatorname{Gr}(\pi) \quad \Leftrightarrow \quad (Ti)(w) = \pi^\sharp(\mu),\ \ \mu \in \ker(\varphi^*) = \operatorname{ann}(TN).$$

Hence, the condition $i^! E \cap TN = 0$ is equivalent to $\pi^\sharp(\operatorname{ann}(TN)) \cap TN = 0$. □

7. Normal bundles and Euler-like vector fields

In this section we will develop some differential geometric machinery, in preparation for our approach to the Weinstein splitting theorem.

7.1. **Normal bundles.** Consider the category of manifold pairs: An object (M, N) in this category is a manifold M together with a submanifold $N \subseteq M$, and a morphism $\Phi\colon (M_1, N_1) \to (M_2, N_2)$ is a smooth map $\Phi\colon M_1 \to M_2$ taking N_1 to N_2. The normal bundle functor ν assigns to (M, N) the vector bundle

$$\nu(M, N) = TM|_N/TN,$$

over N, and to a morphism $\Phi\colon (M_1, N_1) \to (M_2, N_2)$ the vector bundle morphism

$$\nu(\Phi)\colon \nu(M_1, N_1) \to \nu(M_2, N_2).$$

Under composition of morphisms, $\nu(\Phi' \circ \Phi) = \nu(\Phi') \circ \nu(\Phi)$.

Example 7.1 (Tangent functor). Given a pair (M, N), the tangent functor gives a new pair (TM, TN) with a morphism

$$p\colon (TM, TN) \to (M, N)$$

defined by the bundle projection. Applying the normal functor, we obtain an example of a *double vector bundle*

$$\begin{array}{ccc} \nu(TM, TN) & \longrightarrow & \nu(M, N) \\ \downarrow & & \downarrow \\ TN & \longrightarrow & N \end{array}$$

On the other hand, by applying the tangent functor to $\nu(M, N) \to N$ we obtain a similar double vector bundle,

$$\begin{array}{ccc} T\nu(M, N) & \longrightarrow & \nu(M, N) \\ \downarrow & & \downarrow \\ TN & \longrightarrow & N \end{array}$$

There is a canonical identification $T\nu(M, N) \cong \nu(TM, TN)$ identifying these two double vector bundles, in such a way that for morphisms Φ of pairs, $\nu(T\Phi) = T\nu(\Phi)$. See [**?**] for details.

Example 7.2. Suppose X is a vector field on M such that $X|_N$ is tangent to N. Viewed as a section of the tangent bundle $p\colon TM \to M$, it defines a morphism $X\colon (M, N) \to (TM, TN)$, inducing

$$\nu(X)\colon \nu(M, N) \to \nu(TM, TN) = T\big(\nu(M, N)\big).$$

From $p \circ X = \mathrm{id}_M$ we get

$$T\nu(p) \circ \nu(X) = \nu(Tp) \circ \nu(X) = \nu(\mathrm{id}_M) = \mathrm{id}_{\nu(M,N)}\,.$$

That is, $\nu(X)$ is a vector field on $\nu(M, N)$. It is called the *linear approximation* to X along N. In local bundle trivializations, the linear approximation is the first order Taylor approximation in the normal directions.

Lemma 7.3. *For a vector bundle $V \to M$, there is a canonical identification $\nu(V, M) \cong V$.*

Proof. The restriction of $TV \to V$ to $M \subseteq V$ splits into the tangent bundle of the fiber and the tangent space to the base: $TV|_M = V \oplus TM$. Hence $\nu(V, M) = TV|_M/TM = V$. $\square$

7.2. **Tubular neighborhood embeddings.** Given a pair (M, N), Lemma 7.3, applied to $\nu(M, N) \to N$, gives an identification $\nu(\nu(M, N), N) = \nu(M, N)$.

Definition 7.4. A *tubular neighborhood embedding* is a map of pairs

$$\varphi\colon (\nu(M, N), N) \to (M, N)$$

such that $\varphi\colon \nu(M, N) \to M$ is an embedding as an open subset, and the map $\nu(\varphi)$ is the identity.

Definition 7.5. Let X be a vector field on M that is tangent to N. By a *linearization* of the vector field X along N, we mean a tubular neighborhood embedding φ taking $\nu(X)$ to X on a possibly smaller neighborhood of N.

The problem of C^∞-linearizability of vector fields is quite subtle; the main result (for $N = \mathrm{pt}$) is the *Sternberg linearization theorem* [34] which proves existence of linearizations under *non-resonance conditions.*

Example 7.6. The vector field on $\mathbb{R}^2$ given as

$$x\frac{\partial}{\partial x} + (2y + x^2)\frac{\partial}{\partial y}$$

does not satisfy Sternberg's non-resonance conditions, and turns out to be not linearizable.[3]. On the other hand,

$$2x\frac{\partial}{\partial x} + (y + x^2)\frac{\partial}{\partial y}$$

is linearizable.

Example 7.7. A vector field on $\mathbb{R}^n$ of the form

$$\sum_i x_i\frac{\partial}{\partial x_i} + \text{ higher order terms}$$

satisfies the non-resonance condition, and hence is always linearizable.

We will make the following definition.

Definition 7.8. [8] A vector field $X \in \mathfrak{X}(M)$ is called *Euler-like along* N if it is complete, with $X|_N = 0$, and with linear approximation $\nu(X)$ the Euler vector field $\mathcal{E}$ on $\nu(M, N)$.

Remark 7.9. (a) In a submanifold chart, with coordinates $x_1, \ldots, x_n$ on N and $y_1 \ldots, y_k$ in the transverse direction, an Euler-like vector field has the form

$$X = \sum_i y_i\frac{\partial}{\partial y_i} + \sum_i g_i(x, y)\frac{\partial}{\partial y_i} + \sum_j h_j(x, y)\frac{\partial}{\partial x_j},$$

where g_i vanish to second order for $y \to 0$, while h_j vanish to first order.

[3]See http://mathoverflow.net/questions/76971/nice-metrics-for-a-morse-gradient-field-counterexample-request

(b) Another coordinate-free characterization of Euler-like vector fields is as follows [?] Let $\mathcal{I} \subseteq C^\infty(M)$ be the ideal of functions vanishing along N. Its powers $\mathcal{I}^k$ are the functions vanishing to order k along N. Then a complete vector field X is Euler-like along N if and only if for all $f \in \mathcal{I}$, we have that $\mathcal{L}_\mathcal{E} f$ equals f modulo functions vanishing on N to second order (or higher). That is,

$$\mathcal{L}_X - \mathrm{id}\colon \mathcal{I} \to \mathcal{I}^2.$$

More generally, this property implies that

$$\mathcal{L}_X - k\,\mathrm{id}\colon \mathcal{I}^k \to \mathcal{I}^{k+1}$$

for all $k = 0, 1, 2, \ldots$.

An Euler-like vector field determines a tubular neighborhood embedding:

Theorem 7.10. *If $X \in \mathfrak{X}(M)$ is Euler-like along N, then X determines a unique tubular neighborhood embedding $\varphi\colon \nu(M,N) \to M$ such that*

$$\mathcal{E} \sim_\varphi X.$$

Proof. The main point is to show that X is linearizable along N. Start out by picking any tubular neighborhood embedding to assume $M = \nu(M,N)$. Since $\nu(X) = \mathcal{E}$, it follows that the difference $Z = X - \mathcal{E}$ vanishes to second order along N. Let κ_t denote scalar multiplication by t on $\nu(M,N)$, and consider the family of vector fields, defined for $t \neq 0$,

$$Z_t = \frac{1}{t}\kappa_t^* Z$$

Since Z vanishes to second order along N, this is well-defined even at $t = 0$. Let ϕ_t be its (local) flow. [4] On a sufficiently small open neighborhood of N in $\nu(M,N)$, it is defined for all $|t| \le 1$. The flow Ψ_s of the Euler vector field $\mathcal{E}$ is

$$\Psi_s = \kappa_{\exp(-s)}$$

by substitution $t = \exp(s)$ this shows that

$$\frac{d}{dt}\kappa_t^* = t^{-1}\kappa_t^* \circ \mathcal{L}_\mathcal{E}.$$

Consequently,

$$\frac{d}{dt}(tZ_t) = \frac{d}{dt}(\kappa_t^* Z) = \mathcal{L}_\mathcal{E} Z_t = [\mathcal{E}, Z_t].$$

Therefore,

$$\frac{d}{dt}(\phi_t)_*(\mathcal{E} + tZ_t) = (\phi_t)_*(\mathcal{L}_{Z_t}(\mathcal{E} + tZ_t) + [\mathcal{E}, Z_t]) = 0,$$

which shows that $(\phi_t)_*(\mathcal{E} + tZ_t)$ does not depend on t. Comparing the values at $t = 0$ and $t = 1$ we obtain $(\phi_1)_* X = \mathcal{E}$, so that $(\phi_1)^{-1}$ giving the desired linearization on a neighborhood of N. In summary, this shows that there exists a map from a neighborhood of the zero section of $\nu(M,N)$ to a neighborhood of N in M, intertwining the two vector fields $\mathcal{E}$ and X, and hence also their flows. Since X is complete, we may use the flows to extend globally to a tubular neighborhood embedding of the full normal bundle. This proves existence.

[4] Thus $\frac{d}{dt}(\phi_t)_* = (\phi_t)_* \circ \mathcal{L}_{Z_t}$ as operators on tensor fields (e.g., functions, vector fields, and so on).

For uniqueness, suppose that a tubular neighborhood embedding ψ satisfying $\mathcal{E} \sim_\psi X$ is given. Let Ψ_s be the flow of $\mathcal{E}$ and Φ_s the flow of X. We have that $\kappa_t = \Psi_{-\log(t)}$ for $t > 0$; accordingly we define $\lambda_t = \Phi_{-\log(t)}$. Since ν_N is invariant under κ_t for all $t > 0$, its image $U = \psi(\nu_N)$ is invariant under λ_t for all $t > 0$. Furthermore, since $\lim_{t\to 0} \kappa_t$ is the retraction p from ν_N onto $N \in \nu_N$, we have

$$U = \{m \in M | \lim_{t\to 0} \lambda_t(m) \text{ exists and lies in } N \subseteq M\}. \tag{47}$$

We want to give a formula for the inverse map $\psi^{-1} \colon U \to \nu(M,N)$. For all $v \in \nu(M,N)$, with base point $x \in N$, the curve $\kappa_t(v)$ in $\nu(M,N)$ has tangent vector at $t = 0$ equal to v itself (using the identification $T\nu(M,N)|_N = TN \oplus \nu(M,N)$). Hence

$$v = \Big(\frac{d}{dt}\Big|_{t=0} \kappa_t(v)\Big) \mod T_xN.$$

Using $\lambda_t \circ \psi = \psi \circ \kappa_t$, and writing $\psi(v) = m$, this shows,

$$\psi^{-1}(m) = \Big(\frac{d}{dt}\Big|_{t=0} \lambda_t(m)\Big) \mod T_xN \in T_xM/T_xN \tag{48}$$

Formulas (47) and (48) give a description of the tubular neighborhood embedding directly in terms of the flow of X, which proves the uniqueness part. ☐

Example 7.11. Let $V \to M$ be a vector bundle. Its Euler vector field $\mathcal{E}_V$ determines a tubular neighborhood embedding

$$\nu(V,M) \to V.$$

This is just the 'canonical identification' from Lemma 7.3.

Proposition 7.12. *Let $\Phi \colon (M_1,N_1) \to (M_2,N_2)$ be smooth map of pairs. Suppose X_i are Euler-like vector fields for these pairs, and that $X_1 \sim_\Phi X_2$. Then the following diagram, where the vertical maps are the tubular neighborhood embeddings defined by the X_i, commutes:*

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ \uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_2} \\ \nu(M_1,N_1) & \xrightarrow[\nu(\Phi)]{} & \nu(M_2,N_2) \end{array}$$

Proof. Let U_i be the images of the tubular neighborhood embeddings φ_i. Since Φ intertwines the Euler-like vector fields, it follows that $\Phi(U_1) \subseteq U_2$. We need to show that

$$\nu(\Phi) \circ \varphi_1^{-1} = \varphi_2^{-1} \circ \Phi,$$

but this is immediate from the explicit formula for φ_i^{-1}. ☐

7.3. **The Grabowski-Rotkiewicz theorem.** This result has the following remarkable consequence for vector bundles, due to Grabowski-Rotkiewicz [20, Corollary 2.1].

Theorem 7.13 (Grabowski-Rotkiewicz)**.** *Let $V_1 \to M_1$ and $V_2 \to M_2$ be vector bundles, and $\Psi \colon V_1 \to V_2$ a smooth map. Then Ψ is a vector bundle morphism if and only if Ψ intertwines the Euler vector fields.*

Proof. Proposition 7.12 gives a commutative diagram

$$\begin{CD} V_1 @>{\Psi}>> V_2 \\ @A{\varphi_1}AA @A{\varphi_2}AA \\ \nu(V_1, M_1) @>>{\nu(\Psi)}> \nu(V_2, M_2) \end{CD}$$

Here the vertical maps, given as the tubular neighborhood embeddings for the Euler(-like) vector fields, are just the standard identifications of the normal bundle of the zero section inside a vector bundle, with the vector bundle itself. In particular, they are vector bundle isomorphisms. Since the lower horizontal map is a vector bundle map, the upper horizontal map is one also. □

This result shows that a smooth map of vector bundles is a vector bundle morphism if and only if it intertwines the scalar multiplications – the fact that it intertwines additions is automatic. We may thus characterize vector bundles as manifold pairs (V, M) together with a smooth map action $\kappa_t \colon V \to V$ of the multiplicative group $\mathbb{R}_{>0}$ such that

- for all $v \in V$, $\lim_{t \to 0} \kappa_t(v) \in M$,
- the action preserves M, i.e. $\kappa_t \colon (V, M) \to (V, M)$,
- the resulting action $\nu(\kappa_t) \colon \nu(V, M) \to \nu(V, M)$ is the standard scalar multiplication by $t > 0$.

Indeed, letting $\mathcal{E}$ be the vector field on V with flow $s \mapsto \kappa_{\exp(-s)}$, the second condition shows that $\mathcal{E}$ is tangent to M, the last condition shows that $\mathcal{E}$ is Euler-like (in particular, it vanishes along M), and the first condition guarantees that the resulting tubular neighborhood embedding $\nu(V, M) \to V$ is surjective. The manifold V inherits the vector bundle structure via this identification with $\nu(V, M)$. Grabowski-Rotkiewicz [20] also have the following attractive characterization of vector subbundles.

Proposition 7.14. *Let $V \to M$ be a vector bundle. A subset $L \subseteq V$ is a vector subbundle if and only if it is invariant under scalar multiplication κ_t, for all $t \geq 0$ (including $t = 0$)*

Proof. It is a general result (see e.g. [**?**]) that if Q is a manifold and $\Phi \colon Q \to Q$ is a smooth projection (i.e., $\Phi \circ \Phi = \Phi$), then the image $\Phi(Q) \subseteq Q$ is a submanifold. In our case, $\kappa_0 \colon V \to V$ is such a projection, and so is its restriction to L. It follows that $\kappa_0(L) = \kappa_0(V) \cap L = M \cap L$ is a smooth submanifold of L. The Euler vector field of V restricts to L, and is Euler-like along $M \cap L$. Hence, L acquires the structure of a vector bundle over $M \cap L$. Since the inclusion $L \to V$ intertwines Euler vector fields, it is a vector bundle morphism. □

Remark 7.15. One of the main applications of the Grabowski-Rotkiewicz theorem is a simple characterization of *double vector bundles.* A double vector bundle is a commutative square

$$\begin{CD} D @>>> A \\ @VVV @VVV \\ B @>>> M \end{CD}$$

where all maps are vector bundle maps, with suitable compatibility conditions between the horizontal and vertical vector bundle structures. In the original definition, this was given by a long list of conditions for vertical and horizontal addition and multplication. According to Grabowski-Rotkiewicz, the compatibility conditions are equivalent to stating that the horizontal and vertical Euler vector fields (equivalently the horizontal and vertical scalar multiplication) commute! A typical example is the tangent bundle of a vector bundle,

$$\begin{array}{ccc} TV & \longrightarrow & V \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

8. The splitting theorem for Lie algebroids

8.1. **Statement of the theorem.** Our goal in this section is to prove the following result.

