CHAPTER 2

Clifford algebras

1. Exterior algebras

1.1. Definition. For any vector space V over a field $\mathbb{K}$, let $T(V) = \bigoplus_{k\in\mathbb{Z}} T^k(V)$ be the tensor algebra, with $T^k(V) = V \otimes \cdots \otimes V$ the k-fold tensor product. The quotient of $T(V)$ by the two-sided ideal $\mathcal{I}(V)$ generated by all $v \otimes w + w \otimes v$ is the exterior algebra, denoted $\wedge(V)$. The product in $\wedge(V)$ is usually denoted $\alpha_1 \wedge \alpha_2$, although we will frequently omit the wedge sign and just write $\alpha_1\alpha_2$. Since $\mathcal{I}(V)$ is a *graded* ideal, the exterior algebra inherits a grading

$$\wedge(V) = \bigoplus_{k\in\mathbb{Z}} \wedge^k(V)$$

where $\wedge^k(V)$ is the image of $T^k(V)$ under the quotient map. Clearly, $\wedge^0(V) = \mathbb{K}$ and $\wedge^1(V) = V$ so that we can think of V as a subspace of $\wedge(V)$. We may thus think of $\wedge(V)$ as the associative algebra linearly generated by V, subject to the relations $vw + wv = 0$.

We will write $|\phi| = k$ if $\phi \in \wedge^k(V)$. The exterior algebra is *commutative* (in the graded sense). That is, for $\phi_1 \in \wedge^{k_1}(V)$ and $\phi_2 \in \wedge^{k_2}(V)$,

$$[\phi_1, \phi_2] := \phi_1\phi_2 + (-1)^{k_1k_2}\phi_2\phi_1 = 0.$$

If V has finite dimension, with basis $e_1, \ldots, e_n$, the space $\wedge^k(V)$ has basis

$$e_I = e_{i_1} \cdots e_{i_k}$$

for all ordered subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$. (If $k = 0$, we put $e_\emptyset = 1$.) In particular, we see that $\dim \wedge^k(V) = \binom{n}{k}$, and

$$\dim \wedge(V) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Letting $e^i \in V^*$ denote the dual basis to the basis e_i considered above, we define a dual basis to e_I to be the basis $e^I = e^{i_1} \cdots e^{i_k} \in \wedge(V^*)$.

1.2. Universal property, functoriality. The exterior algebra is characterized by its *universal property*: If $\mathcal{A}$ is an algebra, and $f\colon V \to \mathcal{A}$ a linear map with $f(v)f(w) + f(w)f(v) = 0$ for all $v, w \in V$, then f extends uniquely to an algebra homomorphism $f_\wedge\colon \wedge(V) \to \mathcal{A}$. Thus, is $\widetilde{\wedge}(V)$ is another algebra with a homomorphism $V \to \widetilde{\wedge}(V)$, satisfying this universal property, then there is a unique isomorphism $\wedge(V) \to \widetilde{\wedge}(V)$ intertwining the two inclusions of V.

Any linear map $L\colon V \to W$ extends uniquely (by the universal property, applied to L viewed as a map into $V \to \wedge(W)$) to an algebra homomorphism $\wedge(L)\colon \wedge(V) \to \wedge(W)$. One has $\wedge(L_1 \circ L_2) = \wedge(L_1) \circ \wedge(L_2)$. As a special case, taking L to be the zero map $0\colon V \to V$ the resulting algebra homomorphism $\wedge(L)$ is the *augmentation map* (taking $\phi \in \wedge(V)$ to its component in $\wedge^0(V) \cong \mathbb{K}$). Taking L to be the map $v \mapsto -v$, the map $\wedge(L)$ is the *parity homomorphism* $\Pi \in \mathrm{Aut}(\wedge(V))$, equal to $(-1)^k$ on $\wedge^k(V)$.

The functoriality gives in particular a group homomorphism [1]

$$\mathrm{GL}(V) \to \mathrm{Aut}(\wedge(V)),\ g \mapsto \wedge(g)$$

into the group of algebra automorphisms of V. We will often write g in place of $\wedge(g)$, but reserve this notation for invertible transformations since e.g. $\wedge(0) \neq 0$.

