LIE GROUPOIDS AND LIE ALGEBROIDS
LECTURE NOTES, FALL 2017

ECKHARD MEINRENKEN

ABSTRACT. These notes are very much under construction. They contain many errors, and the references are very incomplete. Apologies!

CONTENTS

1. Lie groupoids

Symmetries in mathematics, as well as in nature, are often defined to be invariance properties under actions of groups. Lie groupoids are given by a manifold M of 'objects' together with a type of symmetry of M that is more general than those provided by group actions. For example, a foliation of M provides an example of such a generalized symmetry, but foliations need not be obtained from group actions in any obvious way.

1.1. **Definitions.** The groupoid will assign to any two objects $m_0,\ m_1 \in M$ a collection (possibly empty) of arrows from m_1 to m_0. These arrows are thought of as 'symmetries', but in contrast to Lie group actions this symmetry need not be defined for all $m \in M$ – only pointwise. On the other hand, we require that the collection of all such arrows (with arbitrary end points) fit together smoothly to define a manifold, and that arrows can be composed provided the end point (target) of one arrow is the starting point (source) of the next.

The formal definition of a *Lie groupoid* $\mathcal{G} \rightrightarrows M$ involves a manifold $\mathcal{G}$ of *arrows*, a submanifold $i\colon M \hookrightarrow \mathcal{G}$ of *units* (or *objects*), and two surjective submersions $\mathsf{s}, \mathsf{t}\colon \mathcal{G} \to M$ called *source* and *target* such that

$$\mathsf{t} \circ i = \mathsf{s} \circ i = \mathrm{id}_M .$$

One thinks of g as an arrow from its source $\mathsf{s}(g)$ to its target $\mathsf{t}(g)$, with M embedded as trivial arrows.

$$\begin{tikzcd} \mathsf{t}(g) & & \mathsf{s}(g) \arrow[ll, bend right, "g"'] \end{tikzcd} \tag{1}$$

Using that s, t are submersion, one finds (cf. Exercise 1.1 below) that for all $k = 1, 2, \ldots$ the set of k-*arrows*

$$\mathcal{G}^{(k)} = \{(g_1, \ldots, g_k) \in \mathcal{G}^k |\ \mathsf{s}(g_i) = \mathsf{t}(g_{i+1})\}$$

is a smooth submanifold of $\mathcal{G}^k$, and the two maps $\mathcal{G}^{(k)} \to M$ taking $(g_1, \ldots, g_k)$ to $\mathsf{s}(g_k)$, respectively to $\mathsf{t}(g_1)$, are submersions. For $k = 0$ one puts $\mathcal{G}^{(0)} = M$.

The definition of a Lie groupoid also involves a smooth multiplication map, defined on composable arrows (i.e., 2-arrows)

$$\mathrm{Mult}_{\mathcal{G}}\colon \mathcal{G}^{(2)} \to \mathcal{G}, \quad (g_1, g_2) \mapsto g_1 \circ g_2,$$

such that $\mathsf{s}(g_1 \circ g_2) = \mathsf{s}(g_2),\quad \mathsf{t}(g_1 \circ g_2) = \mathsf{t}(g_1)$. It is thought of as a concatenation of arrows. Note that when picturing this composition rule, it is best to draw arrows from the right to the left.

$$\begin{tikzcd} m_0 & m_1 \arrow[l, bend right, "g_1"'] & m_2 \arrow[l, bend right, "g_2"'] & & m_0 & & m_2 \arrow[ll, bend right, "g_1 \circ g_2"'] \end{tikzcd} \tag{2}$$

Definition 1.1. The above data define a *Lie groupoid* $\mathcal{G} \rightrightarrows M$ if the following axioms are satisfied:

1. **Associativity:** $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ for all $(g_1, g_2, g_3) \in \mathcal{G}^{(3)}$.
2. **Units:** $\mathsf{t}(g) \circ g = g = g \circ \mathsf{s}(g)$ for all $g \in \mathcal{G}$.
3. **Inverses:** For all $g \in \mathcal{G}$ there exists $h \in \mathcal{G}$ such that $\mathsf{s}(h) = \mathsf{t}(g),\ \mathsf{t}(h) = \mathsf{s}(g)$, and such that $g \circ h,\ h \circ g$ are units.

The inverse of an element is necessarily unique (cf. Exercise). Denoting this element by g^{-1}, we have that $g\circ g^{-1}=\mathsf{t}(g),\ \ g^{-1}\circ g=\mathsf{s}(g)$. Inversion is pictured as reversing the direction of arrows.

From now on, when we write $g=g_1\circ g_2$ we implicitly assume that g_1,g_2 are composable, i.e. $\mathsf{s}(g_1)=\mathsf{t}(g_2)$. Let

$$\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})=\{(g,g_1,g_2)\in\mathcal{G}^3|\ g=g_1\circ g_2\}.$$

be the graph of the multiplication map; we will think of $\mathrm{Mult}_{\mathcal{G}}$ as a smooth relation from $\mathcal{G}\times\mathcal{G}$ to $\mathcal{G}$.

Remark 1.2. A groupoid structure on a manifold $\mathcal{G}$ is completely determined by $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}})$, i.e. by declaring when $g=g_1\circ g_2$. Indeed, the units are the elements $m\in\mathcal{G}$ such that $m=m\circ m$. Given $g\in\mathcal{G}$, the source $\mathsf{s}(g)$ and target $\mathsf{t}(g)$ are the unique units for which $g=g\circ\mathsf{s}(g)=\mathsf{t}(g)\circ g$. The inverse of g is the unique element g^{-1} such that $g\circ g^{-1}$ is a unit.

Remark 1.3. One may similarly consider 'set-theoretic' groupoids $\mathcal{G}\rightrightarrows M$, by taking s,t, and $\mathrm{Mult}_{\mathcal{G}}$ to be set maps (with s,t surjective). Such a set-theoretic groupoid is the same as a category with objects M and arrows $\mathcal{G}$, with the property that every arrow is invertible.

Remark 1.4. Let $\mathrm{Inv}_{\mathcal{G}}\colon\mathcal{G}\to\mathcal{G},\ \ g\mapsto g^{-1}$ be the inversion map. As an application of the implicit function theorem, it is automatic that $\mathrm{Inv}_{\mathcal{G}}$ is a diffeomorphism.

Definition 1.5. A *morphism of Lie groupoids* $F\colon\mathcal{H}\to\mathcal{G}$ is a smooth map such that

$$F(h_1\circ h_2)=F(h_1)\circ F(h_2)$$

for all $(h_1,h_2)\in\mathcal{H}^{(2)}$. If F is an inclusion as a submanifold, we say that $\mathcal{H}$ is a Lie subgroupoid of $\mathcal{G}$.

By Remark 1.2, it is automatic that such a morphism takes units of $\mathcal{H}$ to units of $\mathcal{G}$, and that it intertwines the source, target, inversion maps. We will often present Lie groupoid homomorphisms as follows:

$$\begin{tikzcd}
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F"'] & M \arrow[d] \\
\mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & N
\end{tikzcd}$$

Example 1.6. Any manifold M has a 'trivial' Lie groupoid structure, where $m=m_1\circ m_2$ if and only of $m=m_1=m_2$. Given a Lie groupoid $\mathcal{G}\rightrightarrows M$, the units of M with the trivial groupoid structure are a subgroupoid of $\mathcal{G}$.

Example 1.7. If $\mathcal{G}\rightrightarrows M$ is a Lie groupoid, and $m\in M$, the intersection of the source and target fibers

$$\mathcal{G}_m=\mathsf{t}^{-1}(m)\cap\mathsf{s}^{-1}(m)$$

is a Lie group, with group structure induced by the groupoid multiplication. (We will prove later that it's a submanifold.) It is called the *isotropy group of* $\mathcal{G}$ *at* m. Regarding the isotropy group as a Lie groupoid $\mathcal{G}_m \rightrightarrows \{m\}$ (see below), it is a Lie subgroupoid

$$\begin{tikzcd} \mathcal{G}_m \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \{m\} \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Exercise 1.1. Let $\phi_1\colon Q_1 \to M_1,\ \phi_2\colon Q_2 \to M_2$ be two submersions. Given any smooth map $F\colon Q_1 \to M_2$, show that the fiber product

$$Q_2 \ {}_{\phi_2}\!\times_F Q_1 = \{(q_2, q_1)|\ \phi_2(q_2) = F(q_1)\}$$

is a smooth submanifold of $Q_2 \times Q_1$, and the map $Q_2 \ {}_{\phi_2}\!\times_F Q_1 \to M_1$ induced by ϕ_1 is a submersion. Use this to verify that for a Lie groupoid $\mathcal{G}$, the spaces of k-arrows $\mathcal{G}^{(k)}$ are smooth manifolds.

Exercise 1.2. Using the definition, show that inverses of a (Lie) groupoid are unique. In fact, show that if $g \in \mathcal{G}$ is given, and $h_1, h_2 \in \mathcal{G}$ are such that $g \circ h_1$ and $h_2 \circ g$ are units, then $h_1 = h_2$.

Exercise 1.3. Show that the inversion map $\mathrm{Inv}_{\mathcal{G}}$ of a groupoid is a diffeomorphism.

Exercise 1.4. a) Given two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$, show that their direct product becomes a Lie groupoid

$$\mathcal{G} \times \mathcal{H} \rightrightarrows M \times N.$$

b) Show that a smooth map $F\colon \mathcal{H} \to \mathcal{G}$ between Lie groupoids is a Lie groupoid morphism if and only if its graph

$$\mathrm{Gr}(F) = \{(g, h) \in \mathcal{G} \times \mathcal{H}|\ F(h) = g\}$$

is a Lie subgroupoid of the direct product:

$$\begin{tikzcd} \mathrm{Gr}(F) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Gr}(F|_M) \arrow[d] \\ \mathcal{G} \times \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & M \times N \end{tikzcd}$$

1.2. **Examples.**

Example 1.8. A Lie group G is the same as a Lie groupoid with a unique unit, $G \rightrightarrows \mathrm{pt}$. At the opposite extreme, every manifold M can be regarded as a trivial Lie groupoid $M \rightrightarrows M$ where all elements are units.

Example 1.9. Suppose $\pi\colon Q \to M$ is a Lie group bundle, i.e., a locally trivial fiber bundle whose fibers have Lie group structure, in such a way that the local trivializations respect these group structures. (As a special case, any vector bundle is a Lie group bundle, using the additive group structure on the fibers.) Then Q is a groupoid $Q \rightrightarrows M$, with $\mathsf{s} = \mathsf{t} = \pi$, and with the groupoid multiplication $g = g_1 \circ g_2$ if and only if $\pi(g) = \pi(g_1) = \pi(g_2)$ and $g = g_1 g_2$ using the group structure on the fiber.

In the opposite direction, any Lie groupoid $\mathcal{G} \rightrightarrows M$ with $\mathsf{s} = \mathsf{t}$ defines a *family of Lie groups*: A surjective submersion with a fiberwise group structure such that the fiberwise multiplication depends smoothly on the base point. In general, it is *not* a Lie group bundle since there need not be local trivializations. In fact, the groups for different fibers need not even be isomorphic as Lie groups, or even as manifolds.

Example 1.10. For any manifold M, one has the *pair groupoid*

$$\mathrm{Pair}(M) = M \times M \rightrightarrows M,$$

with a unique arrow between any two points m', m (labeled by the pair itself). The composition is necessarily

$$(m', m) = (m_1', m_1) \circ (m_2', m_2) \quad \Leftrightarrow \quad m_1' = m,\ m_1 = m_2',\ m_2 = m.$$

The units are given by the diagonal embedding $M \hookrightarrow M \times M$, and the source and target of (m', m) are m and m', respectively. Note that the isotropy groups $\mathcal{G}_m$ of the pair groupoid are trivial.

The target and anchor of any Lie groupoid $\mathcal{G} \rightrightarrows M$ combine into a Lie groupoid morphism

(3)
$$\begin{tikzcd} \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\mathsf{t},\mathsf{s})"'] & M \arrow[d] \\ \mathrm{Pair}(M) \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

This groupoid morphism (t, s) is sometimes called the (groupoid) *anchor*; it is related to the anchor of Lie algebroids as we will see below.

Example 1.11. Another natural Lie group associated to any manifold M is the *fundamental groupoid*

$$\Pi(M) \rightrightarrows M,$$

consisting of homotopy classes of paths $[\gamma]$ relative to fixed end points. The source and target maps are $\mathsf{s}([\gamma]) = \gamma(0)$, $\mathsf{t}([\gamma]) = \gamma(1)$, and the groupoid multiplication is concatenation of paths. The natural map $\Pi(M) \to \mathrm{Pair}(M)$ is a local diffeomorphism; if M is simply connected it is a global diffeomorphism. [1]

Example 1.12. Given points $m_0, m_1 \in M$ and a diffeomorphism ϕ from an open neighborhood of m_1 to an open neighborhood m_0, with $\phi(m_1) = m_0$, let $j_k(\phi)$ denote its *k-jet*. Thus, $j_k(\phi)$ is the equivalence class of ϕ among such diffeomorphisms, where $j_k(\phi) = j_k(\phi')$ if the Taylor expansions of ϕ, ϕ' in local coordinates centered m_1, m_0 agree up to order k. The set of such triples $(m_0, j_k(\phi), m_1)$ is a manifold $J_k(M, M)$, and with the obvious composition of jets it becomes a Lie groupoid

$$J_k(M, M) \rightrightarrows M.$$

[1] If M is connected, and $\widetilde{M}$ is a simply connected covering space, with covering map $\widetilde{M} \to M$, one has a Lie groupoid homomorphism $\mathrm{Pair}(\widetilde{M}) = \Pi(\widetilde{M}) \to \Pi(M)$. By homotopy lifting, this map is surjective. Let Λ be the discrete group of deck transformations of $\widetilde{M}$, i.e., diffeomorphisms covering the identity map on M. Then $M = \widetilde{M}/\Gamma$, and $\Pi(M) = \Pi(\widetilde{M})/\Gamma = (\widetilde{M} \times \widetilde{M})/\Gamma$, a quotient by the diagonal action.

For $k = 0$, this is just the pair groupoid; for $k = 1$, the elements of the groupoid $J_1(M, M) \rightrightarrows M$ are pairs of elements $m_0, m_1 \in M$ together with an isomorphism $T_{m_1}M \to T_{m_0}M$. The natural maps

$$\cdots \to J_k(M, M) \to J_{k-1}(M, M) \to \cdots \to J_0(M, M) = \mathrm{Pair}(M)$$

are morphisms of Lie groupoids. (These groupoids may be regarded as finite-dimensional approximations of the *Haefliger groupoid of* M, consisting of germs of local diffeomorphisms. The latter is not a Lie groupoid since it is not a manifold.)