Theorem 8.1. *Let $(E, \mathsf{a}, [\cdot,\cdot])$ be a Lie algebroid over M, and $N \subseteq M$ a submanifold transverse to the anchor. Then there exists a tubular neighborhood embedding $\varphi\colon \nu(M,N) \to U \subseteq M$ with an isomorphism of Lie algebroids,*

$$\begin{array}{ccc} p^! i^! E & \longrightarrow & E|_U \subseteq E \\ \downarrow & & \downarrow \\ \nu(M,N) & \xrightarrow[\varphi]{} & U \subseteq M \end{array}$$

Here $p\colon \nu(M,N) \to N$ and $i\colon N \to M$ are the projection and inclusion.

See (25) for the definition of a pull-back of a Lie algebroid under a smooth map transverse to the anchor. In this case, we have that

$$i^! E = \mathsf{a}^{-1}(TN),$$

and

$$p^! i^! E = i^! E \times_{TN} T\nu(M,N).$$

Remark 8.2. If the normal bundle is *trivial*, $\nu(M,N) = N \times S$ where S is a vector space, then $p^! i^! E$ is simply the direct product $i^! E \times TS$. Hence, we obtain an isomorphism

$$i^! E \times TS \cong E|_U$$

which justifies the name 'splitting theorem'. Note that this isomorphism also shows

$$\mathsf{a}(i^! E) \times TS = \mathsf{a}(E)|_U.$$

Hence, the leaves (if any) of the singular distribution $\mathsf{a}(E)|_U$ are of the product form $L \times S$, where L is a leaf of the singular distribution $i^! E$.

Remark 8.3. We can use this to show that there exists an integral submanifold (leaf) through every given $m \in M$. Indeed, take $N \subseteq M$ to be any submanifold passing through m, with

$$T_m M = \mathsf{a}(E_m) \oplus T_m N.$$

Taking N smaller if necessary, we can assume that N is transverse to a everywhere, and that $\nu(M,N) = N \times S$ as above. Then $\mathsf{a}(i^! E)_m = 0$, so that $i^! E$ has the single point $\{m\}$ as an integral submanifold. We conclude that $\{m\} \times S$ is an integral submanifold of $E|_U$.

8.2. **Normal derivative.** The key idea in the proof of the splitting theorem 8.1 is to choose a section $\epsilon \in \Gamma(E)$ such that the vector field $X = \mathsf{a}(\epsilon)$ is Euler-like. The tubular neighborhood embedding φ will be defined by X, and the bundle map lifting φ will be determined by the choice of ϵ.

To prove the existence of ϵ, we need yet another characterization of Euler-like vector fields. Let $V \to M$ be a vector bundle. If a section $\sigma \in \Gamma(V)$ vanishes along $N \subseteq M$, then by applying the normal functor to $\sigma\colon (M, N) \to (V, M)$, and recalling $\nu(V, M) = V$, we obtain a map

$$\mathrm{d}^N \sigma\colon \nu(M, N) \to V|_N$$

called the *normal derivative* of σ along N. Using partitions of unity, it is easy to see that *any* bundle map $\nu(M, N) \to V|_N$ arises in this way, as the normal derivative of a section.

Remark 8.4. The normal derivative of σ can be characterized in several other ways. for example, note that for $\tau \in \Gamma(V^*)$, the restriction $\mathrm{d}\langle\sigma, \tau\rangle|_N \in \Gamma(T^*M|_N)$ vanishes on vector tangent to N, hence it is a section of $\mathrm{ann}(TN)$, and is tensorial in τ. Hence

$$\mathrm{d}\langle\sigma, \cdot\rangle|_N \in \Gamma(\mathrm{ann}(TN) \otimes V|_N),$$

and this is the normal derivative. If $\sigma_1, \ldots, \sigma_r$ is a local frame of sections of V, so that $\sigma = \sum_i f_i \sigma_i$ with $f_i|_N = 0$, then

$$\mathrm{d}^N \sigma = \sum_i \mathrm{d}f_i \otimes \sigma_i|_N.$$

Example 8.5. If X is a vector field vanishing along N, then X is Euler-like if and only if its normal derivative

$$\mathrm{d}^N X\colon \nu(M, N) \to TM|_N$$

defines a splitting of the quotient map $TM|_N \to \nu(M, N)$. To see this, let $f \in C^\infty(M)$ with $f|_N = 0$. Then

$$\langle \mathrm{d}^N X,\ \mathrm{d}f|_N\rangle = \mathrm{d}\langle X,\ \mathrm{d}f\rangle|_N = \mathrm{d}X(f)|_N$$

This coincides with $\mathrm{d}f|_N$ if and only if $X(f) = f$ modulo functions vanishing to second order.

8.3. **Anchored vector bundles.** For the following considerations, the Lie bracket on sections of E does not play a role, hence we will work in the more general context of *anchored vector bundles*. An anchored vector bundle is a vector bundle $E \to M$ together with a bundle map $\mathsf{a}\colon E \to TM$, called the *anchor*. Morphism of anchored vector bundles are defined in the obvious way. We denote by $\mathrm{Aut}_{AV}(E)$ the bundle automorphisms $\widehat{\Phi}$ (with base map Φ) compatible with the anchor, i.e.

$$\mathsf{a} \circ \widehat{\Phi} = T\Phi \circ \mathsf{a};$$

its Lie algebra is denoted $\mathfrak{aut}_{AV}(E)$ and consists of infinitesimal vector bundle automorphisms $D\colon \Gamma(E) \to \Gamma(E)$, with corresponding vector feld X, such that $\mathsf{a}(D\sigma) = [X, \mathsf{a}(\sigma)]$. In this section, we prefer to regard the elements of $\mathfrak{aut}(E)$ has vector fields $\widehat{X}$ on the total space of E, homogeneous of degree 0 and with base vector field X. The compatibility with the anchor $\mathsf{a}\colon E \to TM$ is then expressed as the property

$$\widetilde{X} \sim_{\mathsf{a}} X_T, \tag{49}$$

where $X_T \in \mathfrak{X}(TM)$ is the tangent lift of X. Given a submanifold $N \subseteq M$ that is transverse to a, we can define a 'pull-back' $i^! E = \mathsf{a}^{-1}(TN)$; it is an anchored subbundle of E.

Proposition 8.6. *There exists a section $\epsilon \in \Gamma(E)$ such that $\epsilon|_N = 0$, and with $\mathsf{a}(\epsilon)$ Euler-like.*

Proof. The transversality condition means precisely that the map $E|_N \to \nu(M, N)$, given by the anchor map to $TM|_N$ followed by the quotient map, is surjective. Its kernel is the subbundle $i^!E$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i^!E & \longrightarrow & E|_N & \longrightarrow & \nu(M,N) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathsf{a}} & & \big\downarrow{\scriptstyle \mathsf{a}} & & \big\downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & TN & \longrightarrow & TM|_N & \longrightarrow & \nu(M,N) & \longrightarrow & 0 \end{array}$$

Choose a section $\epsilon \in \Gamma(E)$ with $\epsilon|_N = 0$, whose normal derivative $\mathrm{d}^N\epsilon \colon \nu(M, N) \to E_N$ splits the map $E|_N \to \nu(M, N)$. Its image under the anchor is a vector field $X = \mathsf{a}(\epsilon)$, with $X|_N = 0$, such that $\mathrm{d}^N X$ splits the map $TM|_N \to \nu(M, N)$. That is, $\mathsf{a}(\epsilon)$ is Euler-like. ☐

The definition of $i^!E$ generalizes to any smooth map $\Phi \colon N \to M$ that is transverse to E. Indeed, transversality implies that the fiber product $\Phi^!E = E \times_{TM} TN$ is a vector bundle over N, with anchor given by projection to TN. It comes with a morphism of anchored vector bundles $\Phi^!E \to E$.

Lemma 8.7. *Suppose (E, a) is an anchored vector bundle over M, and $N \subseteq M$ is a submanifold transverse to a. Then there is a canonical isomorphism (of double vector bundles)*

$$\nu(E, i^!E) \cong p^!i^!E,$$

where $i \colon N \to M$ is the inclusion and $p \colon \nu(M, N) \to N$ is the projection.

Proof. By applying the normal functor to $(E, i^!E) \to (TM, TN)$, we obtain a double vector bundle

$$\begin{array}{ccc} \nu(E, i^!E) & \longrightarrow & \nu(TM, TN) \\ \big\downarrow & & \big\downarrow \\ i^!E & \longrightarrow & TN \end{array}$$

A dimension count shows that this is a fiber product diagram. But the fiber product of $i^!E$ and $\nu(TM, TN) = T\nu(M, N)$ over TN s just the pullback $p^!i^!E$. ☐

8.4. **Proof of the splitting theorem for Lie algebroids.** For any Lie algebroid, the anchor map on sections has a canonical lift

$$\begin{array}{ccc} & & \mathfrak{aut}_{\mathcal{LA}}(E) \\ & \overset{\widetilde{\mathsf{a}}}{\nearrow} & \big\downarrow \\ \Gamma(E) & \xrightarrow[\mathsf{a}]{} & \mathfrak{X}(M) \end{array}$$

here $\widetilde{\mathsf{a}}(\sigma) = [\sigma, \cdot]$, viewed as an infinitesimal Lie algebroid automorphism. In particular, our Euler-like vector field $X = \mathsf{a}(\epsilon)$ gets lifted to $\widetilde{X} = \widetilde{\mathsf{a}}(\epsilon)$. We will think of $\widetilde{X}$ as a linear vector field on E (that is, $\widetilde{X}$ is homogeneous of degree 0).

Lemma 8.8. *The vector field $\widetilde{X}$ is Euler-like for $(E, i^!E)$.*

Proof. We have to show that $\nu(\widetilde{X})$ is the Euler vector field for $\nu(E, i^!E) \to i^!E$. Since $\widetilde{X}$ preserves the anchor, we have that $\widetilde{X} \sim_a X_T$, the tangent lift of X. Hence, under the map $\mathsf{a}\colon (E, i^!E) \to (TM, TN)$,

$$\nu(\widetilde{X}) \sim_{\nu(\mathsf{a})} \nu(X_T) = \nu(X)_T = \mathcal{E}_T$$

where $\mathcal{E}_T \in \mathfrak{X}(T\nu(M,N))$ is the tangent lift of $\mathcal{E} \in \mathfrak{X}(\nu(M,N)$. But the tangent lift of an Euler vector field on a vector bundle $V \to M$ is just the Euler vector field of $TV \to TM$. We conclude that $\nu(\widetilde{X})$ is $\nu(\mathsf{a})$-related to the Euler vector field of $\nu(TM, TN) \to TN$. But by the pullback diagram 8.7, the bundle $\nu(E, i^!E) \to i^!E$ is just the pull-back of the bundle $\nu(TM, TN) \to TN$ under $\mathsf{a}\colon i^!E \to TN$. We conclude that $\nu(\widetilde{X})$ is an Euler vector field. $\square$

We are now in position to prove the splitting theorem for Lie algebroids, Theorem 8.1.

Proof of the splitting theorem. Let $\Phi_s, \widetilde{\Phi}_s$ be the flow of $X, \widetilde{X}$, respectively. Put

$$\lambda_t = \Phi_{-\log(t)},\ \widetilde{\lambda}_t = \widetilde{\Phi}_{-\log(t)}.$$

Let $\phi\colon \nu(M,N) \to U$ be the tubular neighborhood embedding defined by X. Since $\widetilde{\lambda}_t$ covers the flow of λ_t, it is defined over $E|_U$ even for $t = 0$. Consider the following diagram, defined for all $0 \le t \le 1$,

$$\begin{array}{ccccc} E|_U & \xrightarrow[\cong]{} & \lambda_t^!(E|_U) & \xrightarrow[\cong]{} & \phi^!\lambda_t^!(E|_U) \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow[\mathrm{id}]{} & U & \xrightarrow[\phi^{-1}]{} & \nu(M,N) \end{array}$$

Here the first upper horizontal arrow is given by the Lie algebroid morphism

$$E|_U \to \lambda_t^!(E|_U) \subseteq TU \times E|_U, \quad v \mapsto (\mathsf{a}(v),\ \widetilde{\lambda}_t(v)).$$

This map is an isomorphism for all t: If $t > 0$, this is clear since $\widetilde{\lambda}_t$ is an isomorphism then. If $t = 0$ we note that it is an isomorphism along $N \subseteq U$, hence also on some neighborhood of N, and using e.g. $\lambda_0 = \lambda_0 \circ \lambda_t$ we conclude that it is an isomorphism over all of U.

We hence obtain a family of Lie algebroid isomorphisms $\phi^!\lambda_t^!(E|_U) \to E|_U$, all with the base map ϕ. For $t = 0$, we have

$$\lambda_0 \circ \phi = \phi \circ \kappa_0 = i \circ p,$$

so we obtain the desired Lie algebroid isomorphism

$$\widetilde{\phi}\colon p^!i^!(E|_U) \to E|_U,$$

with base map ϕ. $\square$

Remark 8.9. The map $\widetilde{\phi}$ can itself be regarded as a tubular neighborhood embedding. Indeed, under the isomorphism $\nu(E, i^!E) \cong p^!i^!E = \mathbb{T}\nu(M,N) \times_{TN} (i^!E)$, the inverse of the tubular neighborhood embedding is given by

$$E|_U \to \nu(E, i^!E) \cong T\nu(M,N) \times_{TN} (i^!E), \quad v \mapsto \big(T\phi^{-1}(\mathsf{a}(v)), \widetilde{\lambda_0}(v)\big)$$

(where we regard $\widetilde{\lambda_0}$ as a map to $i^!E \subseteq E|_U$). Indeed, this is the unique map of anchored vector bundles, with base map ϕ^{-1}, such that the $i^!E$-component is $\widetilde{\lambda_0}(v)$. But this is just the description of $\widetilde{\phi}^{-1}$.

8.5. **The Stefan-Sussmann theorem.** The idea of proof of the splitting theorem for Lie algebroids also works for anchored vector bundles, provided that they satisfy the following condition.

Definition 8.10. [8] An anchored vector bundle (E, a) is called *involutive* if $\Gamma(\mathsf{a}(E)) \subseteq \mathfrak{X}(M)$ is closed under Lie brackets.

For example, Lie algebroids are involutive, due to the property $[\mathsf{a}(\sigma), \mathsf{a}(\tau)] = \mathsf{a}([\sigma, \tau])$ for sections $\sigma, \tau \in \Gamma(E)$. Courant algebroids are involutive as well.

Remark 8.11. Stefan [33] and Sussmann [36] defined a 'singular distribution' on a manifold M to be a subset $D \subseteq TM$ spanned locally by a finite collection of vector fields. They developed necessary and sufficient conditions of integrability for such singular distributions; in terms of the submodule $\mathcal{D} \subseteq \mathfrak{X}(M)$ of vector fields taking values in D. However, their results contain some errors that were corrected by Balan [4].

Androulidakis-Skandalis [3] take a 'singular distribution' to be a locally finitely generated submodule $\mathcal{C} \subseteq \mathfrak{X}(M)$, and call it 'integrable' if $\mathcal{C}$ is involutive. Our definition of involutive anchored vector bundles is very similar to this viewpoint.

Theorem 8.12. [8] *Let (E, a) be an involutive anchored vector bundle over M, and $N \subseteq M$ a submanifold transverse to the anchor. Then there exists a tubular neighborhood embedding $\phi\colon \nu(M, N) \to U \subseteq M$, which is the base map for an isomorphism of anchored vector bundles,*

$$p^! i^! E \to E|_U.$$

The discussion for Lie algebroids in Remark 8.3 applies to the more general setting, and shows that every point of $m \in M$ is contained in a leaf S of $\mathsf{a}(E)$.

The proof of this theorem is parallel to that for Lie algebroids, once the following result is established:

Proposition 8.13. [8] *An anchored vector bundle (E, a) is involutive if and only if the map $\mathsf{a}\colon \Gamma(E) \to \mathfrak{X}(M)$ lifts to a map $\widetilde{\mathsf{a}}\colon \Gamma(E) \to \mathfrak{aut}_{AV}(E)$. In this case, one can arrange that the lift satisfies*

$$\widetilde{\mathsf{a}}(f\sigma)\tau = f\widetilde{\mathsf{a}}(\sigma)\tau - (\mathsf{a}(\tau)f)\sigma \tag{50}$$

for all $\sigma, \tau \in \Gamma(E)$.