As another application of the universal property, suppose V_1, V_2 are two vector spaces, and define $\wedge(V_1) \otimes \wedge(V_2)$ as the tensor product of graded algebras. This tensor product contains $V_1 \oplus V_2$ as a subspace, and satisfies the universal property of the exterior algebra over $V_1 \oplus V_2$. Hence there is a unique algebra isomorphism

$$\wedge(V_1 \oplus V_2) \to \wedge(V_1) \otimes \wedge(V_2)$$

intertwining the inclusions of $V_1 \oplus V_2$. It is clear that this isomorphism preserves gradings.

For $\alpha \in V^*$, define the contraction operators $\iota(\alpha) \in \mathrm{End}(\wedge(V))$ by $\iota(\alpha)1 = 0$ and

$$\iota(\alpha)(v_1 \wedge \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i \rangle\ v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k. \tag{1}$$

On the other hand, for $v \in V$ we have the operator $\epsilon(v) \in \mathrm{End}(\wedge V)$ of exterior multiplication by v. These satisfy the relations

$$\begin{aligned} \iota(v)\epsilon(w) + \epsilon(w)\epsilon(v) &= 0,\\ \iota(\alpha)\iota(\beta) + \iota(\beta)\iota(\alpha) &= 0,\\ \iota(\alpha)\epsilon(v) + \epsilon(v)\iota(\alpha) &= \langle \alpha, v\rangle. \end{aligned} \tag{2}$$

For later reference, let us also observe that the kernel of $\iota(\alpha)$ is the exterior algebra over $\ker(\alpha) \subset V$; hence $\bigcap_{\alpha\in V^*} \ker(\iota(\alpha)) = 0$.

2. Clifford algebras

2.1. Definition and first properties. Let V be a vector space over $\mathbb{K}$, with a symmetric bilinear form $B\colon V \times V \to \mathbb{K}$ (possibly degenerate).

[1] If $\mathcal{A}$ is any algebra, we denote by $\mathrm{End}(\mathcal{A})$ (resp. $\mathrm{Aut}(\mathcal{A})$) the vector space homomorphisms (res. automorphisms) $\mathcal{A} \to \mathcal{A}$, while $\mathrm{End}_{\mathrm{alg}}(\mathcal{A})$ (resp. $\mathrm{Aut}_{\mathrm{alg}}(V)$) denotes the set of algebra homomorphisms (resp. group of algebra automorphisms).

DEFINITION 2.1. The *Clifford algebra* $\mathrm{Cl}(V;B)$ is the quotient

$$\mathrm{Cl}(V;B) = T(V)/\mathcal{I}(V;B)$$

where $\mathcal{I}(V;B) \subset T(V)$ is the two-sided ideal generated by all

$$v \otimes w + w \otimes v - B(v,w)1, \quad v,w \in V$$

Clearly, $\mathrm{Cl}(V;0) = \wedge(V)$. It is not obvious from the definition that $\mathrm{Cl}(V;B)$ is non-trivial, but this follows from the following Proposition.

PROPOSITION 2.2. *The inclusion $\mathbb{K} \to T(V)$ descends to an inclusion $\mathbb{K} \to \mathrm{Cl}(V;B)$. The inclusion $V \to T(V)$ descends to an inclusion $V \to \mathrm{Cl}(V;B)$.*

PROOF. Consider the linear map

$$f\colon V \to \mathrm{End}(\wedge(V)),\ v \mapsto \epsilon(v) + \tfrac{1}{2}\iota(B^\flat(v)).$$

and its extension to an algebra homomorphism $f_T\colon T(V) \to \mathrm{End}(\wedge(V))$. The commutation relations (2) show that $f(v)f(w) + f(w)f(v) = B(v,w)1$. Hence f_T vanishes on the ideal $\mathcal{I}(V;B)$, and therefore descends to an algebra homomorphism

$$f_{\mathrm{Cl}}\colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge(V)), \tag{3}$$

i.e. $f_{\mathrm{Cl}} \circ \pi = f_T$ where $\pi\colon T(V) \to \mathrm{Cl}(V;B)$ is the projection. Since $f_T(1) = 1$, we see that $\pi(1) \neq 0$, i.e. the inclusion $\mathbb{K} \hookrightarrow T(V)$ descends to an inclusion $\mathbb{K} \hookrightarrow \mathrm{Cl}(V;B)$. Similarly, from $f_T(v).1 = v$ we see that the inclusion $V \hookrightarrow T(V)$ descends to an inclusion $V \hookrightarrow \mathrm{Cl}(V;B)$. $\square$