Example 1.13. Given an smooth action of a Lie group G on M, one has the *action groupoid* or transformation groupoid $\mathcal{G} \rightrightarrows M$. It may be defined as the subgroupoid of the direct product of groupoids $G \rightrightarrows \mathrm{pt}$ and $\mathrm{Pair}(M) \rightrightarrows M$, consisting of all $(g, m', m) \in G \times (M \times M)$ such that $m' = g.m$. Using the projection $(g, m', m) \mapsto (g, m)$ to identify $\mathcal{G} \cong G \times M$, the product reads as

$$(g, m) = (g_1, m_1) \circ (g_2, m_2) \quad \Leftrightarrow \quad g = g_1 g_2,\ m = m_2,\ m_1 = g_2.m_2.$$

Note that the isotropy groups $\mathcal{G}_m$ of the action groupoid coincide with the stabilizer groups of the G-action, G_m.

Example 1.14. Given a surjective submersion $\pi\colon M \rightrightarrows N$, one has a *submersion groupoid*

$$M \times_N M \rightrightarrows M$$

given as the fiber product with itself over N. The groupoid structure is as a subgroupoid of the pair groupoid $\mathrm{Pair}(M)$. For the special case of a principal G-bundle $\pi\colon P \to N$, the submersion groupoid is identified with $P \times G$; the groupoid structure is that of an action groupoid.

Example 1.15. Let $P \to M$ be a principal G-bundle. Let $\mathcal{G}(P)$ be the set of triples (m', m, ϕ) where $m, m' \in M$ and $\phi\colon P_m \to P_{m'}$ is a G-equivariant map between the fibers over $m, m' \in M$. Put $\mathsf{s}(m', m, \phi) = m$, $\mathsf{t}(m', m, \phi) = m'$, and define the composition by

$$(m'_1, m_1, \phi_1) \circ (m'_2, m_2, \phi_2) = (m', m, \phi)$$

whenever $\phi = \phi_1 \circ \phi_2$ and $(m'_1, m_1) \circ (m'_2, m_2) = (m', m)$ (as for the pair groupoid). We will call the resulting groupoid

$$\mathcal{G}(P) \rightrightarrows M$$

the *Atiyah algebroid* of $\mathcal{P}$; it is also known as the *gauge groupoid*. Equivalently, we may regard $\mathcal{G}(P)$ as the quotient

$$\mathcal{G}(P) = \mathrm{Pair}(P)/G,$$

of the pair groupoid by the diagonal action. As a special case, for a vector bundle $\mathcal{V} \to M$ one has an Atiyah algebroid $\mathcal{G}(\mathcal{V})$ of its frame bundle, given more directly as the set of linear isomorphisms from one fiber of $\mathcal{V}$ to another fiber. Note that the Atiyah algebroid of the tangent bundle TM is the same as the first jet groupoid $J_1(M, M)$.

Exercise 1.5. Let $\mathcal{G} \subseteq \mathbb{R} \times \mathrm{Mat}_{\mathbb{R}}(3)$ be the 4-dimensional submanifold consisting of all (t, B) such that

$$B + B^\top + t\ B^\top B = 0.$$

Show that

$$(t, B) = (t_1, B_1) \circ (t_2, B_2)$$

if and only if

$$t = t_1 = t_2, \quad B = B_1 + B_2 + t\ B_1 B_2$$

defines a Lie groupoid structure $G \rightrightarrows \mathbb{R}$, with $\mathsf{s} = \mathsf{t}$ given by projection $\mathcal{G} \to \mathbb{R}$. Identify the Lie groups $\mathcal{G}_t$ given as the fibers of this projection. (Hint: For $t \neq 0$, consider $A = tB + I$.)

Exercise 1.6. Let $\mathfrak{d}$ be a Lie algebra, and $\mathfrak{g}, \mathfrak{h} \subseteq \mathfrak{d}$ two Lie subalgebras such that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Let D, G, H be the corresponding simply connected Lie groups, and $i\colon G \to D$, $j\colon H \to D$ the group homomorphisms exponentiating the inclusions of $\mathfrak{g}, \mathfrak{h}$ into $\mathfrak{d}$. Let

$$\Gamma = \{(h, g, g', h') \in H \times G \times G \times H |\ j(h)i(g) = i(g)j(h)\}.$$

In other words, $\mathcal{G}$ is the fiber product of $H \times G$ with $G \times H$ over D, relative the natural maps from these spaces to D. Put

$$(h, g, g', h') = (h_1, g_1, g_1', h_1') \circ (h_2, g_2, g_2', h_2')$$

if and only if

$$h_1' = h_2, \quad h = h_2, \quad h' = h_2', \quad g = g_1 g_2, \quad g' = g_1' g_2'.$$

Show that this defines the structure of a Lie groupoid $\mathcal{G} \rightrightarrows H$. Exchanging the roles of G and H the same space has a groupoid structure $\mathcal{G} \rightrightarrows G$, and the two structures are compatible in the sense that they define a *double Lie groupoid*. Try to invent such a compatibility condition of two groupoid structures, and verify that it is satisfied in this example. (See Lu-Weinstein [?])

Exercise 1.7. Let X be a vector field on a manifold M. If X is complete, then the flow $\Phi_t(m)$ of any $m \in M$ is defined for all t, and one obtains a group action $\Phi\colon \mathbb{R} \times M \to M$. For an incomplete vector field, Φ is defined on a suitable $U \subseteq \mathbb{R} \times M$.

2. Foliation groupoids

2.1. **Definition, examples.** A foliation $\mathcal{F}$ of a manifold M may be defined to be a subbundle $E \subseteq TM$ satisfying the Frobenius condition: for any two vector fields X, Y taking values in E, their Lie bracket again takes values in E. Given such a subbundle, one obtains a decomposition of M into *leaves* of the foliation, i.e., maximal connected injectively immersed submanifolds. The quotient space $M/\sim$, where two points are considered equivalent if they lie in the same leaf, is called the *leaf space*. Locally, a foliation looks very simple: For every $m \in M$ there exists a chart (U, ϕ) centered at m, with $\phi\colon U \to \mathbb{R}^n$, such that the tangent map $T\phi$ takes $E|_U$ to the tangent bundle of the projection $\mathrm{pr}_{\mathbb{R}^q}\colon \mathbb{R}^n \to \mathbb{R}^q$ to the last q coordinates. Such an adapted chart is called a *foliation chart*. For a foliation chart, every $U_a = \phi^{-1}(\mathbb{R}^{n-q} \times \{a\})$ for $a \in \mathbb{R}^q$ is an open subset of a leaf. Globally, the situation can be much more complicated, since U_a, U_b for $a \neq b$ might belong to the same leaf. Accordingly, the leaf space of a foliation can be extremely complicated.

Example 2.1. For any surjective submersion $\pi\colon P \to B$, the bundle $\ker(T\pi) \subseteq TP$ defines a foliation, with leaves the fibers $\pi^{-1}(b)$. If the fibers are connected, then $P/\sim$ is just B itself. If the fibers are disconnected, the leaf space can be a non-Hausdorff manifold. (E.g., take $P = \mathbb{R}^2\backslash\{0\}$ with π projection to the first coordinate; here $P/\sim$ is the famous 'line with two origins'.)

Example 2.2. Given a diffeomorphism $\Phi\colon M \to M$ of a manifold, one can form the mapping torus as the associated bundle

$$M_\Phi = \mathbb{R} \times_{\mathbb{Z}} M,$$

where $\mathbb{R}$ is regarded as a principal $\mathbb{Z}$-bundle over $\mathbb{R}/\mathbb{Z}$, and the action of $\mathbb{Z}$ is generated by Φ. That is, it is the quotient of $\mathbb{R} \times M$ under the equivalence relation generated by $(t, m) \sim (t+1, \Phi(m))$. The 1-dimensional foliation of $M \times \mathbb{R}$, given as the fibers under projection to M, is invariant under the $\mathbb{Z}$-action, and so it descends to a 1-dimensional foliation of the mapping torus. If some point $m \in M$ is fixed under some power Φ^N, then the corresponding leaf in the mapping torus is a circle winding N times around the mapping torus. But if m is not a fixed point, then the corresponding leaf is diffeomorphic to $\mathbb{R}$.

Example 2.3. Given a manifold M with a foliation $\mathcal{F}$, and any proper action of a discrete group Λ preserving this foliation, the quotient M/Λ inherits a foliation. For example, given a connected manifold B, with base point b_0, and any action of the fundamental group $\Lambda = \pi_1(B, b_0)$ on another manifold Q, the foliation of $M = \widetilde{B}\times Q$ given by the fibers of the projection to Q is Λ-invariant for the diagonal action, and hence the associated bundle

$$M/\Lambda = \widetilde{B} \times_\Lambda Q$$

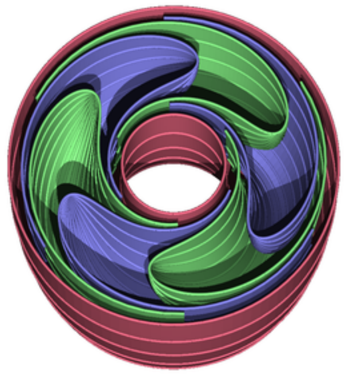
inherits a foliation. Note that the leaves of this foliations are coverings of B.

Example 2.4. Consider the foliation of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by the vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ whose lift to $\mathbb{R}^2$ is $\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$. If c is a rational number, then the flow of X is periodic, and space of leaves of the foliation is a manifold (a circle). If c is irrational, then the space $T^2/\sim$ of leaves is quite pathological: its only open subsets are the empty set and the entire space. (This example may also be regarded as a mapping torus, where Φ is given by a rotation of the standard circle by a fixed $2\pi c$.)

Example 2.5. We next describe a 2-dimensional foliation of the 3-sphere, known as the *Reeb foliation*. Consider S^3 as the total space of the Hopf fibration

$$\pi\colon S^3 \to S^2$$

(realized for example as the quotient map from $\mathrm{SU}(2) = S^3$ to its homogeneous space $\mathbb{C}P(1) \cong S^2$). The pre-image of the equator on S^2 is a 2-torus $T^2 \subseteq S^3$, and this will be one leaf of the foliation. The pre-image of the closed upper hemisphere is a solid 2-torus bounded by T^2, and similarly for the closed lower hemisphere. Thus, S^3 is obtained by gluing two solid 2-tori along their boundary. Note that this depends on the choice of gluing map: what is the 'small circle' with respect to one of the solid tori becomes the 'large circle' for the other, and vice versa. Now, foliate the interiors of these solid 2-tori as in the following picture (from wikipedia):



More specifically, this foliation of the interior of a solid torus is obtained from a translation invariant foliation of the interior of a cylinder $Z = \{(x, y, z)|\ x^2 + y^2 < 1\}$, for example given by the hypersurfaces

$$z = \exp\Big(\frac{1}{1-(x^2+y^2)}\Big) + a,\ \ a \in \mathbb{R}$$

for $a \in \mathbb{R}$. The Reeb foliation has a unique compact leaf (the 2-torus), while all other leaves are diffeomorphic to $\mathbb{R}^2$.

2.2. **Monodromy and holonomy.** We will need the following notions.

Definition 2.6. Let $\mathcal{F}$ be a foliation of M, of codimension q.

(a) A path (resp. loop) in M that is contained in a single leaf of the foliation $\mathcal{F}$ is called a *foliation path* (resp., foliation loop).
(b) A q-dimensional submanifold N is called a *transversal* if N is transverse to all leaves of the foliation. That is, for all $m \in N$, the tangent space T_mN is a complement to the tangent space of the foliation.

Given points m, m' in the same leaf L, and transversals N, N' through these points, then any leaf path γ from m to m' determines the germ at m of a diffeomorphism

$$\phi_\gamma \colon N \to N',$$

taking m to m'. Indeed, given $m_1 \in N_1$ sufficiently close to m, there exists a foliation path γ_1 close to γ, and with end point in N'. This end point m_1' is independent of the choice of γ_1, as long as it stays sufficiently close to γ. This germ ϕ_γ is unchanged under homotopies of γ. It is also independent of the choice of transversals, since we may regard a sufficiently small neighborhood of m in N as the 'local leaf space' for M near m, and similarly for m'. One calls ϕ_γ the *holonomy* of the path γ. Intuitively, the holonomy of a path measures how the foliation 'twists' along γ. In the special case $m = m'$, we may take $N = N'$ and obtain a map from the fundamental group of the leaf, $\pi_1(L, m)$, to the group of germs of diffeomorphism of N fixing m.

2.3. **The monodromy and holonomy groupoids.**

Definition 2.7. Let $\mathcal{F}$ be a foliation of M, and $m \in M$.

(a) The monodromy group of $\mathcal{F}$ at m is the fundamental group of the leaf $L \subseteq M$ through m:
$$\mathrm{Mon}(\mathcal{F}, m) = \pi_1(L, m).$$
(b) The holonomy group of $\mathcal{F}$ at m is the image of the homomorphism from $\mathrm{Mon}(\mathcal{F}, m)$ into germs of diffeomorphisms of a local transversal through m. It is denoted
$$\mathrm{Hol}(\mathcal{F}, m).$$

In other words, $\mathrm{Mon}(\mathcal{F}, m)$ consists of homotopy classes of foliation loops bases at m, while $\mathrm{Hol}(\mathcal{F}, m)$ consists of holonomy classes. $\mathrm{Hol}(\mathcal{F}, m)$ is the quotient of $\mathrm{Mon}(\mathcal{F}, m)$ by the classes of foliation loops having trivial holonomy.

Definition 2.8. Let $\mathcal{F}$ be a foliation of M.

(a) The *monodromy groupoid*
$$\mathrm{Mon}(\mathcal{F}) \rightrightarrows M,$$
consists of triples $(m', m, [\gamma])$, where $m, m' \in M$ and $[\gamma]$ is the homotopy class of a foliation path γ from $m = \gamma(0)$ to $m' = \gamma(1)$. The groupoid structure is induced by the concatenation of foliation paths.

(b) The *holonomy groupoid*
$$\mathrm{Hol}(\mathcal{F}) \rightrightarrows M,$$
is defined similarly, but taking $[\gamma]$ to be the holonomy class of a foliation path γ.