Proof. Given such a lift $\widetilde{\mathsf{a}}$, the submodule $\mathsf{a}(\Gamma(E))$ is involutive because

$$[\mathsf{a}(\sigma), \mathsf{a}(\tau)] = \mathsf{a}\big(\widetilde{\mathsf{a}}(\sigma)\tau\big)$$

(by definition of $\mathfrak{aut}_{AV}(E)$). In the other direction, one constructs $\widetilde{\mathsf{a}}$ with the help of a connection. (See [8].) □

Having chosen such a lift, and having chosen a section $\epsilon \in \Gamma(E)$ such that $X = \mathsf{a}(\epsilon)$ is Euler-like, one proves as in the case of Lie algebroids that $\widetilde{X} = \mathsf{a}(\epsilon)$ is Euler-like. The same approach as for Lie algebroids, using the flow of $\widetilde{X}$, gives an isomorphism of anchored vector bundles $p^! i^! E \to E|_U$.

8.6. **The Weinstein splitting theorem.** We begin with a statement of the theorem.

Theorem 8.14 (Weinstein splitting theorem [39]). *Let (M,π) be a Poisson manifold, and $m\in M$. There exists a system of local coordinates $q^1,\dots,q^k,p_1,\dots,p_k,y^1,\dots,y^r$ centered at m in which π takes on the following form:*

$$\pi=\sum_{i=1}^k\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i}+\frac{1}{2}\sum_{i,j=1}^r c^{ij}(y)\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j},$$

where $c^{ij}=-c^{ji}$ are smooth functions with $c^{ij}(0)=0$.

Thus, π splits into a sum $\pi=\pi_S+\pi_N$ on $S\times N$, where $S\cong\mathbb{R}^{2k}$ with the standard non-degenerate Poisson structure π_S, and $N=\mathbb{R}^r$ with a Poisson structure π_N having a critical point at $y=0$. The remarkable fact is that one can eliminate any 'cross-terms'. Of course, the transverse Poisson structure π_N can still be quite complicated.

A direct consequence of the splitting theorem is the existence of a *symplectic foliation*:

Corollary 8.15 (Symplectic leaves). *Let M be a Poisson manifold, and $m\in M$. Then there exists a unique maximal (injectively immersed) integral submanifold $S\subseteq M$ of the singular distribution* $\operatorname{ran}(\pi^\sharp)\subseteq TM$.

Proof. In the model, it is immediate that the submanifold $S\subseteq M$ given by $y^i=0$ is a symplectic leaf. (The passage from local integrability to global integrability merely involves patching of the local solutions, and is the same as in the standard proofs of Frobenius' theorem.) □

To give a coordinate-free formulation of Weinstein's theorem, let

$$\mathsf{S}:=\pi^\sharp(T^*_mM).$$

By definition, $\pi_m\in\wedge^2\mathsf{S}\subseteq\wedge^2T_mM$, defining a constant bivector field $\pi_S\in\Gamma(\wedge^2T\mathsf{S})$. It is non-degenerate, corresponding to a symplectic form on S. Let $N\subseteq M$ be a submanifold through m, with the property that

$$T_mM=T_mN\oplus\mathsf{S}.$$

As we saw in Example 3.41, taking N smaller if necessary, it is a cosymplectic submanifold and hence inherits a Poisson structure π_N (see Proposition 6.7). This is referred to as the *transverse Poisson structure*. The coordinate-free formulation of the splitting theorem is as follows:

Theorem 8.16 (Weinstein splitting theorem, II). *The Poisson manifold (M,π) is Poisson diffeomorphic near $m\in M$ to the product of Poisson manifolds,*

$$N\times\mathsf{S}$$

where S is the symplectic vector space $\operatorname{ran}(\pi^\sharp_m)$*, and N is a transverse submanifold as above, equipped with the transverse Poisson structure. More precisely, there exists a Poisson diffeomorphism between open neighborhoods of m in M and of $(m,0)$ in $N\times\mathsf{S}$, taking m to $(m,0)$, and with differential at m equal to the given decomposition $T_mM\to T_mN\oplus\mathsf{S}$.*

Weinstein's theorem has been generalized by Frejlich-Mărcuţ [17] to a normal form theorem around arbitrary cosymplectic submanifolds $N\subseteq M$. Their result is best phrased using some Dirac geometry. Recall again that any cosymplectic manifold inherits a Poisson structure π_N such that

$$\operatorname{Gr}(\pi_N)=i^!\operatorname{Gr}(\pi_M)$$

(as subbundles of $\mathbb{T}N$). See Proposition 6.7. On the other hand, the vector bundle

$$\mathsf{V} = \pi^\sharp(\mathrm{ann}(TN)) \to N$$

has a fiberwise symplectic structure ω_{V}, defined by the restriction of π to $\mathrm{ann}(TN)$. Since it is a complement to TN in $TM|_N$, we will identify

$$\mathsf{V} \cong \nu(M, N);$$

in particular, the projection map will be denoted $p\colon \mathsf{V} \to N$.

As for any symplectic vector bundle, it is possible to find a closed 2-form ω on the total space of V, such that $\omega(v, \cdot) = 0$ for $v \in TN$, and such that ω pulls back to the given symplectic form on the fibers. (A particularly nice way of obtaining such a 2-form is the 'minimal coupling' construction of Sternberg [35] and Weinstein [38].)

Remark 8.17 (Minimal coupling). The following construction is due to Sternberg [35] and Weinstein [38]. Let $P \to B$ be a principal G-bundle, and Q is a manifold with an invariant closed r-form ω_Q. Then the associated bundle $P \times_G Q$ has a fiberwise form induced by ω_Q. One may wonder if it is possibly to extend ω_Q to a global closed r-form on $P \times_G Q$ which pulls back to the given forms on the fibers.

The minimal coupling gives such a construction if $r = 2$, and the 2-form ω_Q admits a moment map $\Phi_Q\colon Q \to \mathfrak{g}^*$, that is, $\iota_{\xi_Q}\omega_Q = -\langle \mathrm{d}\Phi, \xi\rangle$. (Actually, the construction generalizes to arbitrary r, provided that ω_Q has an equivariant extension in the sense of de Rham theory.) Choose a principal connection $\theta \in \Omega^1(P, \mathfrak{g})$; thus θ has the equivariance property $\mathcal{A}_g^*\theta = \mathrm{Ad}_g\,\theta$ for $g \in G$ (where $\mathcal{A}_g$ denotes the action of g on P), and $\iota(\xi_P)\theta = \xi$ (where ξ_P are the generating vector fields for the action). Then the 2-form

$$\omega_Q - \mathrm{d}\langle \theta, \Phi\rangle \in \Omega^2(P \times Q)$$

is G-invariant for the diagonal action. In fact it is G-basic, by the calculation

$$\iota(\xi_Q)\langle \mathrm{d}\theta, \Phi\rangle = -\mathrm{d}\iota(\xi_Q)\langle \theta, \Phi\rangle = -\mathrm{d}\langle \Phi, \xi\rangle = \iota(\xi_Q)\omega_Q$$

(where we used Cartan's identity and the invariance of $\langle \theta, \Phi\rangle \in \Omega^1(P \times Q)$). Hence it descends to a closed 2-form

$$\widetilde{\omega}_Q \in \Omega^2(P \times_G Q)$$

pulls back to ω_Q on the fibers. As a special case, one can apply this construction to symplectic vector bundles $\mathsf{V} \to M$. Any such vector bundle is an associated bundle

$$\mathsf{V} = P \times_{\mathrm{Sp}(2k)} \mathbb{R}^{2k}$$

where P is the associated symplectic frame bundle, and $\mathbb{R}^{2k}$ has the standard symplectic structure. That is, the fiberwise symplectic form extends to a global closed 2-form on the total space of V.

Consider the pull-back Dirac structure $p^!\,\mathrm{Gr}(\pi_N) \subseteq \mathbb{T}\mathsf{V}$. It is not the graph of a Poisson structure (see Proposition 6.7), indeed

$$p^!\,\mathrm{Gr}(\pi_N) \cap T\mathsf{V} = \ker(Tp).$$

However, once we take a gauge transformation by $\omega \in \Omega^2(\mathsf{V})$, the resulting

$$\mathcal{R}_\omega\big(p^!\,\mathrm{Gr}(\pi_N)\big) \subseteq \mathbb{T}\mathsf{V} \tag{51}$$

is transverse to $T\mathsf{V}$ near $N \subseteq \mathsf{V}$, hence it defines a Poisson structure π_{V} on a neighborhood of N. In fact, this Poisson structure agrees with π along N, in terms of the identification

$$TM|_N = TN \oplus \mathsf{V} \cong \mathsf{V}|_N.$$

The Poisson structures for different choices of ω are related by the Moser method (Theorem 5.7).

Theorem 8.18 (Frejlich-Mărcuţ [17])**.** *Let $N \subseteq M$ be a cosymplectic submanifold of a Poisson manifold, with normal bundle V. Define a Poisson structure π_{V} on a neighborhood of N in V, as explained above. Then there exists a tubular neighborhood embedding $\mathsf{V} \hookrightarrow M$ which is a Poisson map on a possibly smaller neighborhood of $N \subseteq \mathsf{V}$.*

If the bundle V admits a trivialization $\mathsf{V} = N \times \mathsf{S}$, then the 2-form ω can simply be taken as pull-back of ω_{S} under projection to the second factor. Furthermore, $p^! \operatorname{Gr}(\pi_N) = \operatorname{Gr}(\pi_N) \times T\mathsf{S}$ in this case, with

$$\mathcal{R}_\omega(p^! \operatorname{Gr}(\pi_N)) = \operatorname{Gr}(\pi_N) \times \operatorname{Gr}(\omega_{\mathsf{S}}).$$

In particular, we recover the Weinstein splitting theorem.

For the proof of the Frejlich-Marcut theorem, we will use:

Lemma 8.19. *Let $N \subseteq M$ be cosymplectic. Then there exists a 1-form $\alpha \in \Omega^1(M)$, with $\alpha|_N = 0$, such that the vector field $X = \pi^\sharp(\alpha)$ is Euler-like.*

Proof. This is a special case of Proposition **??**, applied to the cotangent Lie algebroid $E = T^*_\pi M$, with the section $\epsilon \in \Gamma(T^*_\pi M)$ interpreted as a 1-form. Indeed, the condition that N is cosymplectic means in particular that $\pi^\sharp \colon T^*M \to TM$ is transverse to N. □

Proof of the Frejlich-Mărcuţ theorem 8.18, after [8]. Choose a 1-form α as in the Lemma. The Euler-like vector field $X = \pi^\sharp(\alpha)$ gives a tubular neighborhood embedding

$$\psi \colon \nu(M, N) \to M,$$

with $\mathcal{E} \sim_\psi X$. Using this embedding, we may assume $M = \nu(M, N)$ is a vector bundle, with $X = \mathcal{E}$ the Euler vector field. Let Φ_s be the flow, and $\kappa_t = \Phi_{-\log(t)}$ as before. Consider the infinitesimal automorphism

$$(\mathrm{d}\alpha, X) \in \Omega_{cl}(M) \rtimes \mathfrak{X}(M) \cong \mathfrak{aut}_{CA}(\mathbb{T}M)$$

defined by $\sigma = X + \alpha \in \Gamma(\operatorname{Gr}(\pi))$. By Proposition 5.4, the corresponding 1-parameter group of automorphisms is

$$(-\omega_t, \Phi_t) \in \Omega_{cl}(M) \rtimes \operatorname{Diff}(M) \cong \operatorname{Aut}_{CA}(\mathbb{T}M),$$

where

$$\omega_t = -\mathrm{d} \int_0^t (\Phi_s)_* \alpha \; \mathrm{d}s = -\mathrm{d} \int_0^t (\Phi_{-s})^* \alpha \; \mathrm{d}s = \mathrm{d} \int_{\exp(t)}^1 \frac{1}{v} \kappa_v^* \alpha \; \mathrm{d}v.$$

Since σ is a section of $\operatorname{Gr}(\pi)$, the action $\mathcal{R}_{\omega_t} \circ \mathbb{T}\Phi_t$ of this 1-parameter group preserves $\operatorname{Gr}(\pi)$. That is,

$$\mathcal{R}_{\omega_t}\big((\Phi_{-t})^! \operatorname{Gr}(\pi)\big) = \operatorname{Gr}(\pi)$$

for all $t \geq 0$. Consider the limit $t \to -\infty$ in this equality. Since α vanishes along N, the family of forms $\frac{1}{v}\kappa_v^*\alpha$ extends smoothly to $v = 0$. Hence $\omega := \omega_{-\infty}$ is well-defined:

$$\omega = \mathrm{d} \int_0^1 \frac{1}{v}\ \kappa_v^*\alpha\ \mathrm{d}v.$$

On the other hand,

$$\Phi_\infty = \kappa_0 = i \circ p$$

where $p \colon \nu(M, N) \to N$ is the projection, and $i \colon N \to M$ is the inclusion. Thus

$$\mathcal{R}_\omega(p^! i^! \operatorname{Gr}(\pi)) = \operatorname{Gr}(\pi).$$

Since $i^! \operatorname{Gr}(\pi) = \operatorname{Gr}(\pi_N)$, the left hand side is $\mathcal{R}_\omega(p^! \operatorname{Gr}(\pi_N))$, which coincides with the model Poisson structure π_{V} near N. $\square$

Remark 8.20. A similar argument can be used to prove more general normal form theorems for Dirac structures whose anchor is transverse to a submanifold $N \subseteq M$.

9. The Karasev-Weinstein symplectic realization theorem

9.1. **Symplectic realizations.**
Our starting point is the following definition, due to Weinstein.

Definition 9.1. [39] A *symplectic realization* of a Poisson manifold (M, π_M) is a symplectic manifold (P, ω_P), with associated Poisson structure π_P, together with a surjective submersion $\varphi \colon P \to M$ such that

$$\varphi \colon (P, \pi_P) \to (M, \pi_M)$$

is a Poisson map .

Remark 9.2. In Weinstein's original definition, it is not required that φ is a surjective submersion. For instance, the inclusion of a symplectic leaf would be a symplectic realization in the more general sense. The definition above is what Weinstein calls a *full* symplectic realization. We will drop 'full' to simplify the terminology.

Examples 9.3.
(a) (See [14].) Let $M = \mathbb{R}^2$ with the Poisson structure $\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. A symplectic realization is given by $P = T^*\mathbb{R}^2$ with the standard symplectic form $\omega = \sum_{i=1}^2 \mathrm{d}q^i \wedge \mathrm{d}p_i$, and

$$\varphi(q^1, q^2, p_1, p_2) = (q^1, q^2 + p_1 q^1).$$

To check that this is indeed a realization, we calculate the Poisson brackets:

$$\{q^1, q^2 + p_1 q^1\}_P = \{q^1, p_1\}_P q^1 = q^1,$$

corresponding to $\{x, y\}_M = x$.

(b) Let G be a Lie group, with Lie algebra $\mathfrak{g}$. The space $M = \mathfrak{g}^*$, with the Lie-Poisson structure, has a symplectic realization

$$\varphi \colon T^*G \to \mathfrak{g}^*,$$

where T^*G has the standard symplectic structure, and the map φ is given by left trivialization. (The first example (a) may be seen as a special case, using that $x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ is a linear Poisson structure, corresponding to a 2-dimensional Lie algebra.)

(c) Let (P, ω_P) be a symplectic manifold, with a proper, free action preserving the symplectic form ω_P. Then the Poisson structure π_P descends to a Poisson structure π_M on the quotient space $M = P/G$. (Indeed, smooth functions on M are identified with G-invariant smooth functions on P, and these are a Poisson subalgebra of $C^\infty(P)$.) The manifold P is then a symplectic realization of M. (Example (b) is a special case, with G acting on T^*G by the cotangent lift of the left-multiplication.)
(d) Let M be a manifold with the zero Poisson structure. Then the cotangent bundle, with its standard symplectic structure, and with φ the cotangent projection
$$\tau \colon T^*M \to M,$$
is a symplectic realization.
(e) Let M be a manifold with a symplectic structure. Then $P = M$, with φ the identity map, is a symplectic realization.
(f) Every symplectic manifold P can be regarded as a symplectic realization of $M = \mathrm{pt}$ with the zero Poisson structure.