The Proposition shows that V is a subspace of $\mathrm{Cl}(V;B)$. We may thus characterize $\mathrm{Cl}(V;B)$ as the unital associative algebra, with generators $v \in V$ and relations

$$vw + wv = B(v,w)1, \quad v,w \in V. \tag{4}$$

Let $T(V)$ carry the $\mathbb{Z}_2$-grading

$$T^{\bar{0}}(V) = \bigoplus_{k=0}^{\infty} T^{2k}(V),\ T^{\bar{1}}(V) = \bigoplus_{k=0}^{\infty} T^{2k+1}(V).$$

(Here $\bar{k}$ denotes $k \mod 2$.) Since the elements $v \otimes w + w \otimes v - B(v,w)1$ are even, the ideal $\mathcal{I}(V;B)$ is $\mathbb{Z}_2$ graded, i.e. it is a direct sum of the subspaces $\mathcal{I}^{\bar{k}}(V;B) = \mathcal{I}(V;B) \cap T^{\bar{k}}(V)$ for $k = 0,1$. Hence the Clifford algebra inherits a $\mathbb{Z}_2$-grading,

$$\mathrm{Cl}(V;B) = \mathrm{Cl}^{\bar{0}}(V;B) \oplus \mathrm{Cl}^{\bar{1}}(V;B).$$

The two summands are spanned by products $v_1 \cdots v_k$ with k even, respectively odd. From now on, commutators $[\cdot,\cdot]$ in the Clifford algebra $\mathrm{Cl}(V;B)$ will denote $\mathbb{Z}_2$-graded commutators. (We will write $[\cdot,\cdot]_{\mathrm{Cl}}$ if there is risk of confusion.) In this notation, the defining relations for the Clifford algebra become

$$[v,w] = B(v,w), \quad v,w \in V.$$

If $\dim V = n$, and e_i are an orthogonal basis of V, then (using the same notation as for the exterior algebra), the products

$$e_I = e_{i_1} \cdots e_{i_k}, \quad I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\},$$

with the convention $e_\emptyset = 1$, span $\mathrm{Cl}(V; B)$. We will see in Section 2.4 that the e_I are a basis.

2.2. Universal property, functoriality. The Clifford algebra is characterized by the following by a universal property:

PROPOSITION 2.3. *Let $\mathcal{A}$ be an associative unital algebra, and $f\colon V \to \mathcal{A}$ a linear map satisfying*

$$f(v_1)f(v_2) + f(v_2)f(v_1) = B(v_1, v_2) \cdot 1, \quad v_1, v_2 \in V.$$

Then f extends uniquely to an algebra homomorphism $\mathrm{Cl}(V; B) \to \mathcal{A}$.

PROOF. By the universal property of the tensor algebra, f extends to an algebra homomorphism $f_{T(V)}\colon T(V) \to \mathcal{A}$. The property $f(v_1)f(v_2) + f(v_2)f(v_1) = B(v_1, v_2) \cdot 1$ shows that f vanishes on the ideal $\mathcal{I}(V; B)$, and hence descends to the Clifford algebra. Uniqueness is clear, since the Clifford algebra is generated by elements of V. □

Suppose B_1, B_2 are symmetric bilinear forms on V_1, V_2, and $f\colon V_1 \to V_2$ is a linear map such that

$$B_2(f(v), f(w)) = B_1(v, w), \quad v, w \in V_1.$$

Viewing f as a map into $\mathrm{Cl}(V_2; B_2)$, the universal property provides a unique extension

$$\mathrm{Cl}(f)\colon \mathrm{Cl}(V_1; B_1) \to \mathrm{Cl}(V_2; B_2).$$

For instance, if $F \subset V$ is an isotropic subspace of V, there is an algebra homomorphism $\wedge(F) = \mathrm{Cl}(F) \to \mathrm{Cl}(V; B)$. Clearly, $\mathrm{Cl}(f_1 \circ f_2) = \mathrm{Cl}(f_1) \circ \mathrm{Cl}(f_2)$. Taking $V_1 = V_2 = V$, and restricting attention to invertible linear maps, one obtains a group homomorphism

$$\mathrm{O}(V; B) \to \mathrm{Aut}(\mathrm{Cl}(V; B)), \ g \mapsto \mathrm{Cl}(g).$$