Proposition 2.9. $\mathrm{Mon}(\mathcal{F})$ *and* $\mathrm{Hol}(\mathcal{F})$ *are (possibly non-Hausdorff) manifolds.*

Sketch. Here is a sketch of the construction of charts, first for the monodromy groupoid. Given $(m', m, [\gamma]) \in \mathrm{Mon}(\mathcal{F})$, choose local transversals $N,\ N'$ through $m,\ m'$: that is, q-dimensional submanifolds transverse to the foliation, where q is the codimension of $\mathcal{F}$. Let $\phi_\gamma \colon N \to N'$ be the diffeomorphism germ determined by $[\gamma]$. Choosing a germ of a diffeomorphism $N \to \mathbb{R}^q$, which we may think of as *transverse* coordinates at m, we then also obtain transverse coordinates near m'. These sets of transverse coordinates may be completed to local foliation charts at m and m'. We hence obtain $2(n-q)+q = 2n-q$-dimensional charts for $\mathrm{Mon}(\mathcal{F})$. A similar construction works for $\mathrm{Hol}(\mathcal{F})$. ☐

Remark 2.10. (a) To see why the groupoids $\mathrm{Mon}(\mathcal{F})$ or $\mathrm{Hol}(\mathcal{F})$ are sometimes non-Hausdorff, suppose $g \in \mathrm{Mon}(M, m)$ is a non-trivial element of the monodromy group. It is represented by a non-contractible loop γ in the leaf through m. Then it can happen that γ is approached through loops γ_n in nearby leaves, but the γ_n are all contractible. Then the elements $g_n \in \mathrm{Mon}(M, m_n)$ (with $m_n = \gamma_n(0)$) satisfy $g_n \to g$, but since $g_n = m_n$ (constant loops) they also satisfy $g_n \to m$. This non-uniqueness of limits then implies that $\mathrm{Mon}(M)$ is not Hausdorff. Similarly phenomena appear for the holonomy groupoid.

(b) There is no simple relationship, in general, between the Hausdorff properties of the holonomy and monodromy groupoids of a foliation $\mathcal{F}$. Indeed, it can happen that two points of $\mathrm{Mon}(\mathcal{F})$ not admitting disjoint open neighborhoods get identified under the quotient map to $\mathrm{Hol}(\mathcal{F})$. On the other hand, it can also happen that two distinct points of $\mathrm{Hol}(\mathcal{F})$ do not admit disjoint open neighborhoods, even if they have pre-images in $\mathrm{Mon}(\mathcal{F})$ have disjoint open neighborhoods. (The images of the latter under the quotient map need no longer be disjoint.)

Exercise 2.1. (From Crainic-Fernandes [**?**].) Let $M = \mathbb{R}^3\backslash\{0\}$ be foliated by the fibers of the projection $(x, y, z) \mapsto z$. Is the monodromy groupoid Hausdorff? What about the holonomy groupoid?

Exercise 2.2. For the Reeb foliation of S^3, show that the holonomy groupoid coincides with the monodromy groupoid, and is non-Hausdorff.

Exercise 2.3. Think of S^3 has obtained by gluing two solid 2-tori as before, and let $M \subseteq S^3$ be the open subset obtained by removing the central circle of each of the solid 2-tori. Show that the monodromy groupoid is Hausdorff, but the holonomy groupoid is non-Hausdorff.

Exercise 2.4. Similar to S^3, the product $S^2 \times S^1$ is obtained by gluing to solid 2-tori, given as the pre-images of the closed upper/lower hemispheres under the projection to S^2. Foliate these solid 2-tori as for the Reeb foliation. Show that the monodromy groupoid is non-Hausdorff, but the holonomy groupoid is Hausdorff.

2.4. **Appendix: Haefliger's approach.** A cleaner definition of holonomy proceeds as follows (following Haefliger [**?**]): Let $\mathcal{F}$ be a given codimension q foliation of M. A foliated manifold can be covered by foliation charts $\phi\colon U \to \mathbb{R}^{n-q} \times \mathbb{R}^q$, i.e. the pre-images $\phi^{-1}(\mathbb{R}^{n-q} \times \{y_0\})$ for $y_0 \in \mathbb{R}^q$ are tangent to the leaves. There exists a topology on M, called the *foliation topology*, generated by such pre-images. Put differently, the foliation charts become local homeomorphisms if we give $\mathbb{R}^{n-q}$ its standard topology and $\mathbb{R}^q$ the discrete topology. The connected components of M for the foliation topology are exactly the leaves of M, and the continuous paths in M for the foliation topology are the foliation paths.

Let $\widetilde{M}$ be the set of all $(m, [\psi])$, where $m \in M$, and where $[\psi]$ is the germ of a smooth map $\psi\colon U \to \mathbb{R}^q$, where U is an open neighborhood of m, and ψ is a submersion whose fibers are tangent to leaves. Given a foliation chart (U, ϕ), one obtains a subset $\widetilde{U}$ of $\widetilde{M}$ consisting of germs of $[\psi]$ at points of U, where ψ is ϕ followed by projection. Give $\widetilde{M}$ the topology generated by all $\widetilde{U}$. Then the natural projection $\widetilde{M} \to M$ is a local homeomorphism relative to the foliation topology on M.

Let $\gamma\colon [0, 1] \to M$ be a continuous path for the foliation topology (a foliation path in M), with end points $m = \gamma(0)$ and $m' = \gamma(1)$. Since $\widetilde{M} \to M$ is a covering, γ to paths in $\widetilde{M}$, defining a map $\pi^{-1}(m) \to \pi^{-1}(m')$ between fibers of $\widetilde{M}$. Two such paths, with the same end points, are said to define the same *holonomy* if they determine the same map.

3. Properties of Lie groupoids

3.1. **Orbits and isotropy groups.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Define a relation $\sim$ on M, where

$$m \sim m' \Leftrightarrow \exists g \in \mathcal{G}\colon \mathsf{s}(g) = m,\ \mathsf{t}(g) = m'.$$

Lemma 3.1. *The relation $\sim$ is an equivalence relation.*

Proof. Transitivity follows from the groupoid multiplication, symmetry follows from the existence of inverses, reflexivity follows since elements of M are units (so, $m \circ m = m$) □

The equivalence classes of the relation $\sim$ are called the *orbits* of the Lie groupoid. The equivalence class of the element $m \in M$ is denoted

$$\mathcal{G} \cdot m \subseteq M.$$

Let us also recall the definition of isotropy groups

$$\mathcal{G}_m = \mathsf{s}^{-1}(m) \cap \mathsf{t}^{-1}(m).$$

Example 3.2. For a Lie group G with a Lie group action on M, the orbits and isotropy groups of the action groupoid $\mathcal{G} = G \times M$ are just the usual ones for the G-action:

$$\mathcal{G} \cdot m = G \cdot m, \quad \mathcal{G}_m = G_m.$$

Example 3.3. For a foliation $\mathcal{F}$ of M, the orbits of both $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$ and $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ are the leaves of the foliation $\mathcal{F}$, while the isotropy groups $\mathcal{G}_m$ are the monodromy groups and holonomy groups, respectively.

We will show later in this section that the orbits are injectively immersed submanifolds of M, while the isotropy groups are embedded submanifolds of $\mathcal{G}$. Note that for any given m, the orbit can be characterized as

$$\mathcal{G} \cdot m = \mathsf{t}(\mathsf{s}^{-1}(m)),$$

(or also as $\mathsf{s}(\mathsf{t}^{-1}(m))$). The isotropy group $\mathcal{G}_m$ is the fiber of m under the map $\mathsf{s}^{-1}(m) \to G \cdot m$. Since is a submersion, the fibers $\mathsf{s}^{-1}(m)$ are embedded submanifolds of $\mathcal{G}$, of dimension $\dim \mathcal{G} - \dim M$. Hence, to show that the orbits and stabilizer group are submanifolds, it suffices to show that the restriction of t to any source fiber $\mathsf{s}^{-1}(m)$ has constant rank. This will be proved as Proposition 3.7 below.

3.2. **Bisections.** A *bisection* of a Lie groupoid is a submanifold $S \subseteq \mathcal{G}$ such that both t, s restrict to diffeomorphisms $S \to M$. For example, M itself is a bisection. The name indicates that S can be regarded as a section of both s and t. We will denote by

$$\Gamma(\mathcal{G})$$

the set of all bisections. It has a group structure, with the multiplication given by

$$S_1 \circ S_2 = \mathrm{Mult}_{\mathcal{G}}((S_1 \times S_2) \cap \mathcal{G}^{(2)}).$$

That is, $S_1 \circ S_2$ consists of all products $g_1 \circ g_2$ of composable elements with $g_i \in S_i$ for $i = 1, 2$. The identity element for this multiplication is the unit bisection M, and the inverse is given by $S^{-1} = \mathrm{Inv}_{\mathcal{G}}(S)$. This group of bisections comes with a group homomorphism

$$\Gamma(\mathcal{G}) \to \mathrm{Diff}(M), \quad S \mapsto \Phi_S$$

where $\Phi_S = \mathsf{t}|_S \circ (\mathsf{s}|_S)^{-1}$.

Remark 3.4. Alternatively, a bisection of $\mathcal{G}$ may be regarded as a section $\sigma \colon M \to \mathcal{G}$ of the source map s such that its composition with the target map t is a diffeomorphism of Φ. The definition as a submanifold has the advantage of being more 'symmetric'.

Examples 3.5.
(a) For a Lie group $G \rightrightarrows \mathrm{pt}$, regarded as a Lie groupoid, a bisection is simply an element of G, and $\Gamma(G) = G$ as a group.
(b) For a vector bundle $V \to M$, regarded as groupoid $V \rightrightarrows M$, a bisection is the same as a section. More generally, this is true for any bundle of Lie groups.
(c) For the 'trivial' groupoid $M \rightrightarrows M$ the only bisection is M itself. The resulting group $\Gamma(M)$ consists of only the identity element.
(d) For the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$, a bisection is the same as the graph of a diffeomorphism of M. This identifies $\Gamma(\mathcal{G}) \cong \mathrm{Diff}(M)$.

(e) Let $P \to M$ be a principal G-bundle. A bisection of Atiyah groupoid $\mathcal{G}(P) \rightrightarrows M$ is the same as a principal bundle automorphism $\Phi_P \colon P \to P$. That is,

$$\Gamma(\mathcal{G}) = \mathrm{Aut}(P).$$

(f) Given a G-action on M, a bisection of the action groupoid is a smooth map $f \colon M \to G$ for which the map $m \mapsto f(m).m$ is a diffeomorphism.

The group of bisections has three natural actions on $\mathcal{G}$:

- Left multiplication:
$$\mathcal{A}^L_S(g) = h \circ g,$$
with the unique element $h \in S$ such that $\mathsf{s}(h) = \mathsf{t}(g)$. Namely, $h = ((\mathsf{s}|_S)^{-1} \circ \mathsf{t})(g)$. This has the property
$$\mathsf{s} \circ \mathcal{A}^L_S = \mathsf{s}, \quad \mathsf{t} \circ \mathcal{A}^L_S = \Phi_S \circ \mathsf{t}. \tag{4}$$
- Right multiplication:
$$\mathcal{A}^R_S(g) = g \circ (h')^{-1},$$
with the unique element $h' \in S$ such that $\mathsf{s}(h') = \mathsf{s}(g)$. Namely, $h' = (\mathsf{s}|_S)^{-1}(\mathsf{s}(g))$. We have that
$$\mathsf{t} \circ \mathcal{A}^R_S = \mathsf{t}, \quad \mathsf{s} \circ \mathcal{A}^R_S = \Phi_S \circ \mathsf{s}. \tag{5}$$
- Adjoint action:
$$\mathrm{Ad}_S(g) = h \circ g \circ (h')^{-1},$$
with h, h' as above. Note that the adjoint action is by groupoid automorphisms, restricting to the map Φ_S on units.

Examples 3.6. Diffeomorphisms of a manifold give a natural action on the pair groupoid $\mathrm{Pair}(M)$. For a principal G-bundle, the group $\mathrm{Aut}(P)$ of principal bundle automorphisms naturally acts by automorphisms of the Atiyah groupoid. In both cases, the natural action is the adjoint action.

3.3. **Local bisections.** In general, there may not exist a global bisection passing through a given point $g \in \mathcal{G}$. However, it is clear that one can always find a *local bisection* $S \subseteq M$, that is, t, s restrict to local diffeomorphisms to open subsets $\mathsf{t}(S) = V$, $\mathsf{s}(S) = U$ of M. Any local bisection defines a diffeomorphism between these open subsets:

$$\Phi_S = \mathsf{t}|_S \circ (\mathsf{s}|_S)^{-1} \colon U \to V,$$

with inverse defined by the local bisection $S^{-1} = \mathrm{Inv}_{\mathcal{G}}(S)$. We have the left, right, and adjoint actions defined as diffeomorphisms

$$\mathcal{A}^L_S \colon \mathsf{t}^{-1}(U) \to \mathsf{t}^{-1}(V), \quad g \mapsto h \circ g,$$
$$\mathcal{A}^R_S \colon \mathsf{s}^{-1}(U) \to \mathsf{s}^{-1}(V), \quad g \mapsto g \circ (h')^{-1},$$

where, for a given element $g \in \mathcal{G}$, we take h, h' to be the unique elements in S such that $\mathsf{s}(h) = \mathsf{t}(g)$, $\mathsf{s}(h') = \mathsf{s}(g)$. These satisfy the relations (4), (5) as before, and hence we also have an adjoint action defined as a diffeomorphism

$$\mathrm{Ad}_S \colon \mathsf{s}^{-1}(U) \cap \mathsf{t}^{-1}(U) \to \mathsf{s}^{-1}(V) \cap \mathsf{t}^{-1}(V), \quad g \mapsto h \circ g \circ (h')^{-1},$$

extending the map Φ_S on units. As an application of local bisections, we can now prove

Proposition 3.7. *For any Lie groupoid $\mathcal{G} \rightrightarrows M$ and any $m \in M$, the restriction of* t *to the source fiber* $\mathsf{s}^{-1}(m)$ *has constant rank.*

Proof. To show that the ranks of
$$\mathsf{t}|_{\mathsf{s}^{-1}(m)} \colon \mathsf{s}^{-1}(m) \to M$$
at given points $g, g' \in \mathsf{s}^{-1}(m)$ coincide, let S be a local bisection containing the element $g' \circ g^{-1}$, and let $U = \mathsf{s}(S),\ V = \mathsf{t}(S)$. The diffeomorphism
$$\mathcal{A}_S^L \colon \mathsf{t}^{-1}(U) \to \mathsf{t}^{-1}(V)$$
takes g to g'. Since $\mathsf{s} \circ \mathcal{A}_S^L = \mathsf{s}$, it restricts to a diffeomorphism on each s fiber. Since furthermore $\mathsf{t} \circ \mathcal{A}_S^L = \Phi_S \circ \mathcal{A}_S^L$, we obtain a commutative diagram
$$\begin{tikzcd}
\mathsf{t}^{-1}(U) \cap \mathsf{s}^{-1}(m) \arrow[r, "\mathcal{A}_S^L|_{\mathsf{s}^{-1}(m)}"] \arrow[d, "\mathsf{t}|_{\mathsf{s}^{-1}(m)}"'] & \mathsf{t}^{-1}(V) \cap \mathsf{s}^{-1}(m) \arrow[d, "\mathsf{t}|_{\mathsf{s}^{-1}(m)}"] \\
U \arrow[r, "\Phi_S"'] & V
\end{tikzcd}$$
where the horizontal maps are diffeomorphisms, and the upper map takes g to g'. Hence, the ranks of the vertical maps at g, g' coincide. $\square$

Corollary 3.8. *For every $m \in \mathcal{G}$, the orbit $\mathcal{G} \cdot m$ is an injectively immersed submanifold of M, while the isotropy group $\mathcal{G}_m$ is an embedded submanifold of $\mathcal{G}$, hence is a Lie group. In fact, all fibers of the map*
$$\mathsf{t}, \mathsf{s} \colon \mathcal{G} \to \mathrm{Pair}(M)$$
are embedded submanifolds.