Does every Poisson manifold admit a symplectic realization? Before addressing this question, let us first consider the opposite problem: when does a symplectic structure descend under a surjective submersion.

Proposition 9.4. *Let (P, π_P) be a Poisson manifold, and $\varphi \colon P \to M$ a surjective submersion with connected fibers.*
(a) *Then π_P descend to a bivector field π_M on M if and only if*
$$\mathrm{ann}(\ker T\varphi) \subseteq T^*P$$
is a subalgebroid of the cotangent Lie algebroid.
(b) *(Libermann's theorem [26].) If π_P is the Poisson structure for a symplectic form ω_P, then π_P descends if and only if the ω_P-orthogonal distribution to $\ker(T\varphi)$ is involutive in the sense of Frobenius.*

Proof. (a) "⇒". Suppose π_P descends to π_M. Then π_M necessarily is a Poisson structure, and φ is a Poisson map. Then
$$[\mathrm{d}\,\varphi^* f,\ \mathrm{d}\,\varphi^* g] = \mathrm{d}\,\{\varphi^* f,\ \varphi^* g\}_P = \mathrm{d}\,\varphi^*\{f,\ g\}_M,$$
for all $f, g \in C^\infty(M)$. Since the bundle $\mathrm{ann}(\ker T\varphi)$ is spanned by all $\mathrm{d}\,\varphi^* f$ with $f \in C^\infty(M)$, this shows that $\mathrm{ann}(\ker T\varphi)$ is a Lie subalgebroid.

"⇐". If $\mathrm{ann}(\ker T\varphi)$ is a Lie subalgebroid, it follows that for all $f, g \in C^\infty(M)$, $\mathrm{d}\{\varphi^* f,\ \varphi^* g\}_P$ vanishes on $\ker(T\varphi)$. Since the fibers of φ are connected, this means that the function $\{\varphi^* f,\ \varphi^* g\}_P$ is fiberwise constant, and hence is the pull-back of a function on M. Taking this function to be the definition of $\{f, g\}_M$, it follows that there is a unique bilinear form $\{\cdot,\cdot\}_M$ on M such that
$$\{\varphi^* f,\ \varphi^* g\}_P = \varphi^*\{f, g\}_M$$
for all $f, g \in C^\infty(M)$. Since $\{\cdot,\cdot\}_P$ is a Poisson structure, it follows that $\{\cdot,\cdot\}_M$ is a Poisson structure, and the identity above shows that φ is a Poisson map.

(b) The map $\pi_P^\sharp \colon T^*P \to TP$ is a Lie algebroid isomorphism, taking $\mathrm{ann}(\ker(T\varphi))$ to the ω_P-orthogonal bundle of $\ker(T\varphi)$. The latter being a Lie subalgebroid is equivalent to Frobenius integrability. □

Libermann's theorem shows that if $\varphi\colon P \to M$ is a symplectic realization, then the foliation given by the φ-fibers is symplectically orthogonal to another foliation.

Example 9.5. Let (P, ω_P) be a symplectic manifold with a free, proper G-action. As we saw, $M = P/G$ inherits a Poisson structure, and the quotient map is a symplectic realization. In this case, the transverse distribution is given by the ω-orthogonal spaces to the G-orbit directions:

$$\{v \in T_pP |\ \forall \xi \in \mathfrak{g}\colon \omega_P(\xi_P(p), v) = 0\}$$

If the action admits an equivariant moment map $\Phi\colon P \to \mathfrak{g}^*$, then this foliation is given exactly by the level sets of Φ. Indeed, for v tangent to a level set, and any $\xi \in \mathfrak{g}$,

$$\omega_P(\xi_P(p), v) = -\iota(v)\, \mathrm{d}\langle \Phi, \xi\rangle = 0.$$

Note that the moment map Φ is again a Poisson map (possibly up to reversing the sign of the Poisson structure – this depends on sign conventions). The assumption that G acts freely means that it is a submersion. That is, Φ (viewed as a map to $\Phi(M) \subseteq \mathfrak{g}^*$) provides a symplectic realization of $\mathfrak{g}^*$. (The assumption on existence of a moment map is not very restrictive; at least locally, on a neighborhood of a G-orbit, a moment map always exists.)

This is a good time to state the Karasev-Weinstein theorem.

Theorem 9.6 (Karasev [23], Weinstein [39])**.** *Let (M, π) be a Poisson manifold. Then there exists a symplectic manifold (P, ω), with an inclusion $i\colon M \hookrightarrow P$ as a Lagrangian submanifold, and with two surjective submersions $\mathsf{t}, \mathsf{s}\colon P \to M$ such that $\mathsf{t} \circ i = \mathsf{s} \circ i = \mathrm{id}_M$, and*

- t *is a Poisson map,*
- s *is an anti-Poisson map,*
- *The* t*-fibers and* s*-fibers are* ω*-orthogonal.*

In fact, much more is true: There exists a structure of a local symplectic groupoid on P, having s, t as the source and target maps, and i as the inclusion of units. We postpone the discussion of the multiplicative structure to Section **??** below. Let us first illustrate the theorem for some of the Examples 9.3.

Examples 9.7.

(a) For $M = \mathbb{R}^2$ with $\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, and $P = T^*(R^2)$ with the standard symplectic form,

$$\mathsf{t}(q^1, q^2, p_1, p_2) = (q^1, q^2 + p_1 q^1), \qquad \mathsf{s}(q^1, q^2, p_1, p_2) = (q^1 \exp(p_1), q^2).$$

(Note in particular that s is anti-Poisson, and that the component functions of t Poisson commute with the component functions of s.) Here i is the inclusion as the zero section.

(b) For $M = \mathfrak{g}^*$, we may take $P = T^*G$, with t the left trivialization, s the right trivialization, and i the inclusion as units.

(c) For M with the zero Poisson structure, we take $P = T^*M$, with $\mathsf{t} = \mathsf{s} = \tau$ the cotangent projection and i the inclusion as units.

(d) For a symplectic manifold M, the choice of $P = M$ with $\varphi = \mathrm{id}$ is a symplectic realization, but it does not have the properties described in the Karasev-Weinstein theorem 9.6. Instead we may take $P = M \times M^-$, where the minus sign signifies the opposite symplectic structure. Here t is projection to the first factor, s is projection to the second factor, and i is the diagonal inclusion.

The three conditions that t be Poisson, s anti-Poisson, and the t- and s-fibers being symplectically orthogonal can be combined into a single condition that the map

$$(\mathsf{t},\mathsf{s})\colon P\to M\times M^-$$

be Poisson. Here M^- indicates M with the opposite Poisson structure. We also have the following Dirac-geometric characterization of the condition.

Lemma 9.8 (Frejlich-Mărcuţ [18]). *Let (P,ω) be a symplectic manifold, $i\colon M\to P$ a Lagrangian submanifold, and $\mathsf{t},\mathsf{s}\colon P\to M$ two surjective submersions such that $\mathsf{t}\circ i=\mathsf{s}\circ i=\mathrm{id}_M$. Let π_M be a Poisson structure on M. Then $(\mathsf{t},\mathsf{s})\colon(P,\pi_P)\to(M,\pi_M)\times(M,-\pi_M)$ is Poisson if and only if*

$$\mathcal{R}_\omega(\mathsf{t}^!\,\mathrm{Gr}(\pi))=\mathsf{s}^!\,\mathrm{Gr}(\pi),$$

as Dirac structures in $\mathbb{T}P$.

Proof. We will write $\mathsf{s}_*=T\mathsf{s}$, and similar, to simplify notation.

"$\Leftarrow$". Observe $\ker(\mathsf{t}_*)\subseteq\mathsf{t}^!\,\mathrm{Gr}(\pi)$, $\ker(\mathsf{s}_*)\subseteq\mathsf{s}^!\,\mathrm{Gr}(\pi)$. For $v\in\ker(\mathsf{t}_*)$ and $\omega\in\ker(\mathsf{s}_*)$, we have

$$\omega(v,w)=\langle v+\iota_v\omega,\ w\rangle=\langle\mathcal{R}_\omega(v),w\rangle=0,$$

since both w and $\mathcal{R}_\omega(v)$ are in the Lagrangian subbundle $\mathsf{s}^!\,\mathrm{Gr}(\pi)\subseteq\mathbb{T}P$. This shows that the subbundles $\ker(\mathsf{t}_*),\ \ker(\mathsf{s}_*)\subseteq TP$ are ω-orthogonal; for dimension reasons $\ker(\mathsf{t}_*)$ is exactly the ω-orthogonal bundle to $\ker(\mathsf{s}_*)$. We next show that s is anti-Poisson. Given $\mu\in T^*_{\mathsf{s}(p)}M$, define $v\in T_pP$ by

$$\iota_v\omega=\mathsf{s}^*\mu.$$

That is,

$$v=-\pi_P^\sharp(\mathsf{s}^*\mu)\in T_pP.$$

Since $\iota_v\omega=\mathsf{s}^*\mu$ pairs to zero with all vectors of $\ker(\mathsf{s}_*)$, it follows that v is in the ω-orthogonal space to $\ker(\mathsf{s}_*)$. Hence $v\in\ker(\mathsf{t}_*)$. But

$$-\pi_P^\sharp(\mathsf{s}^*\mu)+\mathsf{s}^*\mu=v+\iota_v\omega=\mathcal{R}_\omega(v)\in\mathcal{R}_\omega(\mathsf{t}^!\,\mathrm{Gr}(\pi))=\mathsf{s}^!\,\mathrm{Gr}(\pi)$$

is s-related to some element of $\mathrm{Gr}(\pi)_{\mathsf{s}(p)}$. Since the T^*P-component is $\mathsf{s}^*\mu$, that element must be $\pi^\sharp(\mu)+\mu\in\mathrm{Gr}(\pi)$. This shows that

$$\mathsf{s}_*(-\pi_P^\sharp(\mathsf{s}^*\mu))=\pi^\sharp(\mu),$$

hence s is anti-Poisson. A similar argument shows that t is Poisson.

"$\Rightarrow$" For the converse, suppose that $(\mathsf{t},\mathsf{s})\colon P\to M\times M$ is a Poisson map with respect to $(\pi_M,-\pi_M)$. Equivalently, for all $v,w\in T_pP,\ \mu\in T^*_{\mathsf{t}(p)}M,\ \nu\in T^*_{\mathsf{s}(p)}M$

$$\iota(v)\omega=-\mathsf{t}^*\mu\quad\Rightarrow\quad\mathsf{t}_*v=\pi^\sharp(\mu),\quad\mathsf{s}_*v=0, \tag{52}$$

$$\iota(w)\omega=-\mathsf{s}^*\nu\quad\Rightarrow\quad\mathsf{t}_*w=0,\qquad\mathsf{s}_*w=-\pi^\sharp(\nu) \tag{53}$$

Consider the direct sum decompositions

$$\mathsf{t}^!\,\mathrm{Gr}(\pi)=\ker(\mathsf{t}_*)\oplus(\mathsf{t}^!\,\mathrm{Gr}(\pi)\cap\mathrm{Gr}(\pi_P)) \tag{54}$$

$$\mathsf{s}^!\,\mathrm{Gr}(\pi)=\ker(\mathsf{s}_*)\oplus(\mathsf{s}^!\,\mathrm{Gr}(\pi)\cap\mathrm{Gr}(\pi_P)). \tag{55}$$

Elements in the second summand of (54) are of the form $v + \mathsf{t}^*\mu$, with v uniquely determined by $\iota_v\omega = -\mathsf{t}^*\mu$. Elements in the second summand of (55) are of the form $w + \mathsf{s}^*\nu$ with $\iota(w)\omega = -\mathsf{s}^*\nu$.

Let $v + \mathsf{t}^*\mu \in (\mathsf{t}^! \operatorname{Gr}(\pi) \cap \operatorname{Gr}(\pi_P))$. The property $\iota_v\omega = -\mathsf{t}^*\mu$ shows $\mathcal{R}_\omega(v + \mathsf{t}^*\mu) = v$, by (52) this lies in $\ker(\mathsf{s}_*)$. Hence

$$\mathcal{R}_\omega(\mathsf{t}^! \operatorname{Gr}(\pi) \cap \operatorname{Gr}(\pi_P)) = \ker(\mathsf{s}_*).$$

Similarly, $\mathcal{R}_{-\omega}$ is an isomorphism from the second summand of (55) to the first summand of (54); equivalently,

$$\mathcal{R}_\omega(\ker \mathsf{t}_*) = (\mathsf{s}^! \operatorname{Gr}(\pi) \cap \operatorname{Gr}(\pi_P))).$$

This shows $\mathcal{R}_\omega(\mathsf{t}^! \operatorname{Gr}(\pi)) = \mathsf{s}^! \operatorname{Gr}(\pi)$; in fact, $\mathcal{R}_\omega$ interchanges the two summands in the decompositions (54) and (55). ☐

9.2. **The Crainic-Mărcuţ formula.** Our proof of Theorem 9.6 will use an explicit construction of the realization due to Crainic-Mărcuţ [13], with later simplifications due to Frejlich-Mărcuţ [18]. As the total space P for the symplectic realization, we will take a suitable open neighborhood of M inside the cotangent bundle

$$\tau\colon T^*M \to M.$$

Definition 9.9 (Crainic-Mărcuţ [13])**.** Let (M, π) be a Poisson manifold. A vector field $X \in \mathfrak{X}(T^*M)$ is called a *Poisson spray* if it homogeneous of degree 1 in fiber directions, and for all $\mu \in T^*M$,

$$(T_\mu\tau)(X_\mu) = \pi^\sharp(\mu).$$

The homogeneity requirement means $\kappa_t^* X = tX$, where κ_t is fiberwise multiplication by $t \neq 0$. In local coordinates, for a given Poisson structure

$$\pi = \frac{1}{2}\sum_{ij} \pi^{ij}(q)\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j}, \tag{56}$$

a Poisson spray is of the form

$$X = \sum_{ij} \pi^{ij}(q)\, p_i\, \frac{\partial}{\partial q^j} + \frac{1}{2}\sum_{ijk} \Gamma_k^{ij}(q)\, p_i\, p_j\, \frac{\partial}{\partial p^k} \tag{57}$$

where p_i are the cotangent coordinates, and $\Gamma_k^{ij} = \Gamma_k^{ji}$ are functions.

Lemma 9.10. *Every Poisson manifold (M, π) admits a Poisson spray.*

Proof. In local coordinates, Poisson sprays can be defined by the formula above (e.g., with $\Gamma_k^{ij} = 0$). To obtain a global Poisson spray, one patches these local definitions together, using a partition of unity on M. ☐

Let X be a Poisson spray, and Φ_t its local flow. Since X vanishes along $M \subseteq T^*M$, there exists an open neighborhood of M on which the flow is defined for all $|t| \leq 1$. On such a neighborhood, put

$$\omega = \int_0^1 (\Phi_s)_*\omega_{\text{can}}\ \mathrm{d}s,$$

where ω_{can} is the standard symplectic form of the cotangent bundle.