We will usually just write g in place of $\mathrm{Cl}(g)$. For example, the involution $v \mapsto -v$ lies in $O(V; B)$, hence it defines an involutive algebra automorphism Π of $\mathrm{Cl}(V; B)$ called the *parity automorphism*. The ± 1 eigenspaces are the even and odd part of the Clifford algebra, respectively.

Suppose again that $(V_, B_1)$ and (V_2, B_2) be two vector spaces with symmetric bilinear forms, and consider the direct sum $(V_1 \oplus V_2, \ B_1 \oplus B_2)$. Then

$$\mathrm{Cl}(V_1 \oplus V_2; B_1 \oplus B_2) = \mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$$

as $\mathbb{Z}_2$-graded algebras. This follows since $\mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$ satisfies the universal property of the Clifford algebra over $(V_1 \oplus V_2; B_1 \oplus B_2)$. In particular, if $\mathrm{Cl}(n, m)$ denotes the Clifford algebra for $\mathbb{K}^{n,m}$ we have

$$\mathrm{Cl}(n, m) = \mathrm{Cl}(1, 0) \otimes \cdots \otimes \mathrm{Cl}(1, 0) \otimes \mathrm{Cl}(0, 1) \otimes \cdots \otimes \mathrm{Cl}(0, 1),$$

with $\mathbb{Z}_2$-graded tensor products.

2.3. The Clifford algebras $\mathrm{Cl}(n,m)$**.** Consider the case $\mathbb{K}=\mathbb{R}$. For n,m small one can determine the algebras $\mathrm{Cl}(n,m)=\mathrm{Cl}(\mathbb{R}^{n,m})$ by hand.

PROPOSITION 2.4. *For* $\mathbb{K}=\mathbb{R}$*, one has the following isomorphisms of the Clifford algebras* $\mathrm{Cl}(n,m)$ *with* $n+m\le 2$*, as ungraded algebras:*

$$\begin{aligned}\mathrm{Cl}(0,1)&\cong\mathbb{C}\\ \mathrm{Cl}(1,0)&\cong\mathbb{R}\oplus\mathbb{R},\\ \mathrm{Cl}(0,2)&\cong\mathbb{H},\\ \mathrm{Cl}(1,1)&\cong\mathrm{Mat}_2(\mathbb{R}),\\ \mathrm{Cl}(2,0)&\cong\mathrm{Mat}_2(\mathbb{R}).\end{aligned}$$

Here $\mathbb{C}$ *and* $\mathbb{H}$ *are viewed as algebras over* $\mathbb{R}$*, and* $\mathrm{Mat}_2(\mathbb{R})=\mathrm{End}(\mathbb{R}^2)$ *is the algebra of real* 2×2*-matrices.*

PROOF. By the universal property, an algebra $\mathcal{A}$ of dimension 2^{n+m} is isomorphic to $\mathrm{Cl}(n,m)$ if there exists a linear map $f\colon\mathbb{R}^{n,m}\to\mathcal{A}$ satisfying $f(e_i)f(e_j)+f(e_j)f(e_i)=\pm\delta_{ij}$, with a plus sign for $i\le n$ and a minus sign for $i>n$. We will describe these maps for $n+m\le 2$. For $(n,m)=(0,1)$ we take $f\colon\mathbb{R}^{0,1}\to\mathbb{C}$, $\frac{1}{\sqrt{2}}e_1\mapsto i=\sqrt{-1}$. For $(n,m)=(1,0)$, we use $f\colon\mathbb{R}^{1,0}\to\mathbb{R}\oplus\mathbb{R}$, $e_1\mapsto\frac{1}{\sqrt{2}}(1,-1)$. For $(n,m)=(0,2)$ we use

$$f(e_1)=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix},\quad f(e_2)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

(The matrices represent the first two of the standard unit quaternions $i,j,k=ij\in\mathcal{H}$.) For $(n,m)=(1,1)$ the relevant map is

$$f(e_1)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad f(e_2)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

The case $(n,m)=(2,0)$ is left as an exercise. □

The full classification of the Clifford algebras $\mathrm{Cl}(n,m)$ may be found in the book by Lawson-Michelsohn [**?**] or in the monograph by Budinich-Trautman [**?**]. The Clifford algebras exhibit a remarkable mod 8 periodicity,

$$\mathrm{Cl}(n+8,m)\cong\mathrm{Mat}_{16}(\mathrm{Cl}(n,m))\cong\mathrm{Cl}(n,m+8)$$

which is related to the mod 8 periodicity in real K-theory [**?**].