For the last part, we have that $(\mathsf{t}, \mathsf{s})^{-1}(m', m)$ is a submanifold because it coincides with the fiber of m' under the surjective submersion $\mathsf{s}^{-1}(m) \to \mathcal{G} \cdot m$. (The fiber is empty if $m' \notin \mathcal{G} \cdot m$.) Note that (t, s) does not have constant rank, in general.

3.4. **Transitive Lie groupoids.** A G-action on a manifold is called *transitive* if it has only a single orbit: $G \cdot m$. The definition carries over to Lie groupoids:

Definition 3.9. A Lie groupoid is called *transitive* if it has only one orbit: $\mathcal{G} \cdot m = M$.

Here are some examples:

- The pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$ is transitive.
- The jet groupoids $J^k(M, M) \rightrightarrows M$ are transitive.
- The homotopy groupoid $\Pi(M) \rightrightarrows M$ is transitive if and only is M is connected.
- For an action of a Lie group G on M, the action groupoid $G \ltimes M \rightrightarrows M$ is transitive if and only if the G-action on M is transitive.
- For any Lie groupoid $\mathcal{G} \rightrightarrows M$, and any orbit $i \colon \mathcal{O} \hookrightarrow M$, the restriction of $\mathcal{G}|_{\mathcal{O}}$ to $\mathcal{O}$ is transitive. Here, the 'restriction' consists of all groupoid elements having source and fiber in $\mathcal{O}$. More precisely,
$$\mathcal{G}|_{\mathcal{O}} = \{(g, x', x) \in \mathcal{G} \times \mathrm{Pair}(\mathcal{O}) |\ \mathsf{s}(g) = x,\ \mathsf{t}(g) = x'\},$$
with the groupoid structure as a subgroupoid of $\mathcal{G} \times \mathrm{Pair}(\mathcal{O})$. Note that $\mathcal{G}_{\mathcal{O}}$ comes with an injective immersion to $\mathcal{G}$, and $\mathcal{G}$ is a disjoint union of all such immersions.

- For any principal G-bundle $\pi\colon P\to M$, the Atiyah groupoid $\mathcal{G}(P)$ is transitive.

It turns out that all these examples are special cases of the last one.

Theorem 3.10. *Suppose $\mathcal{G}\rightrightarrows M$ is a transitive Lie groupoid. Then $\mathcal{G}$ is isomorphic to an Atiyah groupoid $\mathcal{G}(P)$, for a suitable principal G-bundle $P\to M$. The identification depends on the choice of a base point $m_0\in M$.*

Proof. Given m_0, let $G=\mathcal{G}_{m_0}$ be the isotropy group at m_0, and $P=\mathsf{s}^{-1}(m_0)$ the source fiber. The target map gives a surjective submersion

$$\pi=\mathsf{t}|_{\mathsf{s}^{-1}(m_0)}\colon P\to M,\quad p\mapsto \mathsf{t}(p).$$

The group G acts on P by

$$g.p=p\circ g^{-1};$$

this is well-defined since $\mathsf{s}(p)=m_0=\mathsf{s}(g)$ and $\mathsf{s}(g.p)=\mathsf{s}(g^{-1})=\mathsf{t}(g)=m_0$. This action preserves fibers, since $\pi(g.p)=\mathsf{t}(p\circ g^{-1})=\mathsf{t}(p)=\pi(p)$. The action is free, since $g.p=p$ means $p\circ g^{-1}=p$, hence $g=m_0$ as an element of G, which is the identity of $G=\mathcal{G}_{m_0}$. Conversely, given two points $p',p\in P$ in the same fiber, i.e. $\mathsf{t}(p')=\mathsf{t}(p)$, the element $g=(p')^{-1}\circ p$ is well-defined, lies in $\mathcal{G}_{m_0}=G$, and satisfies $p'=p\circ g^{-1}$. This shows that P is a principal G-bundle.[2]

It remains to identify $\mathcal{G}$ with the Atiyah groupoid of P. Let $\phi\in\mathcal{G}$ be given. Left multiplication by ϕ gives a map

$$P_{\mathsf{s}(\phi)}\to P_{\mathsf{t}(\phi)},\quad p\mapsto \phi\circ p,$$

which commutes with the principal G-action given by multiplication from the right. This defines an injective smooth map $F\colon\mathcal{G}\to\mathcal{G}(P)$. It is clear that F is a groupoid homomorphism. The inverse map is constructed as follows: given $\psi\in\mathcal{G}(P)$, choose $p\in P_{\mathsf{s}(\psi)}$, then the element $\phi=\psi(p)\circ p^{-1}\in\mathcal{G}$ is defined, and independent of the choice of p. Clearly, $F(\phi)=\psi$. $\square$

Example 3.11. For a homotopy groupoid $\Pi(M)\rightrightarrows M$ over a connected manifold M, the choice of a base point m_0 defines the fundamental group $\mathcal{G}_{m_0}\cong\pi_1(M,m_0)$. The bundle P is the universal covering $\widetilde{M}$ of M (with respect to m_0), regarded as a principal $\pi_1(M,m_0)$-bundle, and $\Pi(M)$ gets identified as its Atiyah groupoid. In particular, we see that the group $\Gamma(\Pi(M))$ of bisections is the group

$$\mathrm{Aut}(\widetilde{M})=\mathrm{Diff}(\widetilde{M})^{\pi_1(M)}$$

of automorphisms of the covering space $\widetilde{M}$.

Example 3.12. Let $G\times M\rightrightarrows M$ be the action groupoid of a transitive G-action on M. The choice of $m_0\in M$ identifies M with the homogeneous space

$$M\cong G/K.$$

where $K=G_{m_0}$ is the stabilizer. The principal bundle P for this transitive Lie groupoid is G itself, regarded as a principal K-bundle over M. The resulting identification of the action groupoid and the Atiyah groupoid is the map

$$G\times(G/K)\to(G\times G)/K\quad (g,\ aK)\mapsto(ga,a)K,$$

[2] The local triviality is automatic: given a free Lie group action on a manifold, and a surjective submersion onto another manifold such that the orbits are exactly the fibers of the action, the manifold is a principal bundle; local trivializations are obtained from local sections of the submersion.

the inverse map is

$$(G \times G)/K \to G \times (G/K),\ (b,a)K \mapsto (ba^{-1}, aK).$$

Exercise 3.1. Let M be a connected manifold. Show (by giving a counter-example) that the map $\mathrm{Aut}(\widetilde{M}) \to \mathrm{Diff}(M)$ is not always surjective. **Hint:** You can take $M = S^1$.

Exercise 3.2. Show that a Lie groupoid is transitive if and only if the map (t, s) is surjective, and that it must be a submersion in that case.

4. More constructions with groupoids

In this section, we will promote the viewpoint of describing groupoid structures in terms of the graph of the groupoid multiplication. This will require some preliminary background material in differential geometry.

4.1. **Vector bundles in terms of scalar multiplication.** It is a relatively recent observation that vector bundles are uniquely determined by the underlying manifold structures together with the scalar multiplications:

Proposition 4.1 (Grabowski-Rotkievicz)**.** [**?**]

(a) *A submanifold of the total space of a vector bundle $E \to M$ is a vector subbundle if and only if it is invariant under scalar multiplication by all $t \in \mathbb{R}$.*
(b) *A smooth map $E' \to E$ between the total spaces of two vector bundles $E \to M$, $E' \to M'$ is a vector bundle morphism if and only if it intertwines the scalar multiplications by all $t \in \mathbb{R}$.*

That is, the additive structure is uniquely determined by the scalar multiplication.

4.2. **Relations.** A linear relation from a vector space V_1 to a vector space V_2 is a subspace $R \subseteq V_2 \times V_1$. We will think of R as a generalized map from V_1 to V_2, and will write

$$R\colon V_1 \dashrightarrow V_2.$$

We define the kernel and range of R as

$$\ker(R) = \{v_1 \in V_1 \colon (0, v_1) \in R\},$$
$$\mathrm{ran}(R) = \{v_2 \in V_2 \colon \exists v_1 \in V_1, (v_2, v_1) \in R\}.$$

R is called surjective if $\mathrm{ran}(R) = V_2$, and injective if $\ker(R) = 0$.

An actual linear map $A\colon V_1 \to V_2$ can be viewed as a linear relation, by identifying A with its graph $\mathrm{Gr}(A)$; the kernel and range of A as a linear map coincide withe the kernel and relation as a relation. The identity map $\mathrm{id}_V \colon V \to V$ defines the relation

$$\Delta_V = \mathrm{Gr}(\mathrm{id}_V)\colon V \dashrightarrow V$$

given by the diagonal in $V \times V$. Any subspace $S \subseteq V$ can be regarded as a relation $S\colon 0 \dashrightarrow V$. Given a relation $R\colon V_1 \to V_2$, we define the transpose relation $R^\top \colon V_2 \dashrightarrow V_1$ by setting $(v_1, v_2) \in R^\top \Leftrightarrow (v_2, v_1) \in R$. Note that R is the graph of a linear map $A\colon V_1 \to V_2$ if and only if $\dim R = \dim V_1$ and $\ker(R^\top) = 0$. We also define a relation

$$\mathrm{ann}^\natural(R)\colon V_1^* \to V_2^*$$

by declaring that $(\mu_2, \mu_1) \in \mathrm{ann}^\natural(R)$ if and only if $\langle \mu_2, v_2 \rangle = l\mu_1, v_1\rangle$ for all $(v_2, v_1) \in R$; equivalently, it is obtained from the annihilator of R by a sign change in one of the factors of $V_2^* \times V_1^*$. Note that

$$\mathrm{ann}^\natural(\Delta_V) = \Delta_{V^*}.$$

Also, if $A\colon V_1 \to V_2$ is a linear map, and $A^*\colon V_2^* \to V_1^*$ the dual map, then

$$\mathrm{ann}^\natural(\mathrm{Gr}(A)) = \mathrm{Gr}(A^*)^\top.$$

The composition of relations $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$ is the relation

$$R' \circ R\colon V_1 \dashrightarrow V_3,$$

where $(v_3, v_1) \in R' \circ R$ if and only if there exists $v_2 \in V_2$ such that $(v_3, v_2) \in R'$ and $(v_2, v_1) \in R$. This has the property

$$\mathrm{ann}^\natural(R' \circ R) = \mathrm{ann}^\natural(R') \circ \mathrm{ann}^\natural(R)$$

(see [**?**, Lemma A.2]). Note that in general, given smooth families of subspaces R_t, R'_t, the composition $R_t \circ R'_t$ need not have constant dimension, and even if it does it need not depend smoothly on t (as elements of the Grassmannian). For this reason, one often imposes transversality assumptions on the composition.

Definition 4.2. We say that R', R have *transverse composition* if

(a) $\ker(R') \cap \ker(R^\top) = 0$,
(b) $\mathrm{ran}(R) + \mathrm{ran}((R')^\top) = V_2$.

Notice that the first condition in (4.2) means that for $(v_3, v_1) \in R' \circ R$, the element $v_2 \in V_2$ such that $(v_3, v_2) \in R'$, $(v_2, v_1) \in R$ is *unique*. The second condition is equivalent to the condition that the sum $(R' \times R) + (V_3 \times \Delta_{V_2} \times V_1)$ equals $V_3 \times V_2 \times V_2 \times V_1$. The first condition is automatic if $\ker(R') = 0$ or $\ker(R^\top) = 0$, while the second condition is automatic if $\mathrm{ran}(R) = V_2$ or $\mathrm{ran}((R')^\top) = V_2$.

See e.g. [**?**, Appendix A] for further details, as well as the proof of the following dimension formula:

Proposition 4.3. *If $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$ have transverse composition, then*

$$\dim(R' \circ R) = \dim(R') + \dim(R) - \dim V_2.$$

Conversely, if this dimension formula holds, then the composition is transverse provided that at least one of the conditions in Definition 4.2 holds.

Lemma 4.4. *Let $R\colon V_1 \dashrightarrow V_2$ and $R'\colon V_2 \dashrightarrow V_3$ be* surjective *relations, whose transpose relations are* injective. *Then R', R have transverse composition, $R' \circ R$ is surjective, and $(R' \circ R)^\top$ is injective.*

Proof. Transversality of the composition is immediate from the definition 4.2: The first condition follows from injectivity of $R^\top$, the second condition from surjectivity of R. On the other hand, the composition of surjective relations is surjective, while the composition of injective relations is injective. ∎

More generally, we can consider smooth relations between manifolds. A *smooth relation*

$$\Gamma\colon M_1 \dashrightarrow M_2$$

from a manifold M_1 to a manifold M_2, is an (immersed) submanifold $\Gamma \subseteq M_2 \times M_1$. Any smooth $\Phi\colon M_1 \to M_2$ defines such a relation $\mathrm{Gr}(\Phi) \subseteq M_2 \times M_1$, and we have $\mathrm{Gr}(\Phi \circ \Psi) = \mathrm{Gr}(\Phi) \circ \mathrm{Gr}(\Psi)$ (composition of relations). Given another such relation $\Gamma'\colon M_2 \dashrightarrow M_3$, the set-theoretic composition of relations

$$\Gamma' \circ \Gamma = \{(m_3, m_1) |\ \exists m_2 \in M_2\colon (m_3, m_2) \in \Gamma', \quad (m_2, m_1) \in \Gamma\}$$

is a smooth relation if the composition is *transverse*:

Definition 4.5. The composition of smooth relations $\Gamma\colon M_1 \dashrightarrow M_2$ and $\Gamma'\colon M_2 \dashrightarrow M_3$ is *transverse* if for all points of $\Gamma' \diamond \Gamma := (\Gamma' \times \Gamma) \cap (M_3 \times \Delta_{M_2} \times M_1)$ the composition of tangent spaces is transverse.