Lemma 9.11. *The 2-form ω is symplectic along M.*

Proof. For $m \in M \subseteq T^*M$, consider the decomposition

$$T_m(T^*M) = T_mM \oplus T^*_mM.$$

Since the vector field X is homogeneous of degree 1, it vanishes along M. In particular, its flow Φ_t fixes $M \subseteq P$, hence $T\Phi_t$ is a linear transformation of $T_m(T^*M)$. Consequently, $(T_m\Phi_t)(v) = v$ for all $m \in M$ and $v \in T_mM$. Again by homogeneity, the linear approximation along $M \subseteq T^*M$ vanishes: $\nu(X) = 0$ as a vector field on $\nu(T^*M, M) \cong T^*M$. (This is not to be confused with the linear approximation of X at $\{m\}$, which may be non-zero.) Consequently, $\nu(\Phi_t) = \mathrm{id}_{T^*M}$, which shows that

$$(T_m\Phi_t)(w) = w \mod T_mM$$

for all $w \in T^*_mM$. Hence $((\Phi_s)_*\omega_{\mathrm{can}})(v, \cdot) = \omega_{\mathrm{can}}(v, \cdot)$ for all $v \in T_mM$, and therefore

$$\omega(v, \cdot) = \omega_{\mathrm{can}}(v, \cdot).$$

Since TM is a Lagrangian subbundle (in the symplectic sense!) of $T(T^*M)|_M$ with respect to ω_{can}, this implies that the 2-form ω is symplectic along M. $\square$

Theorem 9.12 (Crainic-Mărcuţ [13])**.** *Let $P \subseteq T^*M$ be an open neighborhood of the zero section, with the property that $\Phi_t(m)$ is defined for all $m \in P$ and $|t| \leq 1$, and such that ω is symplectic on P. Let $i\colon M \hookrightarrow P$ be the inclusion as the zero section, and put*

$$\mathsf{s} = \tau, \qquad \mathsf{t} = \tau \circ \Phi_{-1}.$$

Then the symplectic manifold (P, ω) together with the maps $\mathsf{t}, \mathsf{s}, i$ has the properties from the Karasev-Weinstein theorem 9.6.

Proof. [18]. Let $\alpha \in \Omega^1(T^*M)$ be the canonical (Liouville) 1-form. That is, for all $\mu \in T^*M$,

$$\alpha_\mu = (T_\mu\tau)^*\mu.$$

Recall that $\omega_{\mathrm{can}} = -\mathrm{d}\alpha$. In local cotangent coordinates, $\alpha = \sum_i p_i \mathrm{d}q^i$ and $\omega_{\mathrm{can}} = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$. Given a Poisson spray X, observe that

$$X + \alpha \in \Gamma\big(\mathbb{T}(T^*M)\big)$$

is a section of $\tau^!\big(\mathrm{Gr}(\pi)\big) \subseteq \mathbb{T}(T^*M)$. Indeed, the definition of a spray (and of the canonical 1-form α) means precisely that for all $\mu \in T^*M$,

$$X_\mu + \alpha_\mu \;\sim_\tau\; \pi^\sharp(\mu) + \mu \in \mathrm{Gr}(\pi).$$

The infinitesimal automorphism $(\mathrm{d}\alpha, X) \in \mathfrak{aut}\big(\mathbb{T}(T^*M)\big)$ defined by the section $X + \alpha$ preserves $\tau^!\,\mathrm{Gr}(\pi)$. By Proposition 5.3, the (local) 1-parameter group of automorphisms exponentiating $(\mathrm{d}\alpha, X) \in \mathfrak{aut}\big(\mathbb{T}(T^*M)\big)$ is given by $(-\omega_t, \Phi_t)$, where Φ_t is the (local) flow of X, and

$$\omega_t = -\mathrm{d}\int_0^t (\Phi_s)_*\alpha = \int_0^t (\Phi_s)_*\omega_{\mathrm{can}}.$$

We conclude

$$\mathcal{R}_{\omega_t} \circ \mathbb{T}\Phi_t(\tau^!\,\mathrm{Gr}(\pi)) = \tau^!\,\mathrm{Gr}(\pi).$$

Putting $t = 1$ in this identity, and use the definition of $\mathsf{t}, \mathsf{s}, \omega$, together with the fact that $T\Phi_1(E) = (\Phi_{-1})^! E$ for any Dirac structure $E \subseteq \mathbb{T}(T^*M)$, we obtain $\mathcal{R}_\omega \circ \mathsf{t}^! \operatorname{Gr}(\pi) = \mathsf{s}^! \operatorname{Gr}(\pi)$. By Lemma 9.8, this is equivalent to the conditions from the Karasev-Weinstein theorem. □

9.3. **Linear realizations of linear Poisson structures.** Let $V \to M$ be a vector bundle with a linear Poisson structure. The tangent bundle TV and the cotangent bundle T^*V are double vector bundles

$$\begin{array}{ccc} TV & \longrightarrow & TM \\ \downarrow & & \downarrow \\ V & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} T^*V & \longrightarrow & V^* \\ \downarrow & & \downarrow \\ V & \longrightarrow & M \end{array}$$

We denote by κ_t^h, κ_t^v the horizontal and vertical scalar multiplications. By definition, a Poisson spray $X \in \mathfrak{X}(T^*V)$ satisfies $(\kappa_t^v)^* X = tX$; we will call X a *linear Poisson spray* if it is linear in the horizontal direction, $(\kappa_t^h)^* X = X$.

Lemma 9.13. *For every linear Poisson structure on $V \to M$ there exists a linear Poisson spray $X \in \mathfrak{X}(T^*V)$.*

Proof. It suffices to construct a linear Poisson spray for any local trivialization of V; a global linear Poisson spray is then obtained by using a partition of unity on M. Using local vector bundle coordinates, with q^r the coordinates on the base and y_i those on the fiber, we obtain (see (18))

$$\pi = \frac{1}{2} \sum_{ijk} c_{ij}^k(q)\, y_k \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} + \sum_{ir} a_i^r(q) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial q^r}.$$

with a_i^r, c_{ij}^k smooth functions. A corresponding spray is given by

$$X = \sum_{ijk} c_{ij}^k(q)\, y_k \eta^i \frac{\partial}{\partial y_j} + \sum_{ir} a_i^r(q) \eta^i \wedge \frac{\partial}{\partial q^r} - \sum_{ir} a_i^r(q) p_r \frac{\partial}{\partial y_i};$$

here p_r, η^i are the cotangent variables corresponding to q^r, y_i. The horizontal and vertical scalar multiplications are, in coordinates,

$$\kappa_t^h(q, y, p, \eta) = (q, ty, tp, \eta), \quad \kappa_t^v(q, y, p, \eta) = (q, y, tp, t\eta).$$

We hence see that the spray given above is indeed homogeneous of degree 0 in horizontal direction. □

Let Φ_s be the (local) flow of the linear Poisson spray X. The homogeneity properties of X give that

$$\Phi_s \circ \kappa_t^h = \kappa_t^h \circ \Phi_s, \qquad \Phi_s \circ \kappa_t^v = \kappa_t^v \circ \Phi_{st}.$$

The canonical 1-form on T^*V is given by $\alpha = \sum p_r \mathrm{d}q^r + \sum \eta^k \mathrm{d}y_k$; its homogeneity properties are

$$(\kappa_t^h)^* \alpha = t\alpha, \qquad (\kappa_t^v)^* \alpha = t\alpha.$$

Hence, $\omega_{\mathrm{can}} = -\mathrm{d}\alpha$ has similar scaling properties, and for the Crainic-Marcut form $\omega = \int_0^1 (\Phi_s)_* \omega_{\mathrm{can}}\, ds$, one obtains

$$(\kappa_t^h)^* \omega = t\omega, \quad (\kappa_t^v)^* \omega = \omega_t$$

(where ω_t is given by the integral from 0 to t). Passing to the Poisson structure on T^*V (defined on a κ_t^h-invariant neighborhood of the zero section $V \subseteq T^*V$) corresponding to ω, this means that

$$(\kappa_t^h)^* \pi_{T^*V} = \frac{1}{t} \pi_{T^*V}.$$

That is, it is a linear Poisson structure on T^*V, relative to the vector bundle structure $T^*V \to V^*$. The source and target map $\mathsf{s}, \mathsf{t} \colon T^*V \to V$ are compatible with the linear structure as well:

$$\mathsf{s} \circ \kappa_t^h = \kappa_t^h \circ \mathsf{s}, \quad \mathsf{t} \circ \kappa_t^h = \kappa_t \circ \mathsf{t};$$

here κ_t is the scalar multlication for V.

10. Lie groupoids

For a more detailed discussion of Lie groupoids, we refer to the books [29] or [32].

10.1. **Basic definitions and examples.** The definition of a *Lie groupoid* $\mathcal{G} \rightrightarrows M$ involves a manifold $\mathcal{G}$ of *arrows*, a submanifold $i \colon M \hookrightarrow \mathcal{G}$ of *units* (or *objects*), and two surjective submersions $\mathsf{s}, \mathsf{t} \colon \mathcal{G} \to M$ called *source* and *target* such that $\mathsf{t} \circ i = \mathsf{s} \circ i = \mathrm{id}_M$. One thinks of g as an arrow from its source $\mathsf{s}(g)$ to its target $\mathsf{t}(g)$, with M embedded as trivial arrows. Using that t, s are submersions, one finds that the set of *k-arrows*, for $k = 1, 2, \ldots$

$$\mathcal{G}^{(k)} = \{(g_1, \ldots, g_k) \in \mathcal{G}^k |\ \mathsf{s}(g_i) = \mathsf{t}(g_{i+1})\}$$

is a smooth submanifold of $\mathcal{G}^k$. For $k = 0$ one puts $\mathcal{G}^{(0)} = M$. The definition of a Lie groupoid also involves a smooth multiplication map, defined on composable arrows (i.e., 2-arrows)

$$\mathrm{Mult}_{\mathcal{G}} \colon \mathcal{G}^{(2)} \to \mathcal{G}, \quad (g_1, g_2) \mapsto g_1 \circ g_2,$$

such that $\mathsf{s}(g_1 \circ g_2) = \mathsf{s}(g_2), \ \ \mathsf{t}(g_1 \circ g_2) = \mathsf{t}(g_1)$. It is thought of as a concatenation of arrows. Note that when picturing this composition rule, it is best to draw arrows from the right to the left.

Definition 10.1. The above data define a *Lie groupoid* $\mathcal{G} \rightrightarrows M$ if the following axioms are satisfied:

1. **Associativity:** $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ whenever $\mathsf{s}(g_1) = \mathsf{t}(g_2),\ \mathsf{s}(g_2) = \mathsf{t}(g_3)$.
2. **Units:** $\mathsf{t}(g) \circ g = g = g \circ \mathsf{s}(g)$ for all $g \in \mathcal{G}$.
3. **Inverses:** Every $g \in \mathcal{G}$ is invertible: There exists $h \in \mathcal{G}$ such that $\mathsf{s}(h) = \mathsf{t}(g),\ \mathsf{t}(h) = \mathsf{s}(g)$, and such that $g \circ h,\ h \circ g$ are units.

Remark 10.2. The inverse of an element is necessarily unique. Denoting this element by g^{-1}, we have that $g \circ g^{-1} = \mathsf{t}(g), \ \ g^{-1} \circ g = \mathsf{s}(g)$.

Remark 10.3. Note that the entire groupoid structure is encoded in the graph of the multiplication map,

$$\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}}) = \{(g_1 \circ g_2, g_1, g_2) \in \mathcal{G}^3 |\ (g_1, g_2) \in \mathcal{G}^{(2)}\}$$

Thus, the set of all (g, g_1, g_2) such that $g = g_1 \circ g_2$ determines all the structure maps. For instance, the units are recovered as those elements m such that $m = m \circ m$, while the source $\mathsf{s}(g)$ of an element $g \in \mathcal{G}$ is the unique unit m such that $g \circ \mathsf{s}(g)$ is defined.

There are obvious notions of *Lie subgroupoid* of a Lie groupoid, and of a *morphism of Lie groupoids* $\Phi_{\mathcal{G}}\colon \mathcal{G}\to\mathcal{G}'$ from $\mathcal{G}\rightrightarrows M$ to $\mathcal{G}'\rightrightarrows M'$ (with base map $\Phi_M\colon M\to M'$). The graph of a Lie groupoid morphism is a subgroupoid of

$$\mathcal{G}'\times\mathcal{G}\rightrightarrows M'\times M.$$

Example 10.4. A Lie group G is the same as a Lie groupoid with a unique unit, $G\rightrightarrows \mathrm{pt}$. At the opposite extreme, every set M can be regarded as a trivial groupoid $M\rightrightarrows M$ where all elements are units.

Example 10.5. A Lie groupoid for which the source and target map coincide is the same as a bundle of Lie groups: A surjective submersion with a fiberwise group structure such that the fiberwise multiplication depends smoothly on the base point. The fibers need not be isomorphic as Lie groups, or even diffeomorphic.

Example 10.6. For any manifold M, one has the *pair groupoid* $\mathrm{Pair}(M)=M\times M\rightrightarrows M$, with

$$(m',m)=(m_1',m_1)\circ(m_2',m_2)\quad\Leftrightarrow\quad m_1'=m,\ m_1=m_2',\ m_2=m.$$

Here units are given by the diagonal embedding $M\hookrightarrow M\times M$, and the source and target of (m',m) are m and m', respectively. The target and anchor of a Lie groupoid $\mathcal{G}\rightrightarrows M$ combine into a Lie groupoid morphism

$$(\mathsf{t},\mathsf{s})\colon\mathcal{G}\to\mathrm{Pair}(M). \tag{58}$$

This groupoid morphism is sometimes called the (groupoid) *anchor*; it is related to the anchor of Lie algebroids as we will see below.

Example 10.7. Another natural Lie group associated to any manifold M is the *fundamental groupoid* $\Pi(M)$, consisting of homotopy classes of paths $[\gamma]$ relative to fixed end points. The source and target maps are $\mathsf{s}([\gamma])=\gamma(0)$, $\mathsf{t}([\gamma])=\gamma(1)$, and the groupoid multiplication is concatenation of paths. The natural map $\Pi(M)\to\mathrm{Pair}(M)$ is a local diffeomorphism; if M is simply connected it is a global diffeomorphism. [5]

Example 10.8. Given an smooth action of a Lie group G on M, one has the *action groupoid* $\mathcal{G}\rightrightarrows M$. It may be defined as the subgroupoid of $G\times\mathrm{Pair}(M)\rightrightarrows M$, consisting of all $(g,m',m)\in G\times(M\times M)$ such that $m'=g.m$. Using the projection $(g,m',m)\mapsto(g,m)$ to identify $\mathcal{G}\cong G\times M$, the product reads as

$$(g,m)=(g_1,m_1)\circ(g_2,m_2)\quad\Leftrightarrow\quad g=g_1g_2,\ m=m_2,\ m_1=g_2.m_2.$$

Generalizing group actions, one can also consider *groupoid actions* of $\mathcal{G}\rightrightarrows M$ on sets X with a map $\Phi\colon X\to M$. Such an action is given by an action map

$$\mathcal{A}_X\colon\mathcal{G}\ {}_{\mathsf{s}}\!\times_{\Phi} X\to X,\ (g,p)\mapsto g\circ p$$

with $\Phi(g\circ p)=\mathsf{t}(g)$, such that $(g_1\circ g_2)\circ p=g_1\circ(g_2\circ p)$ whenever the compositions are defined, and such that units act trivially: $\Phi(p)\circ p=p$. Similar to the groupoids themselves, the action is determined by its graph $\mathrm{Gr}(\mathcal{A}_X)\subseteq X\times\mathcal{G}\times X$; for example, $\Phi(p)\in M$ is the

[5] If M is connected, and $\widetilde{M}$ is a simply connected covering space, with covering map $\widetilde{M}\to M$, one has a Lie groupoid homomorphism $\mathrm{Pair}(\widetilde{M})=\Pi(\widetilde{M})\to\Pi(M)$. By homotopy lifting, this map is surjective. Let Λ be the discrete group of deck transformations of $\widetilde{M}$, i.e., diffeomorphisms covering the identity map on M. Then $M=\widetilde{M}/\Gamma$, and $\Pi(M)=\Pi(\widetilde{M})/\Gamma=(\widetilde{M}\times\widetilde{M})/\Gamma$, a quotient by the diagonal action.

unique unit such that $p = \Phi(p) \circ p$. Given such a groupoid action, one can again form an action groupoid, as a subgroupoid of $\mathcal{G} \times \mathrm{Pair}(X) \rightrightarrows M \times X$. Sometimes we use subscripts for the action operation, to avoid confusion with the groupoid multiplication.

Every Lie groupoid $\mathcal{G} \rightrightarrows M$ acts on itself by left multiplication $g \circ_L a = l_g(a) = g \circ a$ (here $\Phi = \mathsf{t}$), and by right multiplication $g \circ_R a = r_{g^{-1}}(a) = a \circ g^{-1}$ (here $\Phi = \mathsf{s}$), and it also acts on its units, by $g \circ_M m = \mathsf{t}(g)$ (here $\Phi = \mathrm{id}_M$).