For $\mathbb{K}=\mathbb{C}$ the pattern is simpler. Denote by $\mathbb{C}l(n)$ the Clifford algebra of $\mathbb{C}^n$.

PROPOSITION 2.5. *One has the following isomorphisms of algebras over* $\mathbb{C}$*,*

$$\mathbb{C}l(2m)=\mathrm{Mat}_{2^m}(\mathbb{C}),\quad \mathbb{C}l(2m+1)=\mathrm{Mat}_{2^m}(\mathbb{C})\oplus\mathrm{Mat}_{2^m}(\mathbb{C}).$$

This will become clear later when we discuss the spinor module for Clifford algebras in the split case. The mod 2 periodicity of the Clifford algebras

$$\mathbb{C}l(n+2)\cong\mathrm{Mat}_2(\mathbb{C}l(n))$$

is related to the mod 2 periodicity in complex K-theory [**?**].

2.4. Symbol map and quantization map. Returning to the algebra homomorphism $f_{\mathrm{Cl}}\colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge V)$ (see (3)), given on generators by $f_{\mathrm{Cl}}(v) = \epsilon(v) + \frac{1}{2}\iota(B^\flat(v))$, one defines the *symbol map*,

$$\sigma\colon \mathrm{Cl}(V;B) \to \wedge(V),\ x \mapsto f_{\mathrm{Cl}}(x).1$$

where $1 \in \wedge^0(V) = \mathbb{K}$.

PROPOSITION 2.6. *The symbol map is an isomorphism of vector spaces. In low degrees,*

$$\begin{aligned}
\sigma(1) &= 1\\
\sigma(v) &= v\\
\sigma(v_1v_2) &= v_1 \wedge v_2 + \tfrac{1}{2}B(v_1, v_2),\\
\sigma(v_1v_2v_3) &= v_1 \wedge v_2 \wedge v_3 + \tfrac{1}{2}(B(v_2, v_3)v_1 - B(v_1, v_3)v_2 + B(v_1, v_2)v_3).
\end{aligned}$$

PROOF. Let $e_i \in V$ be an orthogonal basis. Since the operators $f(e_i)$ commute (in the grade sense), we find

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

for $i_1 < \cdots < i_k$. This directly shows that the symbol map is an isomorphism: It takes the element $e_I \in \mathrm{Cl}(V;B)$ to the corresponding element $e_I \in \wedge(V)$. The formulas in low degrees are obtained by straightforward calculation. □

The inverse map is called the *quantization map*

$$q\colon \wedge(V) \to \mathrm{Cl}(V;B).$$

In terms of the basis, $q(e_I) = e_I$. In low degrees,

$$\begin{aligned}
q(1) &= 1,\\
q(v) &= v,\\
q(v_1 \wedge v_2) &= v_1v_2 - \tfrac{1}{2}B(v_1, v_2),\\
q(v_1 \wedge v_2 \wedge v_3) &= v_1v_2v_3 - \tfrac{1}{2}(B(v_2, v_3)v_1 - B(v_1, v_3)v_2 + B(v_1, v_2)v_3).
\end{aligned}$$

If $\mathbb{K}$ has characteristic 0 (so that division by all non-zero integers is defined), the quantization map has the following alternative description.

PROPOSITION 2.7. *Suppose* $\mathbb{K}$ *has characteristic* 0. *Then the quantization map is given by graded symmetrization. That is, for* $v_1, \ldots, v_k \in V$,

$$q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \mathrm{sign}(s) v_{s(1)} \cdots v_{s(k)}.$$

Here $\mathfrak{S}_k$ *is the group of permutations of* $1, \ldots, k$ *and* $\mathrm{sign}(s) = \pm 1$ *is the parity of a permutation* s.