This assumption implies that $\Gamma' \circ \Gamma$ is a submanifold of dimension $\dim \Gamma + \dim \Gamma' - \dim M_2$, and the map to $\Gamma \circ \Gamma$ is a (local) diffeomorphism. It also follows that

$$T(\Gamma' \circ \Gamma) = T\Gamma \circ T\Gamma'.$$

The manifold counterpart to Lemma 4.4 reads as:

Lemma 4.6. *Let $\Gamma\colon M_1 \dashrightarrow M_2$ and $\Gamma'\colon M_2 \dashrightarrow M_3$ be smooth relations, with the property that the projections from Γ, Γ' to their targets is a surjective submersion, while their projection to the source is an injective immersion. Then Γ', Γ have a transverse composition, and the projections from $\Gamma' \circ \Gamma$ to the target and source are a surjective submersion and injective immersion, respectively.*

Finally, we can also consider relations in the category of vector bundles. A $\mathcal{VB}$-relation $\Gamma\colon E_1 \dashrightarrow E_2$ between vector bundles is a vector subbundle of $\Gamma \subseteq E_2 \times E_1$. By Grabowski-Rotkievicz, this is the same as a smooth relation that is invariant under scalar multiplication. The definition of $\mathrm{ann}^\natural(\Gamma)$ generalizes, and the property under compositions extends:

$$\mathrm{ann}^\natural(\Gamma' \circ \Gamma) = \mathrm{ann}^\natural(\Gamma') \circ \mathrm{ann}^\natural(\Gamma).$$

4.3. **Groupoid structures as relations.** The axioms of a Lie groupoid can be phrased in terms of smooth relations, as follows. Let $\Gamma = \mathrm{Gr}(\mathrm{Mult}_{\mathcal{G}}) \subseteq \mathcal{G} \times \mathcal{G} \dashrightarrow \mathcal{G}$ be the graph of the multiplication map. The projection of Γ onto $\mathcal{G}$ given as $(g; g_1, g_2) \mapsto g$ is a surjective submersion, while the map $\Gamma \to \mathcal{G} \times \mathcal{G}, (g; g_1, g_2) \mapsto (g_1, g_2)$ is an embedding (its image is the submanifold $\mathcal{G}^{(2)}$). By Lemma 4.6, it is automatic that the composition of Γ with $\Gamma \times \Delta_{\mathcal{G}}$ and also with $\Delta_{\mathcal{G}} \times \Gamma$ are smooth, and the associativity of the groupoid multiplication is equivalent to the equality

$$\Gamma \circ (\Gamma \times \Delta_{\mathcal{G}}) = \Gamma \circ (\Delta_{\mathcal{G}} \times \Gamma) \tag{6}$$

Similarly, regarding the submanifold of units as a relation $M\colon \mathrm{pt} \to \mathcal{G}$, the condition for units reads as

$$\Gamma \circ (M \times \Delta_{\mathcal{G}}) = \Delta_{\mathcal{G}} = \Gamma \circ (\Delta_{\mathcal{G}} \times M). \tag{7}$$

`This last composition is not transverse, though.` In the next section, we will give a first application of this viewpoint.

4.4. **Tangent groupoid, cotangent groupoid.** For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the tangent bundle becomes a Lie groupoid

$$T\mathcal{G} \rightrightarrows TM,$$

by applying the tangent functor to all the structure maps. For example, the source map is $\mathsf{s}_{T\mathcal{G}} = T\mathsf{s}_{\mathcal{G}}$, and similarly for the target map; and the multiplication map is $\mathrm{Mult}_{T\mathcal{G}} = T\,\mathrm{Mult}_{\mathcal{G}}$ as a map from $(T\mathcal{G})^{(2)} = T(\mathcal{G}^{(2)})$ to $T\mathcal{G}$. The associativity and unit axioms are obtained from those of $\mathcal{G}$, by applying the tangent functor.

In fact, $T\mathcal{G}$ is a so-called $\mathcal{VB}$-groupoid: It is a vector bundle, and all structure maps are vector bundle morphisms.

Definition 4.7. A $\mathcal{VB}$-groupoid is a groupoid $\mathcal{V} \rightrightarrows E$ such that $\mathcal{V} \to \mathcal{G}$ is a vector bundle, and $\mathrm{Gr}(\mathrm{Mult}_{\mathcal{V}})$ is a vector subbundle of $\mathcal{V}^3$.

Using this result, it follows that the units of a $\mathcal{VB}$-groupoid are a vector bundle $E \to M$, and that all groupoid structure maps are vector bundle morphisms. Furthermore, the zero sections of $\mathcal{V}$ defines a subgroupoid $\mathcal{G} \rightrightarrows M$ of $\mathcal{V} \rightrightarrows E$.

Suppose that $\mathcal{W} \rightrightarrows F$ is a $\mathcal{VB}$-subgroupoid of $\mathcal{V} \rightrightarrows E$, with base $\mathcal{H} \rightrightarrows N$ a subgroupoid of $\mathcal{G} \rightrightarrows M$. Then we can form the *quotient $\mathcal{VB}$-groupoid*,

$$\mathcal{V}|_{\mathcal{H}}/\mathcal{W} \rightrightarrows E|_N/F$$

For example, if $\mathcal{H} \subseteq \mathcal{G}$ is a Lie subgroupoid with units $N \subseteq M$, then the normal bundle $\nu(\mathcal{G}, \mathcal{H}) = T\mathcal{G}|_{\mathcal{H}}/T\mathcal{H}$ becomes a Lie groupoid over $\nu(M, N)$,

$$\nu(\mathcal{G}, \mathcal{H}) \rightrightarrows \nu(M, N).$$

The dual of a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ is also a $\mathcal{VB}$-groupoid:

Theorem 4.8. *For any $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$, the dual bundle $\mathcal{V}^*$ has a unique structure of a $\mathcal{VB}$-groupoid such that $\mu = \mu_1 \circ \mu_2$ if and only if*

$$\langle \mu, v\rangle = \langle \mu_1, v_1\rangle + \langle \mu_2, v_2\rangle$$

whenever $v = v_1 \circ v_2$ in $\mathcal{V}$. (Here it is understood that $v, v_1, v_2 \in \mathcal{V}$ have the same base points as μ, μ_1, μ_2, respectively. In particular, these base points must satisfy $g = g_1 \circ g_2$.) The units for this groupoid structure is the annihilator bundle $\mathrm{ann}(E)$.

We call

$$\mathcal{V}^* \rightrightarrows \mathrm{ann}(E)$$

the *dual $\mathcal{VB}$-groupoid* to $\mathcal{V} \rightrightarrows E$.

Proof. Let $\Gamma_{\mathcal{V}} = \mathrm{Gr}(\mathrm{Mult}_{\mathcal{V}})$ be the graph of the groupoid multiplication of $\mathcal{V}$. Then the graph of the proposed groupoid multiplication of $\mathcal{V}^*$ is

$$\Gamma_{\mathcal{V}^*} = \mathrm{ann}^\natural(\mathrm{Gr}(\mathrm{Mult}_{\mathcal{V}})).$$

By applying $\mathrm{ann}^\natural$ the associativity and unit axioms of $\mathcal{V}$, given by (6) and (7) (with $\mathcal{G}$ replaced by $\mathcal{V}$), one obtains the corresponding axioms of $\mathcal{V}^*$. In particular, we see that the elements of $\mathrm{ann}(E)$ act as units. The inversion map for $\mathcal{V}^*$ is just the dual of that of $\mathcal{V}$; their graphs are related by $\mathrm{ann}^\natural$. $\square$

Remark 4.9. (Some details.) As usual, the units, as well as the source and target maps, are uniquely determined by the groupoid multiplication: Suppose $\mu \in \mathcal{V}^*$ is a unit. Then its base point must be a unit in $\mathcal{G}$. Let $v \in E$ (with the same base point), so that $v = v \circ v$. The multiplication rule tells us that $\langle \mu, v\rangle = \langle \mu, v\rangle + \langle \mu, v\rangle$, hence $\langle \mu, v\rangle = 0$. This shows that $\mu \in \operatorname{ann}(E)$. Conversely, if $v, v_1, v_2 \in \mathcal{V}|_M$ with $v = v_1 \circ v_2$ (in particular, all base points coincide) then $v = v_1 + v_2$ modulo E. (Exercise below.) Hence, for $\mu \in \operatorname{ann}(E)$ we obtain $\mu = \mu \circ \mu$, by definition of the multiplication.

Remark 4.10. We might call a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ a *$\mathcal{VB}$-group* if E is the zero vector bundle over pt. For example, the tangent bundle of a Lie group is a $\mathcal{VB}$-group. The dual bundle to a $\mathcal{VB}$-group need not be a group, in general, since $\operatorname{ann}(E) \cong (\mathcal{V}|_e)^*$ with e the group unit of the base $\mathcal{G} \rightrightarrows$ pt is non-trivial unless $\mathcal{V}$ is the zero bundle over $\mathcal{G}$. For the case that $\mathcal{G} = G$ is a Lie group, and $\mathcal{V} = TG$, we find that $\operatorname{ann}(E) = \mathfrak{g}^*$.

Exercise 4.1. Show that if $\mathcal{V}$ is a $\mathcal{VB}$-groupoid, and $v = v_1 \circ v_2$ where the base points are in M, then these base points are all the same, and $v = v_1 + v_2 - \mathsf{s}(v_1)$.

Exercise 4.2. Using the preceding exercise, give explicit formulas for the source and target map of $\mathcal{V}^*$. (Start with $\mathcal{V}^*|_M$.)

4.5. **Prolongations of groupoids.** Given a groupoid $\mathcal{G} \rightrightarrows M$, one can define new groupoids

$$J_k(\mathcal{G}) \rightrightarrows M,$$

the so-called *k-th prolongation* of $\mathcal{G}$. The points of $J_k(\mathcal{G})$ are k-jets of bisections of $\mathcal{G}$. Thinking of a bisection as a section $\sigma\colon M \to \mathcal{G}$ whose composition with the target map is a diffeomorphism, the source fiber of $J_k(\mathcal{G})$ consists of all k-jets of such sections at m, with the property that the composition with t is the k-jet of a diffeomorphism.

- For $k = 0$, one recovers $J_0(\mathcal{G}) = \mathcal{G}$ itself.
- Elements of $J_1(\mathcal{G})$ are pairs (g, W), where $g \in \mathcal{G}$ is an arrow and $W \subseteq T_g\mathcal{G}$ is a subspace complementary to both the source and target fibers. In other words, W is the tangent space to some bisection passing through g. The composition is induced from the composition of bisections, that is,
$$(g, W) = (g_1, W_1) \circ (g_2, W_2)$$
if and only if
$$W = T\,\mathrm{Mult}_{\mathcal{G}}((W_1 \times W_2) \cap T\mathcal{G}^{(2)})$$
(the linearized version for multiplication of bisections.)

The successive prolongations define a sequence of Lie groupoids

$$\cdots \to J_k(\mathcal{G}) \to J_{k-1}(\mathcal{G}) \to \cdots \to J_0(\mathcal{G}) = \mathcal{G}.$$

By applying this construction to the pair groupoid, one recovers the groupoids $J_k(M, M)$ discussed earlier. Prolongations of groupoids were introduced by Ehresmann [3] recently they have been used in the work of Crainic, Salazar, and Struchiner on *Pfaffian groupoids.*

[3] C. Ehresmann, Prolongements des categories differentiables, Topologie et Geom. Differentielle, 6 (1964), 18

4.6. **Pull-backs and restrictions of groupoids.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Given a submanifold $i\colon N \hookrightarrow M$ such that the map $(\mathsf{t},\mathsf{s})\colon \mathcal{G} \rightrightarrows \mathrm{Pair}(M)$ is transverse to $\mathrm{Pair}(N) \subseteq \mathrm{Pair}(M)$, one obtains a new groupoid $i^!\mathcal{G} \rightrightarrows N$ by taking the pre-image

$$i^!\mathcal{G} = (\mathsf{t},\mathsf{s})^{-1}(\mathrm{Pair}(N)).$$

The groupoid multiplication is simply the restriction of that of $\mathcal{G}$. More generally, suppose $f\colon N \to M$ is a smooth map such that the induced map $\mathrm{Pair}(f)\colon \mathrm{Pair}(N) \to \mathrm{Pair}(M)$ is transverse to (t,s). Then we define a pull-back groupoid $f^!\mathcal{G} \rightrightarrows N$ by

$$f^!\mathcal{G} = \{(g,n',n)|\ \mathsf{s}(g) = f(n),\ \mathsf{t}(g) = f(n')\}.$$

Its groupoid structure is that as a subgroupoid of $\mathcal{G} \times \mathrm{Pair}(N)$ over $N \cong \mathrm{Gr}(f) \subseteq M \times N$. Note

$$\dim f^!\mathcal{G} = \dim \mathcal{G} + 2\dim N - 2\dim M,$$

and also that

$$f^!\,\mathrm{Pair}(M) = \mathrm{Pair}(N).$$

By construction, the pull-back groupoid comes with a morphism of Lie groupoids

$$\begin{tikzcd} f^!\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & N \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Under composition of maps

$$(f_1 \circ f_2)^!\mathcal{G} = f_2^! f_1^! \mathcal{G}.$$

provided that the transversality hypotheses are satisfied.

Remark 4.11. In the definition of $f^!\mathcal{G}$, one can weaken the transversality assumption to clean intersection assumptions. However, the dimension formula for $f^!\mathcal{G}$ has to be modified in that case.

4.7. **A result on subgroupoids.**

Theorem 4.12. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{H} \rightrightarrows N$ a set-theoretic subgroupoid. If $\mathcal{H}$ is a submanifold of $\mathcal{G}$, and the source fibers of $\mathcal{H}$ are connected, then $\mathcal{H}$ is a Lie subgroupoid.* [4]

Our proof use the following Lemma from differential geometry (see e.g. [**?**])

Lemma 4.13 (Smooth retractions)**.** *Let Q be a manifold, and $p\colon Q \to Q$ a smooth map such that $p \circ p = p$. Then $p(Q)$ is a submanifold, and admits an open neighborhood in Q on which the map p is s surjective submersion onto $p(Q)$.*

Remark 4.14.
(a) If Q is connected, then $p(Q)$ is connected. If Q is disconnected, then $p(Q)$ can have several connected components of different dimensions.
(b) In general, the smooth retraction p need not be a submersion *globally*, even when Q is compact and connected.