10.2. **Left-invariant and right-invariant vector fields.** Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, the left multiplication by a groupoid element $g \in \mathcal{G}$ with source $\mathsf{s}(g) = m$ and target $\mathsf{t}(g) = m'$ is a diffeomorphism

$$l_g \colon \mathsf{t}^{-1}(m) \to \mathsf{t}^{-1}(m'), \quad a \mapsto g \circ a$$

A vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if it is tangent to the t-fibers and for all $g \in \mathcal{G}$, the restrictions of X to the corresponding t-fibers of $m = \mathsf{s}(g)$, $m' = \mathsf{t}(g)$ are related by l_g. The left-invariant vector fields form a Lie subalgebra $\mathfrak{X}(\mathcal{G})^L$ of the Lie algebra of vector fields on $\mathcal{G}$, and they are a module over $\mathsf{s}^* C^\infty(M)$. Similarly a vector field is called *right-invariant* if it is tangent to all s-fibers, and for all g as above the restriction to the s-fibers of m', m are related by $r_{g^{-1}}$. The right-invariant vector fields form a Lie subalgebra $\mathfrak{X}(\mathcal{G})^R$, and are a module over $\mathsf{t}^* C^\infty(M)$. The flow of a left-invariant vector field is by right translations; in particular, left-invariant vector fields commute with right-invariant ones.

Example 10.9. For the pair groupoid $\mathrm{Pair}(M) = M \times M$, a left -invariant vector field is of the form $(0, X)$ where X is a vector field on M, while right-invariant vector fields are of the form $(X, 0)$. The flow of a left-invariant vector field is $(m', m) \mapsto (m', \Phi_t(m))$ where Φ_t is the flow of X. This is right multiplication by the element $(m, \Phi_t(m)) = (\Phi_t(m), m)^{-1}$. Similarly, a right invariant vector field has flow $(m', m) \mapsto (\Phi_t(m'), m)$.

10.3. **Bisections.** A *bisection* of a Lie groupoid is a submanifold $S \subseteq \mathcal{G}$ such that both t, s restrict to diffeomorphisms $S \to M$. For example, M itself is a bisection. The name indicates that S can be regarded as a section of both s and t. We will denote by $\Gamma(\mathcal{G})$ the set of all bisections. It has a group structure, with the multiplication given by

$$S \circ S' = \mathrm{Mult}_{\mathcal{G}}(S \ {}_{\mathsf{s}}\times_{\mathsf{t}} \ S').$$

The identity for this multiplication is the unit bisection M, and the inverse is given by $S^{-1} = \mathrm{Inv}_{\mathcal{G}}(S) = \{g^{-1} |\ g \in S\}$. This group comes with a homomorphism

$$\Gamma(\mathcal{G}) \to \mathrm{Diff}(M), \quad S \mapsto \Phi_S$$

where for all $g \in S \subseteq G$,

$$\Phi_S(\mathsf{s}(g)) = \mathsf{t}(g).$$

Examples 10.10.
(a) For a Lie group $G \rightrightarrows \mathrm{pt}$, regarded as a Lie groupoid, a bisection is simply an element of G, and $\Gamma(G) = G$ as a group.
(b) For a vector bundle $V \to M$, regarded as groupoid $V \rightrightarrows M$, a bisection is the same as a section. More generally, this is true for any bundle of Lie groups.
(c) For the 'trivial' groupoid $M \rightrightarrows M$ the only bisection is M itself. The resulting group $\Gamma(M)$ consists of only the identity element.
(d) For the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$, a bisection is the same as the graph of a diffeomorphism of M. This identifies $\Gamma(\mathcal{G}) \cong \mathrm{Diff}(M)$.

(e) Let $P \to M$ be a principal G-bundle, and $\mathcal{G} \rightrightarrows M$ the associated (Atiyah) groupoid. A bisection of $\mathcal{G}$ is the same as a principal bundle automorphism $\Phi_P \colon P \to P$. That is, $\Gamma(\mathcal{G}) = \mathrm{Aut}(P)$.

(f) Given a G-action on M, a bisection is a smooth map $f \colon M \to G$ for which the map $m \mapsto f(m).m$ is a diffeomorphism.

The group of bisections has natural actions on $\mathcal{G}$, by left multiplication:

$$\mathcal{A}^L_S(g) = h \circ g,$$

with the unique element $h \in S$ such that $\mathsf{s}(h) = \mathsf{t}(g)$. Similarly, there is an action by right multiplication,

$$\mathcal{A}^R_S(g) = g \circ (h')^{-1},$$

with the unique element $h' \in S$ such that $\mathsf{s}(h') = \mathsf{s}(g)$. The left and right multiplication actions combine into an *adjoint action*,

$$\mathrm{Ad}_S(g) = h \circ g \circ (h')^{-1}$$

by groupoid automorphisms of $\mathcal{G}$. The restriction of the adjoint action to $M \subseteq \mathcal{G}$ is the diffeomorphism Φ_S.

In general, there may not exist a global bisection passing through a given point $g \in \mathcal{G}$. `example?` However, one can always find a *local bisection* $S \subseteq M$, that is, t, s restrict to local diffeomorphisms to open subsets $\mathsf{t}(S) = V$, $\mathsf{s}(S) = U$ of M. The left-action of a local bisection is then defined as a diffeomorphism

$$\mathcal{A}^L_S \colon \mathsf{t}^{-1}(U) \to \mathsf{t}^{-1}(V).$$

Lemma 10.11. *A vector field is left-invariant if and only if for every local bisection S,*

$$X|_{\mathsf{t}^{-1}(U)} \sim_{\mathcal{A}^L_S} X|_{\mathsf{s}^{-1}(U)}.$$

Proof. We have to show that if a vector field X satisfies the above property for all local bisections, then X is in particular tangent to the t-fibers. Consider X_m for $m \in M \subseteq \mathcal{G}$. Let S be a local bisection passing through m; thus m is a fixed point for the action $\mathcal{A}^L_S$. If X is invariant under $\mathcal{A}^L_S$, then X_m must be fixed. Decompose $X_m = X'_m + X''_m$, where X'_m is tangent to the t-fiber, while $X_{m''}$ is tangent to M. The action $\mathcal{A}^L_S$ fixes the fiber $\mathsf{t}^{-1}(m)$ pointwise, hence it takes X'_m to itself. Hence X''_m must be fixed. On the other hand, the action of S maps $U \subseteq M$ to a submanifold of S. We can choose S such that the image of X''_m is no longer tangent to T_mM. Thus, we must have $X''_m = 0$. This shows that X is tangent to t-fibers along M. Since the left-action by local bisections preserves tangent spaces to t-fibers, X is tangent to t-fibers everywhere. □

10.4. **The Lie algebroid of a Lie groupoid.** Let $\mathcal{G}$ be a Lie groupoid, and let

$$A = \nu(\mathcal{G}, M)$$

be the normal bundle of M in $\mathcal{G}$. In the case of a pair groupoid, we identify

$$\nu(\mathrm{Pair}(M), M) \cong TM$$

by the map taking the class of $(v_2, v_1) \in T_M(\mathrm{Pair}(M))$ to $v_1 - v_2$. The groupoid anchor (58) is a map of pairs

$$\mathsf{a}_{\mathcal{G}} = (\mathsf{t}, \mathsf{s}) \colon (\mathcal{G}, M) \to (\mathrm{Pair}(M), M),$$

hence it induces a vector bundle morphism

$$\mathsf{a} = \nu(\mathsf{a}_{\mathcal{G}}) \colon A \to TM.$$

called the *anchor map.* Equivalently, a is the map induced by

$$T\mathsf{s} - T\mathsf{t} \colon T\mathcal{G}|_M \to TM.$$

For $\sigma \in \Gamma(A)$, there are unique vector fields

$$\sigma^L \in \mathfrak{X}(\mathcal{G})^L, \quad \sigma^R \in \mathfrak{X}(\mathcal{G})^R$$

mapping to σ under restriction to M followed by the quotient map $T\mathcal{G}|_M \to A$.

Lemma 10.12. *We have that*

$$\mathsf{a}(\sigma) \sim_i \ \sigma^L - \sigma^R, \quad \sigma^L \sim_{\mathsf{s}} \mathsf{a}(\sigma), \quad \sigma^R \sim_{\mathsf{t}} -\mathsf{a}(\sigma).$$

Proof. Since σ^L is tangent to the t-fibers, we have that

$$\mathsf{a}(\sigma) = (T\mathsf{s} - T\mathsf{t})(\sigma^L|_M) = (T\mathsf{s})(\sigma^L|_M),$$

Similarly, $\mathsf{a}(\sigma) = -T\mathsf{t}(\sigma^R)$. Finally, note that $\sigma^L - \sigma^R$ is tangent to M, hence restricts to a vector field on M. Since

$$T\mathsf{s}((\sigma^L - \sigma^R)|_M) = \mathsf{a}(\sigma),$$

we see that the restriction is $\mathsf{a}(\sigma)$. □

We will use the identification of $\Gamma(A)$ with *left-invariant* vector fields on $\mathcal{G}$ to define a Lie bracket on $\Gamma(A)$. [6]

Lemma 10.13. *With these data,* $(A, \mathsf{a}, [\cdot,\cdot])$ *is a Lie algebroid.*

Proof. Let $f \in C^\infty(M)$ and $\sigma, \tau \in \Gamma(A)$. Using $(f\tau)^L = (\mathsf{s}^* f)\,\tau^L$ and $\sigma^L(\mathsf{s}^* f) = \mathsf{s}^*\big(\mathsf{a}(\sigma) f\big)$ (since $\sigma^L \sim_{\mathsf{s}} \mathsf{a}(\sigma)$) we compute,

$$[\sigma, f\tau]^L = [\sigma^L,\ \mathsf{s}^* f\ \tau^L] = \mathsf{s}^* f\,[\sigma^L, \tau^L] + \mathsf{s}^*(\mathsf{a}(\sigma) f)\tau^L = \big(f[\sigma,\tau] + \big(\mathsf{a}(\sigma) f\big)\,\tau\big)^L,$$

proving $[\sigma, f\tau] = f[\sigma,\tau] + \big(\mathsf{a}(\sigma) f\big)\,\tau$. □

One calls A the Lie algebroid of the Lie groupoid $\mathcal{G}$; conversely, one refers to $\mathcal{G}$ as an *integration of* A.

Lemma 10.14. *The left and right invariant vector fields satisfy the bracket relations, for* $\sigma, \tau \in \Gamma(A)$*:*

$$[\sigma^L, \tau^L] = [\sigma,\tau]^L, \quad [\sigma^L, \tau^R] = 0, \quad [\sigma^R, \tau^R] = -[\sigma,\tau]^R. \tag{59}$$

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[6]Recall that for Lie groups, one uses left-invariant vector fields to ensure that for *matrix* Lie groups, the Lie bracket is the commutator. Let $\theta^L \in \Omega^1(G, \mathfrak{g})$ be the left-Maurer-Cartan form on G. For a matrix Lie group, this is $\theta^L = g^{-1}\mathrm{d}g$, which satisfies the Maurer-Cartan equation $\mathrm{d}\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, with the bracket denoting the matrix commutator. For an abstract Lie group, we may take this equation as the *definition* of the Lie bracket on $\mathfrak{g}$. The left-invariant vector fields are defined by $\iota(\xi^L)\theta^L = \xi$, for $\xi \in \mathfrak{g}$. We obtain, using the Cartan calculus,

$$[\xi,\zeta] = \iota(\zeta^L)\iota(\xi^L)\frac{1}{2}[\theta^L, \theta^L] = -\iota(\zeta^L)\iota(\xi^L)\mathrm{d}\theta^L = \iota([\xi^L, \zeta^L])\theta^L.$$

where we used that., e.g. $\mathcal{L}(\xi^L)\iota(\zeta^L)\theta^L = \mathcal{L}(\xi^L)\zeta = 0$. That is, our sign conventions give $[\xi,\zeta]^L = [\xi^L, \zeta^L]$.

Proof. The bracket relations for the left-invariant vector fields hold by definition; those for the right-invariant vector fields follow since $\sigma^L \sim \mathrm{Inv}_{\mathcal{G}} - \sigma^R$. The fact that left-and right-invariant vector fields commute follows because their flows commute. □

Remark 10.15. One can also understand the construction of the Lie algebroid in terms of bisections. Given a 1-parameter family of embedded submanifolds $S_t \in \Gamma(\mathcal{G})$ with $S_0 = M$, the t-derivative at $t = 0$ is a section of the normal bundle $\nu(\mathcal{G}, M) = A$. In particular, this applies to 1-parameter families of bisections. Conversely, the flow of a left-invariant vector field σ^L defined by $\sigma \in \Gamma(A)$ is a (locally defined) 1-parameter group of bisections[7] Hence, the Lie algebra of $\Gamma(G)$ is given by $\Gamma(A)$, identified with left-invariant vector fields.

Not every Lie algebroid A admits an integration to a Lie groupoid; the precise obstructions to integrability were determined by Crainic-Fernandes [12]. If an integration exists, then (as one can show) there exists a unique integration with connected and simply connected s-fibers; every other integration is then obtained by taking a quotient by a finite group. On the other hand, every Lie algebroid admits an integration to a *local Lie groupoid*. We will not give the precise definition of a local Lie groupoid, for which we refer to [10]. Roughly, a local Lie groupoid is again given by a manifold $\mathcal{G}$ with a smooth submanifold M and two surjective submersions t, s which are projections from $\mathcal{G}$ to M. However, the groupoid multiplication is not necessarily defined on all of $\mathcal{G}^{(2)}$, but only on some neighborhood of M, and the associativity and unit axioms only hold for elements in a sufficiently small neighborhood of M. The first explicit proof of integrability to a local Lie groupoid was given by Crainic-Fernandes, although one might say it is implicit in the integration theorem for Poisson manifolds (by viewing a Lie algebroid in terms of the linear Poisson structure on the dual bundle).

10.5. **Groupoid multiplication via σ^L, σ^R.** For $i = 0, 1, 2$ and $\sigma \in \Gamma(A)$, the vector fields on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$,

$$X^0_\sigma = (-\sigma^R, -\sigma^R, 0), \quad X^1_\sigma = (0, \sigma^L, -\sigma^R), \quad X^2_\sigma = (\sigma^L, 0, \sigma^L). \tag{60}$$

are all tangent to $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$. For instance, the invariance of the graph under the (local) flow of X^1_σ follows from the fact that $g_1 \circ g_2 = (g_1 \circ h^{-1}) \circ (h \circ g_2)$ whenever $\mathsf{s}(h) = \mathsf{t}(g_2)$. The vector fields satisfy bracket relations

$$[X^i_\sigma,\, X^j_\tau] = X^i_{[\sigma,\tau]}\ \delta_{i,j}$$

for $\sigma, \tau \in \Gamma(A)$ and $i, j = 0, 1, 2$. If $\mathcal{G}$ is t-connected, then the graph is generated from $M \subseteq \Lambda$ (embedded as $m \mapsto (m, m, m)$) by the flow of these vector fields. In fact, it is already obtained using the flows of the X^i_σ's for any two of the indices $i \in \{0, 1, 2\}$. For reference in Section 9, let us note the following partial converse.

Proposition 10.16. *Let $(A, \mathsf{a}, [\cdot,\cdot])$ be a Lie algebroid over M. Suppose $i\colon M \to P$ is an embedding, with normal bundle $\nu(P, M) \cong A$, and suppose σ^L, $\sigma^R \in \mathfrak{X}(P)$ are vector fields on P, mapping to $\sigma \in \Gamma(A)$ under the quotient map $TP|_M \to A$, with $\mathsf{a}(\sigma) \sim_i \sigma^L - \sigma^R$, and satisfying the bracket relations* (59). *Then a neighborhood of M in P inherits a structure of a local Lie groupoid integrating A, in such a way that σ^L, σ^R are the left, right invariant vector fields.*

[7]In contrast to Lie groups, the left-invariant vector fields on a groupoid need not be complete. For example, for a pair groupoid $\mathrm{Pair}(M)$ the left-invariant vector fields correspond to arbitrary vector fields on M.