PROOF. By linearity, it suffices to check for the case that the v_j are elements of an orthonormal basis $e_1, \ldots, e_n$ of V, that is $v_j = e_{i_j}$ (the indices i_j need not be ordered or distinct). If the i_j are all distinct, then the e_{i_j} Clifford commute in the graded sense, and the right hand side equals $e_{i_1} \cdots e_{i_k} \in \mathrm{Cl}(V; B)$, which coincides with the left hand side. If any two e_{i_j} coincide, then both sides are zero. □

2.5. $\mathbb{Z}$-filtration. The increasing filtration

$$T_{(0)}(V) \subset T_{(1)}(V) \subset \cdots$$

with $T_{(k)}(V) = \bigoplus_{j \leq k} T^j(V)$ descends to a filtration

$$\mathrm{Cl}_{(0)}(V; B) \subset \mathrm{Cl}_{(1)}(V; B) \subset \cdots$$

of the Clifford algebra, with $\mathrm{Cl}_{(k)}(V; B)$ the image of $T_{(k)}(V)$ under the quotient map. Equivalently, $\mathrm{Cl}_{(k)}(V; B)$ consists of linear combinations of products $v_1 \cdots v_l$ with $l \leq k$ (including scalars, viewed as products of length 0). The filtration is compatible with product map, that is,

$$\mathrm{Cl}_{(k_1)}(V; B)\mathrm{Cl}_{(k_2)}(V; B) \subset \mathrm{Cl}_{(k_1+k_2)}(V; B).$$

Thus, $\mathrm{Cl}(V; B)$ is a *filtered algebra*. Let $\mathrm{gr}(\mathrm{Cl}(V; B))$ be the associated graded algebra.

PROPOSITION 2.8. *The symbol map induces an isomorphism of associated graded algebras*

$$\mathrm{gr}(\sigma)\colon\ \mathrm{gr}(\mathrm{Cl}(V; B)) \to \wedge(V).$$

PROOF. The symbol map and the quantization map are filtration preserving, hence they descend to isomorphisms of the associated graded vector spaces. Let $\pi_{\mathrm{Cl}}\colon T(V) \to \mathrm{Cl}(V; B)$ and $\pi_\wedge\colon T(V) \to \wedge(V)$ be the quotient maps. By definition of the symbol map, the composition $\sigma \circ \pi_{\mathrm{Cl}}\colon T_{(k)}(V) \to \wedge(V)$ coincides with $\pi_\wedge\colon T_{(k)}(V) \to \wedge(V)$ up to lower order terms. Passing to the associated graded maps, this gives

$$\mathrm{gr}(\sigma) \circ \mathrm{gr}(\pi_{\mathrm{Cl}}) = \pi_\wedge.$$

Since π_{Cl} is a surjective algebra homomorphism, so is $\mathrm{gr}(\pi_{\mathrm{Cl}})$. It hence follows that $\mathrm{gr}(\sigma)$ is an algebra homomorphism as well. □

Note that the symbol map $\sigma\colon \mathrm{Cl}(V; B) \to \wedge(V)$ preserves the $\mathbb{Z}_2$-grading. The even (resp. odd) elements of $\mathrm{Cl}(V; B)$ are linear combinations of products $v_1 \cdots v_k$ with k even (resp. odd). The filtration is also compatible with the $\mathbb{Z}_2$-grading, that is, each $\mathrm{Cl}_{(k)}(V; B)$ is a $\mathbb{Z}_2$-graded subspace. In fact,

$$\begin{aligned} \mathrm{Cl}^{\bar{0}}_{(2k)}(V; B) &= \mathrm{Cl}^{\bar{0}}_{(2k+1)}(V; B), \\ \mathrm{Cl}^{\bar{1}}_{(2k+1)}(V; B) &= \mathrm{Cl}^{\bar{1}}_{(2k+2)}(V; B). \end{aligned} \tag{5}$$

2.6. Transposition. An anti-automorphism of an algebra $\mathcal{A}$ is an invertible linear map $f\colon \mathcal{A} \to \mathcal{A}$ with the property $f(ab) = f(b)f(a)$ for all $a, b \in \mathcal{A}$. Put differently, if $\mathcal{A}^{\mathrm{op}}$ is $\mathcal{A}$ with the opposite algebra structure $a \cdot_{\mathrm{op}} b := ba$, an anti-automorphism is an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$.