Proof of Theorem 4.12. We denote by $\mathsf{s},\mathsf{t}\colon \mathcal{G} \to M$ the source and target map of $\mathcal{G}$, by $i\colon M \to \mathcal{G}$ the inclusion of units, and by $\mathrm{Mult}\colon \mathcal{G}^{(2)} \to \mathcal{G}$ the groupoid multiplication. For the corresponding notions of $\mathcal{H}$, we will put a subscript $\mathcal{H}$. We have to show:

[4] By submanifold, we always mean *embedded submanifold*.

(a) N is a submanifold of $\mathcal{H}$,
(b) $\mathsf{s}_\mathcal{H}, \mathsf{t}_\mathcal{H} \colon \mathcal{H} \to N$ are submersions,
(c) $\mathrm{Mult}_\mathcal{H} \colon \mathcal{H}^{(2)} \to \mathcal{H}$ is smooth.

The map $i_\mathcal{H} \circ \mathsf{s}_\mathcal{H} \colon \mathcal{H} \to \mathcal{H}$ is a retraction to the subset $N \subseteq \mathcal{H}$. It is smooth, since it is the restriction of the smooth map $i \circ \mathsf{s} \colon \mathcal{G} \to \mathcal{G}$. Hence, by the Lemma, N is a submanifold. For the rest of this argument, we can and will assume that the $\mathcal{H}$-orbit space of N is connected; hence N (which may be disconnected) has constant dimension.

The Lemma also tells us that there exists an open neighborhood of $i_\mathcal{H}(N)$ in $\mathcal{H}$ on which $\mathsf{s}_\mathcal{H}$ is a submersions onto N. By using a similar argument for $\mathsf{t}_\mathcal{H}$, we see that the same is true for $\mathsf{t}_\mathcal{H}$. In particular, some neighborhood Ω of N in $\mathcal{H}$ becomes a 'local Lie algebroid'. Define a left-action of Ω on $\mathcal{H}$, by the map

$$\Omega \ {}_{\mathsf{s}_\mathcal{H}}\!\times_{\mathsf{t}_\mathcal{H}} \mathcal{H}, \ (k, g) \mapsto k \circ g.$$

Note that $\Omega \ {}_{\mathsf{s}_\mathcal{H}}\!\times_{\mathsf{t}_\mathcal{H}} \mathcal{H}$ is a smooth submanifold of $\mathcal{G}^{(2)}$, due to the fact that $\mathsf{s}_\mathcal{H}$ is a submersion over Ω, and that the action map is smooth since it is the restriction of $\mathrm{Mult}_\mathcal{G}$ to this submanifold.

If $\mathsf{s}_\mathcal{H}$ is a surjective submersion at some point $g \in \mathcal{H}$, then it is also a submersion at $k \circ g$, for any $k \in \Omega$. Since $\mathcal{H}$ is assumed to be source connected, any $g \in \mathcal{H}$ can be written as a product $k_1 \circ \cdots \circ k_N$ with $k_j \in \Omega$. This shows that $\mathsf{s}_\mathcal{H}$ is a submersion, and similarly $\mathsf{t}_\mathcal{H}$ is a submersion. □

If we drop the assumption that $\mathcal{H}$ is source-connected, it need no longer be true that $\mathsf{s}_\mathcal{H}$ is a submersion everywhere:

Example 4.15. Take $\mathcal{G} = \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$ be the 1-dimensional trivial vector bundle over $\mathbb{R}$, regarded as a groupoid with $\mathsf{s}(x, y) = \mathsf{t}(x, y) = x$. Pick a function $y = f(x)$, taking values in positive real numbers, such that the graph of f is a smooth submanifold, but has a vertical tangent at some point (x_0, y_0). Then $\mathcal{H} = \{(x, kf(x)) | \ x \in \mathbb{R}, \ k \in \mathbb{Z}\}$ is a set-theoretic subgroupoid which is not a Lie subgroupoid, since $\mathsf{s}_\mathcal{H}$ is not surjective at (x_0, y_0).

Remark 4.16. If the submanifold $\mathcal{H}$ is a set-theoretic subgroupoid of the Lie groupoid $\mathcal{G}$, with possibly disconnected source fibers, the its 'source component' is still a Lie subgroupoid.

4.8. **Clean intersection of submanifolds and maps.** Some of the subsequent results will depend on intersection properties of maps that are weaker than transversality. The following notion of clean intersection goes back to Bott [**?**, Section 5].

Definition 4.17 (Clean intersections)**.** (a) Two submanifold S_1, S_2 of a manifold M *intersect cleanly* if $S_1 \cap S_2$ is a submanifold, with

$$T(S_1 \cap S_2) = TS_1 \cap TS_2.$$

(b) A smooth map $F \colon N \to M$ between manifolds has *clean intersection with a submanifold* $S \subseteq M$ if $F^{-1}(S)$ is a submanifold of N, with

$$T_n\left(F^{-1}(S)\right) = (T_nF)^{-1}(T_{F(n)}S), \quad n \in N.$$

(c) Two smooth maps $F_1, F_2 \colon N \to M$ *intersect cleanly* if $F_1 \times F_2$ is clean with respect to the diagonal $\Delta_M \subseteq M \times M$.

Remarks 4.18. (a) One can show (see e.g. [**?**]) that at any point of a clean intersection of submanifolds S_1, S_2, there exist local coordinates in which the submanifolds are vector subspaces. One consequence of this is that for any two functions $f_i \in C^\infty(S_i)$, with

$$f_1|_{S_1\cap S_2} = f_2|_{S_1\cap S_2},$$

there exists a smooth function $f \in C^\infty(M)$ with

$$f|_{S_1} = f_1,\ f|_{S_2} = f_2.$$

More generally, given a vector bundle $V \to M$, and two sections of $\sigma_i \in \Gamma(V|_{S_i})$ with $\sigma_1|_{S_1\cap S_2} = \sigma_2|_{S_1\cap S_2}$, there exists $\sigma \in \Gamma(V)$ with $\sigma|_{S_i} = \sigma_i$.

(b) Note that $F\colon N \to M$ is clean with respect to $S \subseteq M$ if and only if its graph $\mathrm{Gr}(F) \subseteq M \times N$ has clean intersection with $S \times N$.

(c) As a special case, if S_1, S_2 have transverse intersection, in the sense that

$$T_mS_1 + T_mS_2 = T_mM$$

for all $m \in S_1 \cap S_2$, then the intersection is clean: it is automatic in this case that the intersection is a submanifold. Similarly, transversality of a map $F\colon N \to M$ to a submanifold $S \subseteq M$, in the sense that

$$\mathrm{ran}(T_nF) + T_{F(n)}S = T_{F(n)}M$$

for all $n \in N$) implies cleanness.

(d) For a clean intersection of submanifolds S_1, S_2, one calls the quantity

$$e = \dim(S_1 \cap S_2) + \dim(M) - \dim(S_1) - \dim(S_2)$$

the *excess* of the clean intersection. Thus, $e = 0$ if and only if the intersection is transverse. Similarly, one defines the excess of a clean intersection of two maps, or of a map with a submanifold.

Given a vector bundle $V \to M$, with a subbundle $W \to N$, we denote by $\Gamma(V, W) \subseteq \Gamma(V)$ the sections of V whose restriction to N takes values in W. As a special case, if 0_N is the zero bundle over N, then $\Gamma(V, 0_N)$ are the sections of V vanishing along N. We have the exact sequence,

$$0 \to \Gamma(V, 0_N) \to \Gamma(V, W) \to \Gamma(W) \to 0.$$

For example, if N is a submanifold of M, then $\Gamma(TM, TN)$ are the vector fields on M that ar tangent to N. Later we will need the following fact:

Lemma 4.19. *Suppose $W_i \to N_i$ are two vector subbundles of a vector bundle $V \to M$. If W_1, W_2 intersect cleanly (as manifolds), then the zero sections intersect cleanly, and $W_1 \cap W_2 \to N_1 \cap N_2$ is a vector subbundle of V. Furthermore, the map*

$$\Gamma(V, W_1) \cap \Gamma(V, W_2) \to \Gamma(W_1 \cap W_2)$$

is surjective.

Proof. The first part follows by using the Grabowski-Rotkievicz theorem, since the intersection is a submanifold by assumption, and since it is invariant under scalar multiplication. In particular, its zero sections $N_1 \cap N_2$ is a submanifold, where the intersection is clean:

$$T(N_1 \cap N_2) = T(W_1 \cap W_2) \cap TM = TW_1 \cap TW_2 \cap TM = TN_1 \cap TN_2.$$

For the second part, given a section $\sigma_{12} \in \Gamma(W_1 \cap W_2)$, extend to sections $\sigma_1 \in \Gamma(W_1)$ and $\sigma_2 \in \Gamma(W_2)$, and use Remark 4.18a to extend to a section σ of V. Then $\sigma \in \Gamma(V, W_1) \cap \Gamma(V, W_2)$, and $\sigma|_{N_1 \cap N_2} = \sigma_{12}$. □

4.9. **Intersections of Lie subgroupoids, fiber products.**

Theorem 4.20 (Clean intersection of Lie subgroupoids)**.** *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and $\mathcal{H}_i \rightrightarrows N_i$, $i = 1, 2$ two Lie subgroupoids with clean intersection. Then $\mathcal{H}_1 \cap \mathcal{H}_2$ is a Lie subgroupoid.*

Proof. The cleanness assumption means that $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2$ is a submanifold, with

$$T\mathcal{H} = T\mathcal{H}_1 \cap T\mathcal{H}_2.$$

Let s be the source map of $\mathcal{G}$, and s' its restriction to $\mathcal{H}$. By Theorem 4.12, the source components of $\mathcal{H}$ are Lie subgroupoids; in particular the space of units $N = N_1 \cap N_2$ is a submanifold. Furthermore, s' is a surjective submersion from some open neighborhood of N inside $\mathcal{H}$ onto N. To see that it is a surjective submersion everywhere on $\mathcal{H}$, we use right translation. Given $g \in \mathcal{G}$, with $\mathsf{s}(g) = m$, $\mathsf{t}(g) = m'$, we have that

$$\mathcal{A}^R_{g^{-1}} \colon \mathsf{s}^{-1}(m) \to \mathsf{s}^{-1}(m').$$

In particular, the tangent map preserves the source fibers, and hence

$$T_g \mathcal{A}^R_{g^{-1}} \colon \ker(T_g \mathsf{s}) \to \ker(T_{m'} \mathsf{s}). \tag{8}$$

Given $g \in \mathcal{H}$, the map $T_g \mathcal{A}^R_{g^{-1}}$ restricts to isomorphisms

$$T_g \mathcal{A}^R_{g^{-1}} \colon T_g \mathcal{H}_i \cap \ker(T_g \mathsf{s}) \to T_{\mathsf{s}(g)} \mathcal{H}_i \cap \ker(T_{m'} \mathsf{s})$$

for $i = 1, 2$, since both $]\mathcal{H}_1, \mathcal{H}_2$ are Lie subgroupoids. Using the clean intersection condition, we see that it restricts to an isomorphism

$$T_g \mathcal{A}^R_{g^{-1}} \colon T_g(\mathcal{H}_1 \cap \mathcal{H}_2) \cap \ker(T_g \mathsf{s}) \to T_{\mathsf{s}(g)}(\mathcal{H}_1 \cap \mathcal{H}_2) \cap \ker(T_{m'} \mathsf{s}),$$

that is,

$$T_g \mathcal{A}^R_{g^{-1}} \colon \ker(T_g \mathsf{s}') \to \ker(T_{m'} \mathsf{s}').$$

This shows that s' has constant rank globally, and likewise for t'. □

Corollary 4.21. *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Lie groupoids, and $F \colon \mathcal{G} \to \mathcal{H}$ a morphism of Lie groupoids. Suppose that $\mathcal{H}' \rightrightarrows N'$ is a Lie subgroupoid of $\mathcal{H}$, and let $\mathcal{G}' \rightrightarrows M'$ be its pre-image. If F is clean with respect to $\mathcal{H}'$, then $\mathcal{G}'$ is a Lie subgroupoid. In particular, this is true if F is transverse to $\mathcal{H}'$.*

Proof. We may regard the pre-image as the intersection of two Lie subgroupoids, $\mathrm{Gr}(F) \cap (\mathcal{H}' \times \mathcal{G}) \subseteq \mathcal{H} \times \mathcal{G}$. □

Corollary 4.22. *Suppose $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are Lie groupoids, and that $F \colon \mathcal{G} \to \mathcal{H}$ is a morphism of Lie groupoids. If F is clean with respect to N, then the kernel*

$$\ker(F) = F^{-1}(N)$$

is a Lie subgroupoid. In particular, this is true if F is transverse to N.

Corollary 4.23. *Let $\mathcal{G}_1 \rightrightarrows M_1, \mathcal{G}_2 \rightrightarrows M_2,\ \mathcal{H} \rightrightarrows N$ be Lie groupoids, and $F_i \colon \mathcal{G}_i \to \mathcal{H},\ i = 1, 2$ morphisms of Lie groupoids. Let*

$$\mathcal{G} = \mathcal{G}_1 \times_{\mathcal{H}} \mathcal{G}_2 \subseteq \mathcal{G}_1 \times \mathcal{G}_2.$$

be the fiber product. If $\mathcal{G}$ is a submanifold, and if, for all $g = (g_1, g_2) \in \mathcal{G}$, the tangent space of $\mathcal{G}$ is the fiber product of $T_{g_1}\mathcal{G}_1$ and $T_{g_2}|G_2$ under the tangent maps of the F_i, then $\mathcal{G}$ is a Lie groupoid. In particular, this is true if F_1, F_2 are transverse.

Proof. We may interpret the fiber product as the pre-image of the diagonal $\Delta_{\mathcal{H}}$ under the map $F_1 \times F_2$. The given assumption just means that $F_1 \times F_2$ is clean with respect to $\Delta_{\mathcal{H}}$. □

Remark 4.24. This result, in the strong version stated here, is due to Bursztyn-Cabrera-del Hoyo [**?**]. Note that conversely, the result for fiber products implies Theorem **??**. Indeed, our proof of the Theorem **??** was motivated by the argument in [**?**].

Example 4.25. The pull-back construction may be re-phrased in these terms: Given $\mathcal{G} \rightrightarrows M$ and $f \colon N \to M$ such that $\mathrm{Pair}(f)$ is transverse to (t, s), we have that

$$f^!\mathcal{G} = \mathrm{Pair}(N) \times_{\mathrm{Pair}(M)} \mathcal{G}.$$

More generally, this holds if the two maps are clean with respect to each other.