Sketch of proof. Since $\sigma^L|_M$ maps to σ under $T\mathcal{G}|_M \to \nu(\mathcal{G}, M)$, the restrictions of the left-invariant vector fields to M span a complement to TM in $TP|_M$. In a particular, on a neighborhood of M they determine a distribution of rank equal to that of A. By the bracket relations, this distribution is Frobenius integrable. A similar argument applies to the vector fields $-\sigma^R$. Taking P smaller if necessary, we can assume that these foliations define surjective submersions $\mathsf{t}, \mathsf{s}\colon P \to M$, with σ^L, σ^R tangent to the respective fibers, and with $\mathsf{t} \circ i = \mathsf{s} \circ i = \mathrm{id}$. Define vector fields X^i_σ in $P \times P \times P$ as above, and embed $M \hookrightarrow P \times P \times P$ by $m \mapsto (m, m, m)$. Along M, hence also on some neighborhood of M inside $P \times P \times P$, the vector fields X^0_σ, X^2_τ span a distribution of rank equal to twice the rank of A. The bracket relations guarantee that this distribution is integrable, hence they define a foliation. Since the intersection of this distribution with the tangent bundle of $M \subseteq P \times P \times P$ is trivial, we conclude that the flow-out of M under these vector fields defines a (germ of a) submanifold $\Lambda \subseteq P \times P \times P$, of dimension $2\,\mathrm{rank}(A) + \dim M = 2\dim(A) - \dim M$ This is our candidate for the graph of the multiplication map.

By construction, $\Lambda \subseteq P \times P \times P$ contains M, and is invariant under the local flow of all vector fields X^0_σ, X^2_σ. In fact, it is also invariant under the local flow of X^1_τ for $\tau \in \Gamma(A)$, since these vector fields are tangent to Λ along M, and hence everywhere since the commute with all X^0_σ, X^2_σ.

Under projection $P \times P \times P \to P \times P,\ (p, p_1, p_2) \mapsto (p_1, p_2)$, the vector fields X^0_σ, X^2_σ are related to $(-\sigma^R, 0)$ and $(0, \sigma^L)$, respectively. Hence, this projection restricts to a diffeomorphism from Λ onto a neighborhood of the diagonal embedding of M in $P^{(2)} = P\ {}_{\mathsf{s}}\!\times_{\mathsf{t}} P \subseteq P \times P$. Taking the inverse map, followed by projection to the first P-factor, defines a multiplication map $\mathrm{Mult}_P\colon P^{(2)} \to P$; strictly speaking it is defined only on some neighborhood of M in $P^{(2)}$. By construction, $\Lambda = \mathrm{Gr}(\mathrm{Mult}_P)$.

Letting $\mathrm{id}_P\colon P \to P$ be the identity relation (given by the diagonal in $P \times \overline{P}$, the associativity of the groupoid multiplication means that

$$\Lambda \circ (\Lambda \times \mathrm{id}_P) = \Lambda \circ (\mathrm{id}_P \times \Lambda)$$

as relations $P \times P \times P \dashrightarrow P$, where the circle means composition of relations. In fact, we can see that both sides are given by

$$\Lambda^{[2]} \subseteq P \times (P \times P \times P),$$

the submanifold generated from the diagonal $M^{[3]}_\Delta$ (consisting of elements (m, m, m, m)) by the action of vector fields of the form

$$(-\sigma^R, -\sigma^R, 0, 0),\ (0, \sigma^L, -\sigma^R, 0),\ (0, 0, \sigma^L, -\sigma^R). \qquad \square$$

11. Symplectic groupoids

11.1. Definition, basic properties.

Definition 11.1 (Weinstein [40])**.** Let (M, π) be a Poisson manifold. A *symplectic groupoid* is a Lie groupoid $\mathcal{G} \rightrightarrows M$, equipped with a symplectic structure ω such that that the graph of the groupoid multiplication $\mathrm{Gr}(\mathrm{Mult}_\mathcal{G})$ is a Lagrangian submanifold of $\mathcal{G} \times \overline{\mathcal{G} \times \mathcal{G}}$.

Here the line indicates the opposite symplectic structure; thus $\mathcal{G} \times \overline{\mathcal{G} \times \mathcal{G}}$ has the symplectic structure $\mathrm{pr}_1^*\,\omega - \mathrm{pr}_2^*\,\omega - \mathrm{pr}_3^*\,\omega$, with pr_i the projection to the i-th factor. 'Lagrangian' is meant in the sense of symplectic geometry: $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$ has the middle dimension $\frac{3}{2}\dim\mathcal{G}$, and the pullback of the symplectic form vanishes.

The condition that $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$ is a Lagrangian submanifold may be rephrased as a multiplicativity condition for the symplectic form. Note that $T\mathcal{G}$ is a groupoid over TM, with the groupoid multiplication $\mathrm{Mult}_{T\mathcal{G}} = T\,\mathrm{Mult}_{\mathcal{G}}$.

Lemma 11.2. *Let ω be a symplectic 2-form on a Lie groupoid $\mathcal{G} \rightrightarrows M$. Then $(\mathcal{G}, \omega)$ is a symplectic groupoid if and only if for all $(v_1, v_2), (w_1, w_2) \in T\mathcal{G}^{(2)}$, with the same base points $(g_1, g_2) \in \mathcal{G}^{(2)}$, we have that*

$$\omega(v_1 \circ v_2,\, w_1 \circ w_2) = \omega(v_1, w_1) + \omega(v_2, w_2). \tag{61}$$

Proof. For $(v_1, v_2) \in T\mathcal{G}^{(2)}$, we have $(v_1 \circ v_2, v_1, v_2) \in T\,\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$. Hence, if $(\mathcal{G}, \omega)$ is a symplectic groupoid, then the multiplicative property (61) follows from

$$0 = \omega_{\mathcal{G}\times\overline{\mathcal{G}\times\mathcal{G}}}\big((v_1 \circ v_2, v_1, v_2), (w_1 \circ w_2, w_1, w_2)\big) = \omega(v_1 \circ v_2,\, w_1 \circ w_2) - \omega(v_1, w_1) - \omega(v_2, w_2).$$

Conversely, if the multiplicative property holds then this calculation shows that $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$ is isotropic in $\mathcal{G} \times \overline{\mathcal{G} \times \mathcal{G}}$. Hence

$$2\dim\mathcal{G} - \dim M = \dim\mathcal{G}^{(2)} = \dim\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}}) \le 3/2\dim\mathcal{G},$$

hence $\dim M \ge \frac{1}{2}\dim\mathcal{G}$. On the other hand, by applying the identity to $v \circ v = v, w \circ w = w \in TM$, we also see that M is isotropic in $\mathcal{G}$, hence $\dim M \le \frac{1}{2}\dim\mathcal{G}$. We have thus proved that the inequalities above are all equalities, hence $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$ is Lagrangian. (At the same time, we have proved that M is Lagrangian in $\mathcal{G}$.) □

Theorem 11.3 (Coste-Dazord-Weinstein [10])**.** *For any symplectic groupoid $(\mathcal{G}, \omega)$ the space M of objects acquires a Poisson structure, in such a way that $\mathsf{t}, \mathsf{s}, i$ satisfy the Karasev-Weinstein conditions from Theorem 9.6. That is, the map*

$$(\mathsf{t}, \mathsf{s})\colon \mathcal{G} \to M \times M^-$$

is Poisson, where M^- stands for M with the opposite Poisson structure $-\pi$. The inversion map of a symplectic groupoid is anti-symplectic.

One calls $(\mathcal{G}, \omega_G)$ a *symplectic groupoid integrating* the Poisson manifold (M, π).

Proof. We have already shown, in the proof of Lemma 11.2, that $M \subseteq \mathcal{G}$ is Lagrangian. If $(v_1, v_2), (w_1, w_2) \in \mathcal{G}^{(2)}$ are tangent to the graph of the inversion map, then $v_1 \circ v_2, w_1 \circ w_2$ are both tangent to M, hence

$$0 = \omega(v_1 \circ v_2, w_1 \circ w_2) = \omega(v_1, v_2) + \omega(w_1, w_2).$$

This shows that the inversion is anti-symplectic. If $v \in \ker(\mathsf{t}_*)$ and $w \in \ker(\mathsf{s}_*)$, with the same base point $g \in \mathcal{G}$, then $v = 0_{t(g)} \circ v$, while $w = w' \circ 0_g$, for suitable $w' \in \ker(\mathsf{s}_*)_{\mathsf{t}(g)}$ defined by this equation. (Namely, w' is the image w under right transation by g^{-1}.) It follows that

$$\omega(v, w) = \omega(0, w') + \omega(v, 0) = 0.$$

Since $\ker\mathsf{s}_*, \ker\mathsf{t}_*$ both have rank $\frac{1}{2}\dim\mathcal{G}$, this shows that they are ω-orthogonals of each other. By Libermann's theorem, we conclude that M admits a Poisson structure such that t is Poisson. Since inversion on $\mathcal{G}$ is anti-symplectic, and interchanges the source and target maps, it follows that s is anti-Poisson relative to the same Poisson structure. □

The basic examples of symplectic groupoids are as follows:

Example 11.4. If (M,ω) is a symplectic manifold (regarded as a Poisson manifold), then the pair groupoid $\mathrm{Pair}(M) = M \times \overline{M} \rightrightarrows M$ is a symplectic groupoid integrating M. Here $\overline{M}$ indicates M with the opposite symplectic structure $-\omega$.

Example 11.5. [10] Suppose G is a Lie group, with Lie algebra $\mathfrak{g}$. Then the cotangent bundle T^*G, with its standard symplectic structure, has the structure of a symplectic groupoid

$$T^*G \rightrightarrows \mathfrak{g}^*$$

where for $\mu_i \in T^*_{g_i}G,\ \ i=1,2$ and $\mu \in T^*_gG$,

$$\mu = \mu_1 \circ \mu_2 \quad \Leftrightarrow \quad g = g_1g_2,\ \ (\mu_1,\mu_2) = (T_{g_1,g_2}\,\mathrm{Mult}_G)^*\mu.$$

Here $\mathrm{Mult}_G\colon G\times G\to G$ is the group multiplication map of the group G. The space of units is $\mathfrak{g}^*$ embedded as the fiber T^*_eG, while source and target map are given by left trivialization, respectively right trivialization. The symplectic groupoid T^*G integrates $\mathfrak{g}^*$ equipped with the Lie-Poisson structure. Indeed, the map t is the symplectic moment map for the cotangent lift of the left G-action on itself, and moment maps are always Poisson maps.

11.2. **Lagrangian bisections.** A bisection $S\subseteq \mathcal{G}$ which is a Lagrangian submanifold is called a *Lagrangian bisection.* The units M are such a Lagrangian bisection; more generally, if S is obtained from M by the flow of a Hamiltonian vector field on $\mathcal{G}$, then S is a Lagrangian bisection as long as the flow is sufficiently small (so that S is a section of both t and s). The Lagrangian bisections form a subgroup $\Gamma_{\mathrm{Lag}}(\mathcal{G})$ of $\Gamma(\mathcal{G})$: If $(v_1,v_2),(w_1,w_2)\in T\mathcal{G}^{(2)}$, with $v_1,w_1\in TS$ and $v_2,w_2\in TS'$, then $v_1\circ v_2,\ w_1\circ w_2\in T(S\circ S')$ satisfy

$$\omega(v_1\circ v_2, w_1\circ w_2) = \omega(v_1,w_1)+\omega(v_2,w_2) = 0,$$

as required. A similar argument shows that the action of $\Gamma_{\mathrm{Lag}}(\mathcal{G})$ on $\mathcal{G}$ (by left multiplication, right multiplication, or adjoint action) is by symplectic transformations.

It is often useful to have a Lagrangian bisection passing through a given element $g\in\mathcal{G}$. This is not always possibly, but one can always finds a *local* Lagrangian bisection:

Lemma 11.6. *For every $g\in\mathcal{G}$, there exists a Lagrangian submanifold $S\subseteq\mathcal{G}$ such that t,s restrict to diffeomorphisms from S onto open subsets $\mathsf{t}(S),\mathsf{s}(S)\subseteq M$.*

Proof. By elementary symplectic geometry, one can find a Lagrangian subspace of $T_g\mathcal{G}$ that is transverse to $\ker(\mathsf{s}^*)_g$. It is then automatic that this Lagrangian subspace is also transverse to $\ker(\mathsf{t}_*)_g$, since $\ker(\mathsf{s}^*)_g$ and $\ker(\mathsf{t}_*)_g$ are ω-orthogonal. Take S to be a Lagrangian submanifold through g, with this Lagrangian subspace as its tangent space at g. Replacing S with a sufficiently small neighborhood of g inside S, it follows that t,s restrict to diffeomorphisms. □

11.3. **The Lie algebroid of a symplectic groupoid.** Suppose $\mathcal{G}\rightrightarrows M$ is a symplectic groupoid integrating the Poisson manifold (M,π). Since M is Lagrangian in $\mathcal{G}$, the map $-\pi^\sharp_{\mathcal{G}}$ takes $\mathrm{ann}(TM)$ onto TM; it hence induces an isomorphism

$$T^*M = T^*\mathcal{G}|_M/\,\mathrm{ann}(TM)\to \nu(M,G) = T\mathcal{G}|_M/TM. \tag{62}$$

Give T^*M the Lie algebroid structure as a cotangent Lie algebroid of M.

Proposition 11.7. *The map (62) is an isomorphism of Lie algebroids. Under this isomorphism, the left-invariant and right-invariant vector fields defined by $\alpha \in \Omega^1(M)$ are given by*

$$\alpha^R = -\pi^\sharp_{\mathcal{G}}(\mathsf{t}^*\alpha), \quad \alpha^L = -\pi^\sharp_{\mathcal{G}}(\mathsf{s}^*\alpha). \tag{63}$$

The satisfy the bracket relations,

$$[\alpha^L, \beta^L] = [\alpha, \beta]^L, \quad [\alpha^R, \beta^R] = -[\alpha, \beta]^R, \quad [\alpha^L, \beta^R] = 0 \tag{64}$$

where the bracket of 1-forms is defined using the Lie algebroid bracket of T^*M.

Proof. The two identities (63) hold *along* $M \subseteq \mathcal{G}$, by definition of the isomorphism. To show that they hold true over all of $\mathcal{G}$, it suffices to show that the vector field $\pi^\sharp_{\mathcal{G}}(\mathsf{s}^*\alpha)$ is left-invariant, while $\pi^\sharp_{\mathcal{G}}(\mathsf{t}^*\alpha)$ is right-invariant. For this, it suffices to verify the invariance under local *Lagrangian* bisections $S \subseteq \mathcal{G}$. The left-translation $\mathcal{A}^L_S$ by a Lagrangian bisections is a symplectic transformation, hence pull-back under such a transformation commutes with $\pi^\sharp_{\mathcal{G}}$. But the 1-form $\mathsf{s}^*\alpha$ is invariant under pull-back by $\mathcal{A}^L_S$, since $\mathsf{s} \circ \mathcal{A}^L_S = \mathsf{s}$. It follows that $\pi^\sharp_{\mathcal{G}}(\mathsf{s}^*\alpha)$ is left-invariant, and similarly that $\pi^\sharp_{\mathcal{G}}(\mathsf{t}^*\alpha)$ is right-invariant. The bracket relations follow since t is Poisson, s is anti-Poisson. They also show in particular that the identification $T^*M \to A$ is a Lie algebroid isomorphism. □

11.4. **The cotangent groupoid.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then the tangent bundle $T\mathcal{G} \rightrightarrows TM$ is a $\mathcal{VB}$-groupoid.

$$\begin{array}{ccc} T\mathcal{G} & \rightrightarrows & TM \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}$$

The cotangent bundle $T^*\mathcal{G}$ has a $\mathcal{VB}$-groupoid structure as well. The groupoid multiplication may be described as follows: $\mu = \mu_1 \circ \mu_2$ if and only if the base points satisfy $g = g_1 \circ g_2$, and for all tangent vectors v, v_1, v_2 at those base points, with $v = v_1 \circ v_2$, we have that

$$\langle \mu, v \rangle = \langle \mu_1, v_1 \rangle + \langle \mu_2, v_2 \rangle.$$

Equivalently, the graph of the groupoid multiplication

$$\mathrm{Gr}(\mathrm{Mult}_{T^*\mathcal{G}}) \subseteq T^*\mathcal{G} \times \overline{T^*\mathcal{G} \times T^*\mathcal{G}}$$

is the annihilator of that of $T\mathcal{G}$, followed by sign changes in the second and third factor. In the notation of (26),

$$\mathrm{Gr}(\mathrm{Mult}_{T^*\mathcal{G}}) = (\mathrm{Gr}(\mathrm{Mult}_{T\mathcal{G}}))^\diamond.$$

It is thus *Lagrangian*, and hence $T^*\mathcal{G}$ is a *symplectic groupoid.* [8] The base of this groupoid is the annihilator of TM inside $T\mathcal{G}$, i.e. the dual of $A = \nu(\mathcal{G}, M)$. We obtain a $\mathcal{VB}$-groupoid

$$\begin{array}{ccc} T^*\mathcal{G} & \rightrightarrows & A^* \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}$$

[8] This generalizes to the fact that the dual of any $\mathcal{LA}$-groupoid is a Poisson groupoid.