The tensor algebra carries a unique involutive anti-automorphism that is equal to the identity on $V \subset T(V)$. It is called the *canonical anti-automorphism* or *transposition*, and is given by

$$(v_1 \otimes \cdots \otimes v_k)^\top = v_k \otimes \cdots \otimes v_1.$$

Since transposition preserves the ideal $\mathcal{I}(V)$ defining the exterior algebra, it descends to an anti-automorphism of the exterior algebra, $\phi \mapsto \phi^\top$. In fact, since transposition is given by a permutation of length $(k-1)+\cdots+2+1 = k(k-1)/2$, we have

$$\phi^\top = (-1)^{k(k-1)/2}\phi, \quad \phi \in \wedge^k(V).$$

Given a symmetric bilinear form B ob V the transposition anti-automorphism of the tensor algebra also preserves the ideal $\mathcal{I}(V; B)$, and hence descends to an anti-automorphism of $\mathrm{Cl}(V; B)$, still called *canonical anti-automorphism* or *transposition*, with

$$(v_1 \cdots v_k)^\top = v_k \cdots v_1.$$

The symbol map and its inverse, the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V; B)$ intertwines the transposition maps for $\wedge(V)$ and $\mathrm{Cl}(V; B)$. This information is sometimes useful for computations.

EXAMPLE 2.9. Suppose $\phi \in \wedge^k(V)$, and consider the square of $q(\phi)$. The element $q(\phi)^2 \in \mathrm{Cl}(V)$ is even, and is hence contained in $\mathrm{Cl}^{\bar{0}}_{(2k)}(V)$. But $(q(\phi)^2)^\top = (q(\phi)^\top)^2 = q(\phi)^2$ since $q(\phi)^\top = q(\phi^\top) = \pm q(\phi)$. It follows that

$$q(\phi)^2 \in q\big(\wedge^0(V) \oplus \wedge^4(V) \oplus \cdots \oplus \wedge^{4r}(V)\big),$$

where r is the largest number with $2r \leq k$.

2.7. Chirality element, trace. Let $\dim V = n$. Then any generator $\Gamma_\wedge \in \det(V) := \wedge^n(V)$ quantizes to given an element $\Gamma = q(\Gamma_\wedge)$. This element (or suitable normalizations of this element) is called the *chirality element* of the Clifford algebra. The square Γ^2 of the chirality element is always a scalar, as is immediate by choosing an orthogonal basis e_i, and letting $\Gamma = e_1 \cdots e_n$. In fact, since $\Gamma^\top = (-1)^{n(n-1)/2}\Gamma$ we have

$$\Gamma^2 = (-1)^{n(n-1)/2} 2^{-n} \prod_{i=1}^n B(e_i, e_i).$$

In the case $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^n$ we can always normalize Γ to satisfy $\Gamma^2 = 1$; this normalization determines Γ up to sign. For any $v \in V$, we have $\Gamma v = (-1)^{n-1} v\Gamma$, as one checks e.g. using an orthogonal basis. (If $v = e_i$, then v anti-commutes with all e_j for $j \neq i$ in the product $\Gamma = e_1 \cdots e_n$, and commutes with e_i. Hence we obtain $n - 1$ sign changes.)

$$\Gamma v = \begin{cases} v\Gamma & \text{if } n \text{ is odd} \\ -v\Gamma & \text{if } n \text{ is even} \end{cases}$$

Thus, if n is odd then Γ lies in the center of $\mathrm{Cl}(V;B)$, viewed as an ordinary algebra. In the case that n is even, we obtain

$$\Pi(x) = \Gamma x \Gamma^{-1},$$

for all $x \in \mathrm{Cl}(V;B)$, i.e. the chirality element *implements* the parity automorphism.

For any $\mathbb{Z}_2$-graded algebra $\mathcal{A}$ and vector space Y, a Y-valued *super-trace* on $\mathcal{A}$ is a linear map $\mathrm{tr}_s \colon \mathcal{A} \to Y$ vanishing on the subspace $[\mathcal{A}, \mathcal{A}]$ spanned by super-commutators: That is, $\mathrm{tr}_s([x,y]) = 0$ for $x, y \in \mathcal{A}$.