4.10. **The universal covering groupoid.** Let $\mathcal{G} \rightrightarrows M$ be a source connected Lie groupoid. The source map $\mathsf{s} \colon \mathcal{G} \to M$ is a surjective submersion. Let $\widetilde{\mathcal{G}}$ be obtained by replacing each source fiber by its universal covering. That is,

$$\widetilde{\mathcal{G}} = \{[\gamma] |\ \gamma \colon [0,1] \to \mathcal{G} \text{ is an } \mathsf{s}\text{-foliation path with } \gamma(0) \in M\},$$

where $[\gamma]$ stands for homotopy classes of s-foliation paths, with fixed end points. (Put differently, letting $\mathcal{F}$ be the s-foliation of $\mathcal{G}$, we take the pre-image of $M \subseteq \mathcal{G}$ under the source map of $\mathrm{Mon}(\mathcal{F}) \rightrightarrows \mathcal{G}$.) The space $\widetilde{\mathcal{G}}$ has a natural structure of a (possibly non-Hausdorff) manifold. The manifold structure is obtained from the inclusion as a submanifold of $\mathrm{Mon}(\mathcal{F})$. We emphasize that $\widetilde{\mathcal{G}}$ may be non-Hausdorff even if $\mathcal{G}$ is Hausdorff. Define source and target maps of $\widetilde{\mathcal{G}}$ by

$$\mathsf{s}([\gamma]) = \gamma(0), \quad \mathsf{t}([\gamma]) = \mathsf{t}(\gamma(1)),$$

and define the groupoid multiplication of composable elements as

$$[\gamma'] \circ [\gamma] = [\gamma'']$$

where γ'' is a concatenation of the path γ with the $\mathcal{A}^R_{\gamma(1)}$-translate of the path γ': Thus

$$\gamma''(t) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2, \\ \gamma'(2t-1) \circ \gamma(1)^{-1} & 1/2 \le t \le 1 \end{cases}$$

Using these definitions, the space $\widetilde{\mathcal{G}}$ is a (possibly non-Hausdorff) Lie groupoid

$$\widetilde{\mathcal{G}} \rightrightarrows M.$$

It comes with a local diffeomorphism

$$\pi \colon \widetilde{\mathcal{G}} \to \mathcal{G},\ [\gamma] \mapsto \gamma(1),$$

which is a morphism of Lie groupoids.

5. Groupoid actions, groupoid representations

5.1. **Actions of Lie groupoids.** Generalizing Lie group actions, one can also consider *groupoid actions* of $\mathcal{G} \rightrightarrows M$ on other manifolds.

Definition 5.1. An action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a manifold Q is given by a map $\Phi\colon Q \to M$ together with an *action map*

$$\mathcal{A}\colon \mathcal{G}\times_M Q \to Q,\ (g,q) \mapsto \mathcal{A}_g(q) = g \cdot q$$

where $\mathcal{G}\times_M Q := \{(g,q)|\ \mathsf{s}(g) = \Phi(q)\}$. These are required to satisfy $\Phi(g \cdot q) = \mathsf{t}(g)$, as well as

$$(g_1 \circ g_2) \cdot q = g_1 \cdot (g_2 \cdot q), \quad m \cdot q = q$$

for $g_i \in \mathcal{G},\ q \in Q,\ m \in M$, and whenever these are defined.

The map Φ is sometimes called a *moment map* of the action (due to some relationship with the moment map in symplectic geometry), sometimes it is called an *anchor*. We will say that $\mathcal{G} \rightrightarrows M$ *acts on Q along M.*

For any $\mathcal{G}$-action, one can define its *orbits in* Q as the equivalence classes under the relation

$$q \sim q' \Leftrightarrow \exists g \in \mathcal{G}\colon q' = g \cdot q.$$

Also, for $q \in Q$ we can define its *isotropy group*

$$\mathcal{G}_q = \{g \in \mathcal{G}|\ g \cdot q = q\}.$$

Example 5.2. For a group G, one always has the trivial action on point. The generalization to groupoids is its action on the units $M = \mathcal{G}^{(0)}$; here $\Phi = \mathrm{id}_M$, and the action is $g \cdot m = m'$ for $\mathsf{s}(g) = m,\ tz(g) = m'$. Note that in general, there is no action of a groupoid on a point.

Example 5.3. Every Lie groupoid $\mathcal{G} \rightrightarrows M$ acts on itself by left multiplication $g \cdot a = l_g(a) = g \circ a$ (here $\Phi = \mathsf{t}$), and by right multiplication $g \cdot a = r_{g^{-1}}(a) = a \circ g^{-1}$ (here $\Phi = \mathsf{s}$). These two actions commute, and combine into an action of $\mathcal{G} \times \mathcal{G} \rightrightarrows M \times M$ on $\mathcal{G}$ (with $(\Phi = (\mathsf{t}, \mathsf{s}))$. On the other hand, there is no natural *adjoint action* of $\mathcal{G}$ on itself (although we do have an 'adjoint action' of the group $\Gamma(\mathcal{G})$ of bisections on $\mathcal{G}$).

Remark 5.4. Given an action of $\mathcal{G} \rightrightarrows M$ on $\Phi\colon Q \to M$, and a morphism of Lie groupoids from $\mathcal{H} \rightrightarrows N$ to $\mathcal{G} \rightrightarrows M$, there is no natural way, in general, of producing an $\mathcal{H}$-action on Q (unless the map on units $N \to M$ is a diffeomorphism). For example, in the case of the $\mathcal{G} \times \mathcal{G}$-action on $\mathcal{G}$, there is no natural way of passing to a diagonal action, in general.

Remark 5.5. Note that any $\mathcal{G}$-action on Q determines an action of the group of bisections $\Gamma(\mathcal{G})$ on Q, where the bisection S takes $q \in Q$ to $g \times q$, for the unique $g \in S$ such that this composition is defined (i.e., $\mathsf{s}(g) = \Phi(q)$). More generally, there is an action of the local bisections,

$$\mathcal{A}_S\colon \Phi^{-1}(U) \to \Phi^{-1}(V)$$

where $U = \mathsf{s}(S),\ V = \mathsf{t}(S)$.

Remark 5.6. A groupoid action is fully determined by its graph $\mathrm{Gr}(\mathcal{A}_Q) \subseteq Q \times (\mathcal{G} \times Q)$; for example, $\Phi(q) \in M$ is the unique unit such that $q = \Phi(q) \cdot q$.

Given a groupoid action of $\mathcal{G} \rightrightarrows M$ on $Q \to M$, one can again form an action groupoid

$$\mathcal{G} \ltimes Q \rightrightarrows Q$$

as the subgroupoid of $\mathcal{G} \times \mathrm{Pair}(Q) \rightrightarrows M \times Q$ consisting of all (g, q', q) such that $q' = g \circ q$. The action groupoid comes with a groupoid morphism

$$\begin{tikzcd} \mathcal{G} \ltimes Q \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Q \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

where the left vertical map is $(g, q) \mapsto g$. The orbits and isotropy groups of the action groupoid are the orbits and isotropy groups for the $\mathcal{G}$-action on Q.

Remark 5.7. Suppose conversely that we are given a groupoid morphism ϕ from $\mathcal{H} \rightrightarrows Q$ to $\mathcal{G} \rightrightarrows M$ such that the map $(\phi, \mathsf{s})\colon \mathcal{H} \to \mathcal{G} \times_M Q$ is a diffeomorphism. Then $\mathcal{H}$ is the action groupoid for a $\mathcal{G}$-action on Q, in such a way that the action map is identified with the target map for $\mathcal{H}$. The stabilizers for the action are the isotropy groups of $\mathcal{H}$.

5.2. **Principal actions.** A special case of a groupoid action is a *principal action*. If G is a Lie group, a principal bundle is a G-manifold P for which there exists a surjective submersion $\kappa\colon P \to B$ onto another manifold B, such that the fibers of κ are exactly the G-orbits, and the action is free. These condition may be restated as the assertion that the map

$$G \times P \to P \times_B P,\ (g, p) \mapsto (g \cdot p, p)$$

is a diffeomorphism. (In other words, the action groupoid $G \ltimes P \rightrightarrows P$ is identified with the foliation groupoid). This definition has a direct analogue for groupoids:

Definition 5.8 (Moerdijk-Mcrun [**?**])**.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A principal $\mathcal{G}$-bundle is given by a manifold P with a surjective submersion $\kappa\colon P \to B$, together with a $\mathcal{G}$-action on P along a map $\Phi\colon P \to M$, such that

(a) $\kappa(g \cdot p) = \kappa(p)$ whenever $\mathsf{s}(g) = \Phi(p)$,
(b) the map

$$\mathcal{G} \ltimes P \to P \times_B P, \quad (g, p) \mapsto (g \cdot p, p)$$

is a diffeomorphism.

Morphisms of principal $\mathcal{G}$-bundles $\kappa_i P_i \to B_i$ are $\mathcal{G}$-equivariant maps $P_1 \to P_2$, intertwining the maps κ_i with respect to some base map $B_1 \to B_2$.

As a consequence of (b), the isotropy groups for a principal action are trivial: $\mathcal{G}_p = \{e\}$ for all $p \in P$.

Example 5.9. $\mathcal{G}$ itself is a principal $\mathcal{G}$-bundle for

$$\Phi(h) = \mathsf{s}(h),\ \kappa(h) = \mathsf{t}(h),\ g.h = hg^{-1}.$$

Given a principal $\mathcal{G}$-bundle $\kappa \to B$, with moment map $\Phi\colon P \to \mathcal{G}^{(0)}$, and any smooth map $f\colon B' \to B$, the fiber product of P with B' over B becomes a principal $\mathcal{G}$-bundle, called the *pull-back*:

$$\kappa'\colon f^*P = B' \times_B P \to B'.$$

Here $\kappa'(b', p) = b',\ \Phi'(b', p) = \Phi(p),\ g.(b', p) = (b', g \cdot p)$.

Example 5.10. As a special case, every smooth map $f\colon B \to \mathcal{G}^{(0)}$ defines a *trivial principal G-bundle* by pulling back $\mathcal{G}$ itself:

$$B \times_{\mathcal{G}^{(0)}} \mathcal{G},$$

with

$$\Phi(b,h) = \mathsf{t}(h), \quad \kappa(b,h) = b, \quad g \cdot (b,h) = (b, hg^{-1}).$$

Principal bundles of this type are regarded as *trivial*.

As in the case of principal bundles for Lie groups, if a principal $\mathcal{G}$-bundle admits a section $\sigma\colon B \to P$, then P is identified with the trivial bundle relative to the map $f = \mathsf{t} \circ \sigma$. Explicitly, this map is

$$B \times_{\mathcal{G}^{(0)}} \mathcal{G} \to P,\ (b,h) \mapsto h^{-1} \cdot \sigma(b).$$

For principal bundles of Lie groups, one has a notion of *associated bundle* $P \times_G S$ for any G-manifold S on which G acts. It is the quotient of $P \times S$ under the diagonal action. For general Lie groupoids $\mathcal{G} \rightrightarrows M$, there is no notion of diagonal action, in general. However, given a $\mathcal{G}$-manifold Q with a $\mathcal{G}$-equivariant submersion $Q \to P$, one has that Q is a principal $\mathcal{G}$-bundle, and its space of orbits may be regarded as an analogue of the associated bundle construction. See Moerdijk-Mcrun for more details.

5.3. **Representations of Lie groupoids.** A (linear) *representation* of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a vector bundle $\pi\colon V \to M$ is a $\mathcal{G}$-action on V along π such that the action map

$$\mathcal{G} \times_M V \to V,\ (g,v) \mapsto g \cdot v$$

is a vector bundle map. Equivalently, the action groupoid

$$\mathcal{G} \ltimes V \rightrightarrows V$$

is a $\mathcal{VB}$-groupoid. A representation of $\mathcal{G}$ on V gives a family of linear isomorphisms between fibers

$$\phi_g\colon V_{\mathsf{s}(g)} \to V_{\mathsf{t}(g)}$$

with the property that $\phi_{g_1 \circ g_2} = \phi_{g_1} \circ \phi_{g_2}$. Recall that for any vector bundle, we defined the Atiyah groupoid $\mathcal{G}(V) \rightrightarrows M$ to consist of triples (m', m, ϕ) with $\phi\colon V_m \to V_{m'}$. We may hence also regard a representation to be a Lie groupoid morphism

$$\mathcal{G} \to \mathcal{G}(V).$$

Examples:

- A linear representation of the pair groupoid $\mathrm{Pair}(M)$ on $\mathcal{V}$ is the same as a trivialization of V. Indeed, the action map gives consistent identifications of the fibers.
- A representation of the homotopy groupoid $\Pi(M)$ on $\mathcal{V}$ is the same as a flat connection on $\mathcal{V}$: Any element of $\Pi(M)$ gives a 'parallel transport'.
- Given a G-action on M, a representation of the action groupoid $G \ltimes M$ is the same as G-equivariant vector bundle $V \to M$ (lifting the given action on the base).
- Given a foliation $\mathcal{F}$ of M, let $\nu(M, \mathcal{F})$ be the normal bundle of the foliation. This normal bundle comes with a natural representation of the holonomy groupoid $\mathrm{Mon}(\mathcal{F}) \rightrightarrows M$: Given $g = (m', m, [\gamma])$, and given transversals N, N' at m, m', the tangent map to the holonomy gives an isomorphism $T_m N \to T_{m'} N'$; under the identification with normal bundles this is the desired representation.

6. Lie algebroids

The infinitesimal counterpart to Lie groupoids was introduced by Pradines in the 1960s.

6.1. **Definitions.**

Definition 6.1. A *Lie algebroid over M* is a vector bundle $A \to M$, together with a Lie bracket $[\cdot,\cdot]$ on its space of sections, such that there exists a vector bundle map

$$\mathsf{a}\colon A \to TM$$

called the *anchor map* satisfying the *Leibniz rule*

$$[\sigma, f\tau] = f[\sigma,\tau] + (\mathsf{a}(\sigma)f)\ \tau$$

for all $\sigma,\tau \in \Gamma(A)$ and $f \in C^\infty(M)$.

Remark 6.2.
(a) If an anchor map satisfying the Leibniz rule exists, it is unique. (Exercise.)
(b) In the original definition, it was also assumed that the anchor map a induces a Lie algebra morphism

$$\mathsf{a}\colon \Gamma(E) \to \mathfrak{X}(M).$$

Later, it was noticed that this is automatic. (Exercise.)

6.2. **Examples.**

Example 6.3. A Lie algebroid over a point $M = \mathrm{pt}$ is the same as a Lie algebra.

Example 6.4. The tangent bundle, with its usual bracket on sections, is a Lie algebroid with a the identity.