Note that Poisson structure on $T^*\mathcal{G}$ is *linear* with respect to the vector bundle structure over $\mathcal{G}$. Hence, the resulting Poisson structure on A^* is linear as well.

Theorem 11.8. *The symplectic groupoid $T^*\mathcal{G} \rightrightarrows A^*$ is an integration of the Poisson manifold A^* with its natural Lie-Poisson structure.*

The proof will require the following Lemma. Given $\sigma \in \Gamma(A)$, let $\phi_\sigma \in C^\infty(A^*)$ be the corresponding linear function, and $\mathsf{s}^*\phi_\sigma \in C^\infty(T^*\mathcal{G})$ the left-invariant function obtained by pull-back. By (63), the corresponding Hamiltonian vector field is the left-invariant vector field on the symplectic groupoid $T^*\mathcal{G} \rightrightarrows A^*$ defined by the 1-form $\mathrm{d}\phi_\sigma \in \Omega^1(A^*)$:

$$-\pi^\sharp_{T^*\mathcal{G}}(\mathrm{d}\mathsf{s}^*\phi_\sigma) = (\mathrm{d}\phi_\sigma)^L.$$

Proposition 11.9. *The left-invariant vector field $(\mathrm{d}\phi_\sigma)^L \in \mathfrak{X}(T^*\mathcal{G})$ is the cotangent lift of the left-invariant vector field $\sigma^L \in \mathfrak{X}(\mathcal{G})$. In particular,*

$$\sigma^L \sim_j (\mathrm{d}\phi_\sigma)^L, \tag{65}$$

where j is the inclusion.

Proof. The canonical decomposition $TA^*|_M = A^* \oplus TM$ dualizes to $T^*A^*|_M = A \oplus T^*M$. Since ϕ_σ is linear with respect to the scalar multiplication on A^*, the 1-form $\mathrm{d}\phi_\sigma$ vanishes on tangent vectors to $M \subseteq A^*$. Hence $(\mathrm{d}\phi_\sigma)|_M$ takes values in the first summand, $A \subseteq T^*A^*|_M$. Using local bundle coordinates for A, it is clear that this restriction is just σ. This proves (**??**), and hence shows that the linear vector field $(\mathrm{d}\phi_\sigma)^L$ is a lift of σ^L. Since it is a Hamiltonian vector field with respect to a linear function $f = \mathsf{s}^*\phi_\sigma \in C^\infty(T^*\mathcal{G})$, it is the cotangent lift. □

Proof of Theorem 11.8. Let $\{\cdot,\cdot\}_1$ be the Poisson structure on A^* induced by the symplectic groupoid $T^*\mathcal{G}$, and $\{\cdot,\cdot\}_2$ the Lie Poisson-structure. Let $\sigma, \tau \in \Gamma(A)$, and $\phi_\sigma, \phi_\tau \in C^\infty(A^*)$ the corresponding functions. Since s is anti-Poisson with respect to $\{\cdot,\cdot\}_1$ we have that

$$\begin{aligned}[(\mathrm{d}\phi_\sigma)^L, (\mathrm{d}\phi_\tau)^L] &= [\pi^\sharp_{T^*\mathcal{G}}(\mathrm{d}\mathsf{s}^*\phi_\sigma),\ \pi^\sharp_{T^*\mathcal{G}}(\mathrm{d}\mathsf{s}^*\phi_\tau)] \\ &= -\pi^\sharp_{T^*\mathcal{G}}(\mathrm{d}\mathsf{s}^*\{\phi_\sigma, \phi_\tau\}_1)\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{d}\phi_{[\sigma,\tau]})^L &= -\pi^\sharp_{T^*\mathcal{G}}(\mathrm{d}\mathsf{s}^*\phi_{[\sigma,\tau]}) \\ &= -\pi^\sharp_{T^*\mathcal{G}}(\mathrm{d}\mathsf{s}^*\{\phi_\sigma, \phi_\tau\}_2)\end{aligned}$$

Since $[\sigma^L, \tau^L] = [\sigma, \tau]^L$, their cotangent lifts satisfy $[(\mathrm{d}\phi_\sigma)^L, (\mathrm{d}\phi_\tau)^L] = (\mathrm{d}\phi_{[\sigma,\tau]})^L$. This shows that the Hamiltonian vector fields corresponding to $\mathsf{s}^*\{\phi_\sigma, \phi_\tau\}_1,\ \mathsf{s}^*\{\phi_\sigma, \phi_\tau\}_2$ are equal. Hence the difference of these functions is locally constant, and in fact is zero since they are linear functions on $T^*\mathcal{G} \to \mathcal{G}$. This shows $\{\phi_\sigma, \phi_\tau\}_1 = \{\phi_\sigma, \phi_\tau\}_2$. □

11.5. **Integration of Poisson manifolds.** Not every Poisson manifold admits an integration to a symplectic groupoid. But one always has a *local* symplectic groupoid integrating the Poisson structure. Local Lie groupoids are a generalization of Lie groupoids, where the groupoid multiplication is only defined on some open neighborhood of the diagonally embedded $M \subseteq \mathcal{G}^{(2)}$. (For details, see [10, Definition III.1.2].)

Theorem 11.10 (Coste-Dazord-Weinstein [10], Karasev [23])**.** *Every Poisson manifold (M,π) admits a local symplectic groupoid $\mathcal{G} \rightrightarrows M$ integrating it.*

In fact, it is shown in [10] that any symplectic realization $\tau\colon P \to M$ for which τ is a retraction onto a Lagrangian submanifold $i\colon M \hookrightarrow P$, can be given the structure of a local symplectic groupoid over M. The following discussion is similar to their approach. It shows that given the data from the Karasev-Weinstein theorem, one automatically gets the local groupoid structure.

Proposition 11.11. *Let (M,π_M) be a Poisson manifold, and (P,ω_P) a symplectic manifold with an inclusion $i\colon M \to P$ as a Lagrangian submanifold, and with two surjective submersions*

$$\mathsf{t},\mathsf{s}\colon P \to M,$$

satisfying the properties of the Karasev-Weinstein theorem 9.6. Replacing P with a smaller neighborhood of M if necessary, it has a unique structure of a local symplectic groupoid, with s and t as the source and target map, and i the inclusion of units.

The idea is to generating the graph of the groupoid multiplication by flows of vector fields, as in Proposition 11.7. For all $\alpha \in \Omega^1(M)$, define vector fields $\alpha^L, \alpha^R \in \mathfrak{X}(P)$ by

$$\alpha^L = -\pi_P^\sharp\big((\mathsf{t},\mathsf{s})^*(0,\alpha)\big), \qquad \alpha^R = -\pi_P^\sharp\big((\mathsf{t},\mathsf{s})^*(\alpha,0)\big). \tag{66}$$

Equivalently, $\alpha^L = -\pi_P^\sharp(\mathsf{s}^*\alpha)$, $\alpha^R = -\pi_P^\sharp(\mathsf{t}^*\alpha)$. Since $(\mathsf{t},\mathsf{s})\colon P \to M \times M^-$ is a Poisson map, we have that

$$\alpha^L \sim_{\mathsf{s}} \pi_M^\sharp(\alpha), \quad \alpha^L \sim_{\mathsf{t}} 0, \quad \alpha^R \sim_{\mathsf{t}} -\pi_M^\sharp(\alpha) \quad \alpha^R \sim_{\mathsf{s}} 0$$

and the bracket relations

$$[\alpha^L,\beta^L] = [\alpha,\beta]^L, \quad [\alpha^R,\beta^R] = -[\alpha,\beta]^R, \quad [\alpha^L,\beta^R] = 0.$$

Furthermore,

$$\omega(\alpha^L,\beta^L) = -\mathsf{s}^*\pi_M(\alpha,\beta), \qquad \omega(\alpha^R,\beta^R) = \mathsf{t}^*\pi_M(\alpha,\beta), \qquad \omega(\alpha^L,\beta^R) = 0. \tag{67}$$

By Proposition 11.7, the bracket relations among the vector fields α^L, α^R give P the structure of a local groupoid, where $\Lambda = \mathrm{Gr}(\mathrm{Mult}_P)$ is obtained as the flow-out of $M \subseteq P \times P \times P$ under $(\alpha^L, 0, \alpha^L)$, $(-\alpha^R, -\alpha^R, 0)$. To conclude the argument that it is a local symplectic groupoid, it remains to check:

Proposition 11.12. *The submanifold Λ is a Lagrangian submanifold (in the sense of symplectic geometry) of $P \times \overline{P \times P}$.*

Proof. Recall that Λ is generated from the isotropic submanifold $M \subseteq P \times P \times P$ by the flow-outs of vector fields of the form

$$(-\alpha^R, -\alpha^R, 0), \quad (\alpha^L, 0, \alpha^L).$$

Indeed, it is already generated by vector fields of this form such that α is exact. Since these are Hamiltonian vector fields, they preserve the symplectic form. It therefore suffices to check that the tangent space to Λ is Lagrangian along M. The latter is spanned by the restrictions of the vector fields of the form above, together with $TM \subseteq T(P \times P \times P)$. Using the relations (67), one checks that this subbundle is isotropic, and since it has rank $3\dim M$ it is Lagrangian. $\square$

11.6. **Integration of Lie algebroids.** Suppose $A \to M$ is a Lie algebroid, and $V = A^*$ the dual bundle with associated linear Poisson structure. Consider the double vector bundle

$$\begin{array}{ccc} T^*V & \longrightarrow & V \\ \downarrow & & \downarrow \\ V^* & \longrightarrow & M \end{array}$$

The choice of a linear Poisson spray defines a symplectic realization $T^*V \supseteq P \rightrightarrows V$, where P is a vector subbundle of $T^*V \to V^*$ over $Q = P \cap V^*$. The source and target map are both vector bundle morphisms, so that we have a local VB-groupoid

$$\begin{array}{ccc} T^*V \supseteq P & \rightrightarrows & V \\ \downarrow & & \downarrow \\ V^* \supseteq Q & \rightrightarrows & M \end{array}$$

The symplectic form on the vector bundle $P \to Q$ is *linear*; by the following Lemma, this means that $P \cong T^*Q$.

Lemma 11.13 (Vector bundles with linear symplectic structures)**.**

(a) *Let $W \to Q$ be a vector bundle, equipped with a symplectic structure ω that is homogeneous of degree 1. Then there is a unique vector bundle isomorphism $W \cong T^*Q$ taking ω to the canonical symplectic form, and restricting to the identity map on Q.*

(b) *If $L \subseteq T^*Q_2 \times T^*Q_1^-$ is a Lagrangian relation, which is also a vector subbundle along $S \subseteq Q_2 \times Q_1$, then*

$$L = \mathrm{ann}(TS)^\diamond := \{(\xi_2, -\xi_1) |\ (\xi_2, \xi_1) \in \mathrm{ann}(TS)\} \subseteq T^*Q_2 \times T^*Q_1^-.$$

Proof. Consider the restriction $\omega|_Q$ to $TW|_Q = TQ \oplus W$, where the first summand corresponds to tangent vectors to base and the second to tangent vectors to the fibers The $\mathbb{R}_{>0}$-action defined by scalar multiplication is trivial on the first summand, and multiplication by t on the second summand. Since ω is homogeneous of degree 1, it follows from this scaling behaviour that the symplectic form restricts to zero on each of the subbundles $TQ, W \subseteq TW|_Q$. Hence both are Lagrangian subbundles, and the symplectic form identifies $W \cong T^*Q$ as vector bundles. Under this isomorphism, ω becomes a linear symplectic structure on T^*Q, which agrees with the canonical symplectic structure ω_{can} along Q. Since both are closed 2-forms that are homogeneous of degree 1, it follows that $\omega = \omega_{\mathrm{can}}$. This proves (a). For (b), let $L^\diamond = \{(\xi_2, -\xi_1) |\ (\xi_2, \xi_1) \in L\}$, as a subbundle of $W = T^*(Q_2 \times Q_1)$. Using $TL^\diamond|_S = TS \oplus L^\diamond \subseteq TW|_Q = TQ \oplus W$, we see that $L^\diamond = \mathrm{ann}(TS)$. □

The application to symplectic groupoids gives:

Proposition 11.14. *Let $\mathcal{H} \rightrightarrows V$ be a symplectic groupoid, which is also a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$, and such that the symplectic form ω on $\mathcal{H}$ is homogeneous of degree 1. Then the identification $\mathcal{H} = T^*\mathcal{G}$ from Lemma 11.13 preserves the groupoid structure. In particular, the Lie algebroid $A = V^*$ coincides with the Lie algebroid of $\mathcal{G}$.*

Proof. The condition that $\mathcal{H}$ is a symplectic groupoid means that $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{H}}) \subseteq \mathcal{H} \times \overline{\mathcal{H} \times \mathcal{H}}$ is a Lagrangian submanifold, and since $\mathcal{H}$ is a $\mathcal{VB}$-groupoid this is a vector subbundle along the submanifold $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}}) \subseteq \mathcal{G} \times \mathcal{G} \times \mathcal{G}$. By the Lemma, it follows that under the identification $\mathcal{H} = T^*\mathcal{G}$, we have $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{H}}) = \mathrm{ann}(T\,\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}}))^{\diamond}$. But this is exactly the standard groupoid structure for $T^*\mathcal{G}$. (For details, see [25, Appendix C].) □

A similar result holds for local symplectic $\mathcal{VB}$-groupoids $P \rightrightarrows V$, with the same proof. We hence conclude that the local symplectic groupoid $P \rightrightarrows V$ constructed above is just the cotangent groupoid $P = T^*Q$.

Let us now explain how to construct an integration of a Lie algebroid $A \to M$, using the integration results for Poisson manifolds. Let $V = A^* \to M$ be the dual bundle, and consider the double vector bundle

$$\begin{array}{ccc} T^*V & \longrightarrow & V \\ \downarrow & & \downarrow \\ V^* & \longrightarrow & M \end{array}$$

The bundle $V \to M$ has a linear Poisson structure. Choose a Poisson spray $X \in \mathfrak{X}(T^*V)$ which is *linear* relative to the vertical vector bundle structure. (This is always possible: e.g., choose X locally in bundle charts for $T^*V \to V^*$, and then patch the local definitions using a partition of unity.) The symplectic realization (P, ω) defined by the Crainic-Marcut theorem [13] is then linear, i.e., $P \subseteq T^*V$ is a vector subbundle for the vertical vector bundle structure, with base an open neighborhood $Q \subseteq V^*$ of M, and the symplectic form ω is homogeneous of degree 1. As explained in [30], one obtains the local groupoid structure on P by describing the left and right invariant vector fields. The base of the local symplectic $\mathcal{VB}$-groupoid $P \rightrightarrows V$ is a local Lie groupoid $Q \rightrightarrows M$, and by Proposition 11.14 it defines an integration of $A = V^*$.

Remark 11.15. Alternatively, given *any* local symplectic groupoid $P \rightrightarrows V$ integrating the Poisson manifold V (e.g., as in [10]), one can always lift the $\mathbb{R}_{>0}$-action on V to a (local) action on P, such that the symplectic form is homogeneous of degree 1. See [6].

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