Example 6.5. Suppose E is a Lie algebra bundle, that is, a vector bundle whose fibers have Lie algebra structures, and with local trivializations respecting the Lie algebra structures. Then E with the pointwise Lie bracket and zero anchor is a Lie algebroid. Conversely, if E is a Lie algebroid with zero anchor, then the fibers inherit Lie brackets such that $[\sigma,\tau](m) = [\sigma(m),\tau(m)]$ for all $m \in M$. However, the Lie algebras for different fibers need not be isomorphic.

Example 6.6. Given a foliation $\mathcal{F}$ of a manifold M, one has the Lie algebroid $T_{\mathcal{F}}M$ given by the tangent bundle of the foliation.

Example 6.7. An action of a Lie algebra $\mathfrak{k}$ on M is, by definition, a vector bundle homomorphism $\mathfrak{k} \to \mathfrak{X}(M),\ X \mapsto X_M$ such that the action map $M \times \mathfrak{k} \to TM,\quad (m,X) \mapsto X_M(m)$ is smooth. It defines an *action Lie algebroid*

$$A = \mathfrak{k} \ltimes M,$$

where the anchor map is given by the action map, and the bracket is the unique extension of the given Lie bracket on constant section determined by the Leibniz rule. That is, if $X, Y\colon M \to \mathfrak{k}$ (viewed as sections of A)

$$[X,Y] = [X,Y]_{\mathfrak{k}} + \mathcal{L}_{\mathsf{a}(X)}Y - \mathcal{L}_{\mathsf{a}(Y)}X.$$

Here the subscript $\mathfrak{k}$ indicates the pointwise bracket $[X,Y]_{\mathfrak{k}}(m) = [X(m),Y(m)]$.

Example 6.8. For any principal K-bundle $\kappa\colon P \to M$, the bundle

$$A(P) = (TP)/K \to M$$

is a Lie algebroid, called the *Atiyah algebroid* of P. The bracket on the space of sections $\Gamma(A)$ is induced from its identification with K-invariant vector field on P, while the anchor map is induced by the bundle projection $T\kappa\colon TP \to TM$. This Lie algebroid fits into an exact sequence of vector bundles (in fact, of Lie algebroids), called the *Atiyah sequence*

$$0 \to \mathfrak{gau}(P) \to A(P) \to TM \to 0$$

where $\mathfrak{gau}(P) = (P \times \mathfrak{k})/K$ is the quotient of the vertical bundle $T_vP \cong P \times \mathfrak{k}$ by K; it is the bundle of infinitesimal gauge transformations. A splitting $j\colon TM \to A(P)$ of the Atiyah sequence is equivalent to a principal bundle connection. Given two vector fields X, Y, since $[j(X), j(Y)]$ is a lift of $[X, Y]$, the difference $[j(X), j(Y)] - j([X, Y])$ is a section of the bundle $\mathfrak{gau}(P)$. The resulting 2-form

$$F \in \Omega^2(M, \mathfrak{gau}(P)), \quad F(X, Y) = [j(X), j(Y)] - j([X, Y])$$

is the curvature form of the connection.

A Lie algebroid whose anchor map is a surjection onto TM is called a *transitive Lie algebroid.* Thus, Atiyah algebroids are transitive.

Example 6.9. Given a hypersurface $N \subseteq M$, there is a Lie algebroid $A \to M$ whose space of sections $\Gamma(A)$ are the vector fields tangent to N. (In local coordinates $x^1, \ldots, x^n$, with N corresponding to $x^n = 0$, the space of such vector fields is generated as a module over $C^\infty(M)$ by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}},\ x^n \frac{\partial}{\partial x^{n-1}}.$$

This Lie algebroid was considered by Melrose for manifolds with boundary, in his so-called b-calculus.

Example 6.10 (Jet prolongations). For any vector bundle $V \to M$, one has its k-th jet prolongation $J^k(V) \to M$. The fiber of $J^k(V)$ at m consists of k-jets of sections $\sigma \in \Gamma(V)$, that is, equivalence classes of sections, where two sections are equivalent if their Taylor expansions up to order k agree. The jet bundles come with a tower of bundle maps

$$\cdots J^k(V) \to J^{k-1}(V) \to \cdots \to J^0(V) = V.$$

For any section σ of V, one obtains a section $j^k(\sigma)$ of $J^k(V)$, and these generate $J^k(V)$ as a $C^\infty(M)$-module. Taking V to be a Lie algebroid (for example, $A = TM$), one finds that $J^k(A)$ has a unique Lie algebroid structure such that

$$[j^k(\sigma), j^k(\tau)] = j^k([\sigma, \tau]), \quad \mathsf{a}(j^k(\sigma)) = \mathsf{a}(\sigma)$$

for all sections $\sigma, \tau \in \Gamma(A)$.

Remark 6.11. Suppose $A \to M$ is a vector bundle with an anchor map $\mathsf{a}\colon A \to TM$, and with a skew-symmetric 'bracket' $[\cdot, \cdot]$ on $\Gamma(A)$ satisfying the Leibnitz rule. Then the 'Jacobiator'

$$\mathrm{Jac}(\sigma_1, \sigma_2, \sigma_3) = [\sigma_1, [\sigma_2, \sigma_3]] + [\sigma_2, [\sigma_3, \sigma_1]] + [\sigma_3, [\sigma_1, \sigma_2]]$$

is $C^\infty(M)$-linear in each entry. (Exercise.) That is, it defines a tensor $\mathrm{Jac} \in \Gamma(\wedge^3 A^*)$. In particular, to verify whether $\mathrm{Jac} = 0$ over some open subset $U \subseteq M$, it suffices to check on generators for $\Gamma(A|_U)$.

6.3. **Lie subalgebroids.** We will make use of the following notation, for a vector bundle $V \to M$ with given subbundle $W \to N$:

$$\Gamma(V, W) = \{\sigma \in \Gamma(V) |\ \sigma|_N \in \Gamma(W)\}.$$

The map $\Gamma(V, W) \to \Gamma(W)$ is surjective, with kernel $\Gamma(V, 0_N)$ the sections of V whose restriction to N vanishes.

Definition 6.12. A subbundle $B \to N$ of a Lie algebroid $A \to M$ is called a *Lie subalgebroid* if it has the following two properties:

(a) $\mathsf{a}(B) \subseteq TN$,
(b) $\Gamma(A, B) \subseteq \Gamma(A)$ is a Lie subalgebra.

Proposition 6.13. *A Lie subalgebroid B of A inherits a unique Lie algebroid structure, in such a way that the restriction map $\Gamma(A, B) \to \Gamma(B)$ preserves Lie brackets.*

Proof. We have to show that $\Gamma(A, 0_N)$ is an ideal in $\Gamma(A, B)$. Let $\sigma \in \Gamma(A, B)$. The space $\Gamma(A, 0_N)$ is spanned by products $f\tau$, where $\tau \in \Gamma(A)$ and where $f \in C^\infty(M)$ vanishes along N. The Leibniz rule

$$[\sigma, f\tau] = f[\sigma, \tau] + (\mathsf{a}(\sigma)f)\ \tau$$

shows that the bracket lies in $\Gamma(A, 0_N)$, since both f and $\mathsf{a}(\sigma)f$ vanish along N. This proves the claim, and hence $\Gamma(B) = \Gamma(A, B)/\Gamma(A, 0_N)$ inherits a Lie bracket. The Leibniz rule for $\Gamma(A)$ implies a Leibniz rule for B, with anchor the restriction of a to B. ☐

Example 6.14. If $N \subseteq M$ is a submanifold, then $TN \subseteq TM$ is a sub-Lie algebroid.

Example 6.15. The tangent bundle to a foliation $\mathcal{F}$ is a Lie subalgebroid $T_\mathcal{F}M \subseteq TM$.

Example 6.16. Let $A \to M$ be a Lie algebroid, and $N \subseteq M$ a submanifold. If $B := \mathsf{a}^{-1}(TN)$ is a submanifold, then it is a vector subbundle (by the GR Lemma), and is in fact a Lie subalgebroid. Its sections are all restrictions $\sigma|_N$ of sections $\sigma \in \Gamma(A)$ such that $\mathsf{a}(\sigma)$ is tangent to N. Letting $i \colon N \to M$ be the inclusion, we will also use the notation

$$i^! A \to N$$

for this Lie algebroid. The condition that $\mathsf{a}^{-1}(TN)$ be a subbundle holds true, for example, if N is transverse to the anchor. (I.e., $\mathsf{a}(A_m) + T_mN = T_mM$ for all $m \in N$.) In this case, we have that

$$\mathrm{rank}(i^! A) = \mathrm{rank}(A) - \dim M + \dim N.$$

The transversality conditions is automatic if A is a transitive Lie algebroid. In the special case $A = TM$, we obtain $i^! A = TN$. More generally, for an Atiyah groupoid $A = A(P)$ of a principal K-bundle, $i^! A(P) = A(P|_N)$.

Example 6.17. Let $A \to M$ be a Lie algebroid, and $N \subseteq M$ a submanifold. Suppose $\ker(\mathsf{a})|_N$ is a submanifold of A. Then it is a vector subbundle, and is in fact a Lie subalgebroid of A (with zero anchor). Its sections are the restrictions of sections of A whose image under a vanishes along N. Note that $\ker(\mathsf{a})|_N$ is a Lie subalgebroid of $i^! A$ (if both are defined). As a special case, for any transitive Lie algebroid $A \to M$ the kernel of the anchor map is a Lie subalgebroid $\ker(\mathsf{a}) \subseteq A$.

Suppose $A \to M$ is a Lie algebroid, and $B \subseteq A$ is an anchored subbundle along $N \subseteq M$, that is, $\mathsf{a}(B) \subseteq TN$. Then B is a Lie subalgebroid if and only if the bracket of any two sections of A which restrict to sections of B, is again a section which restricts to a section of B. Fortunately, it is not necessary to check this condition for *all* sections.

Lemma 6.18. *Suppose $A \to M$ is a Lie algebroid, and $B \subseteq A$ is an anchored subbundle along $N \subseteq M$. Suppose that we are given a subset $\mathcal{R} \subseteq \Gamma(A, B)$, with the property that the image in $\Gamma(B)$ as a $C^\infty(N)$-module. Then B is a Lie subalgebroid if and only if*

$$[\mathcal{R}, \mathcal{R}] \subseteq \Gamma(A, B). \tag{9}$$

Proof. By the Leibnitz rule, the condition (9) on $\mathcal{R}$ is equivalent to a similar property for the $C^\infty(M)$-submodule generated by $\mathcal{R}$. We may hence assume that $\mathcal{R}$ is a $C^\infty(M)$-submodule, which hence surjects onto all of $\Gamma(B)$. Consequently,

$$\Gamma(A, B) = \mathcal{R} + \Gamma(A, 0_N).$$

Since $\Gamma(A, B), \Gamma(A, 0_N) \subseteq \Gamma(A, 0_N)$ (which holds regardless of whether $\Gamma(A, B)$ is a Lie subalgebra), we see that $\Gamma(A, B)$ is a Lie subalgebra if and only if (9) holds. $\square$

6.4. **Intersections of Lie subalgebroids.** In general, the intersection of two Lie subalgebroids need not be a Lie subalgebroid, even if the intersection is smooth:

Example 6.19. Consider two foliations $\mathcal{F}_\pm$ of $\mathbb{R}^2$, given by the curves $y = a \pm x^2$ with $a \in \mathbb{R}$. Let $B_\pm \subseteq T\mathbb{R}^2$ be the tangent bundles of these foliations. Then

$$B_+ \cap B_- = B_+|_S = B_-|_S$$

where $S \subseteq \mathbb{R}^2$ is the y-axis. However, this restriction is not a Lie subalgebroid, since $\mathsf{a}(B_+|_S) \not\subseteq TS$.

However, a clean intersection assumption is all that is needed:

Theorem 6.20 (Clean intersection of Lie subalgebroids)**.** *Suppose $A \to M$ is a Lie algebroid, and $B_1 \to N_1,\ B_2 \to N_2$ are two Lie subalgebroids. If B_1, B_2 intersect cleanly, then $B_1 \cap B_2$ is again a Lie subalgebroid of A.*

Proof. As discussed in Section 4.8, as a clean intersection of two vector subbundles, $B_1 \cap B_2$ is again a vector subbundle, and the map

$$\Gamma(A, B_1) \cap \Gamma(A, B_2) \to \Gamma(B_1 \cap B_2)$$

is *surjective*. Furthermore,

$$\mathsf{a}(B_1 \cap B_2) \subseteq \mathsf{a}(B_1) \cap \mathsf{a}(B_2) \subseteq TN_1 \cap TN_2 = T(N_1 \cap N_2).$$

Since both $\Gamma(A, B_i)$ are Lie subalgebras, their intersection is a Lie subalgebra. Hence, Remark 6.18 applies, and shows that $B_1 \cap B_2$ is a Lie subalgebroid. $\square$

6.5. **Direct products of Lie algebroids.**

Lemma 6.21. *Given two Lie algebroids $A \to M$ and $B \to N$, their direct product*

$$A \times B \to M \times N$$

has a unique Lie algebroid structure, with anchor the direct product of the anchors, in such a way that the map

$$\Gamma(A) \oplus \Gamma(B) \to \Gamma(A \times B),\ (\sigma, \tau) \mapsto \mathrm{pr}_M^* \sigma + \mathrm{pr}_N^* \tau$$

is a Lie algebra homomorphism. Here $\mathrm{pr}_M \colon M \times N \to M$ and $\mathrm{pr}_N \colon M \times N \to N$ are the two projections, and $\mathrm{pr}_M^* \colon \Gamma(A) \to \Gamma(\mathrm{pr}_M^* A) \subseteq \Gamma(A \times B)$ and $\mathrm{pr}_N^* \colon \Gamma(B) \to \Gamma(\mathrm{pr}_N^* B) \subseteq \Gamma(A \times B)$ the pull-back maps.

Proof. Uniqueness is clear since $\Gamma(A) \oplus \Gamma(B)$ generates $\Gamma(A \times B)$ as a module over $C^\infty(M \times N)$.

To show existence, we write the bracket locally: Let $\sigma_1, \ldots, \sigma_k$ be a local basis of sections of A, $\tau_1, \ldots, \tau_l$ a local basis of sections of B. Together, this form a local basis ϵ_i of sections of $A \times B$. Then all brackets $[\epsilon_i, \epsilon_j]$ are determined, and the Leibnitz rule forces us to put

$$[\sum_i f_i \epsilon_i, \sum_j g_j \epsilon_j] = \sum_{ij} \big(f_i g_j\, [\epsilon_i, \epsilon_j] + f_i\, (\mathsf{a}(\epsilon_i) g_j) \epsilon_j - (\mathsf{a}(\epsilon_j) f_i) g_j \epsilon_i \big)$$

for all $f_i, g_j \in C^\infty(M \times N)$. By Remark 6.11, this is a Lie bracket. $\square$