Proposition 2.10. *Suppose $n = \dim V < \infty$. The linear map*

$$\mathrm{tr}_s \colon \mathrm{Cl}(V;B) \to \det(V)$$

given as the quotient map to $\mathrm{Cl}_{(n)}(V;B)/\mathrm{Cl}_{(n-1)}(V;B) \cong \wedge^n(V) = \det(V)$, is a super-trace on $\mathrm{Cl}(V;B)$.

Proof. Let e_i be an orthogonal basis, and e_I the assocated basis of $\mathrm{Cl}(V;B)$. Then $\mathrm{tr}_s(e_I) = 0$ unless $I = \{1, \ldots, n\}$. The product e_I, e_J is of the form $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. Hence $\mathrm{tr}_s(e_I e_J) = 0 = \mathrm{tr}_s(e_J e_I)$ unless $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, n\}$. But in case $I \cap J = \emptyset$, e_I, e_J super-commute: $[e_I, e_J] = 0$. □

The Clifford algebra also carries an *ordinary trace*, vanishing on ordinary commutators.

Proposition 2.11. *The formula*

$$\mathrm{tr} \colon \mathrm{Cl}(V;B) \to \mathbb{K}, \quad x \mapsto \sigma(x)_{[0]}$$

defines an (ordinary) trace on $\mathrm{Cl}(V;B)$, that is $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in \mathrm{Cl}(V;B)$. For $\dim V < \infty$, the trace and the super-trace are related by the formula,

$$\mathrm{tr}_s(\Gamma x) = \mathrm{tr}(x)\,\Gamma_\wedge$$

where $\Gamma = q(\Gamma_\wedge)$ is the chirality element in the Clifford algebra defined by a choice of generator of $\det(V)$.

Proof. Again, we use an orthogonal basis e_i of V. The definition gives $\mathrm{tr}(e_\emptyset) = 1$, while $\mathrm{tr}(e_I) = 0$ for $I \neq \emptyset$. Consider a product $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. The set K is non-empty (i.e. $\mathrm{tr}(e_I e_J) = 0$) unless $I = J$, but in the latter case the trace property is trivial. To check the formula relating trace and super-trace we may assume $\Gamma_\wedge = e_I$ with $I = \{1, \ldots, n\}$. For $x = e_J$ we see that $\mathrm{tr}_s(\Gamma x)$ vanishes unless $J = \emptyset$, in which case it is $\Gamma_\wedge$. □

2.8. Lie derivatives and contractions. Let V be a vector space, and $\alpha \in V^*$. Then the map $\iota(\alpha)\colon V \to \mathbb{K},\ v \mapsto \langle \alpha, v\rangle$ extends uniquely to a degree -1 derivation of the tensor algebra $T(V)$, called *contraction*, by

$$\iota(\alpha)(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i\rangle\ v_1 \otimes \cdots \widehat{v_i} \cdots \otimes v_k$$

The contraction operators preserve the ideal $\mathcal{I}(V)$ defining the exterior algebra, and descend to the contraction operators on $\wedge(V)$. Given a symmetric bilinear form B on V, the contraction operators also preserve the ideal $\mathcal{I}(V;B)$ since

$$\iota(\alpha)(v_1 \otimes v_2 + v_2 \otimes v_1 - B(v_1, v_2)) = 0.\ \ v_1, v_2 \in V.$$

It follows that $\iota(\alpha)$ descends to an odd derivation of $\mathrm{Cl}(V;B)$ of filtration degree -1, with

$$\iota(\alpha)(v_1 \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i\rangle v_1 \cdots \widehat{v_i} \cdots v_k. \tag{6}$$

Similarly, any $A \in \mathfrak{gl}(V) = \mathrm{End}(V)$ extends to a derivation L_A of degree 0 on $T(V)$, called *Lie derivative*:

$$L_A(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} v_1 \otimes \cdots \otimes L_A(v_i) \otimes \cdots \otimes v_k.$$

L_A preserves the ideal $\mathcal{I}(V)$, and hence descends to a derivation of $\wedge(V)$. If $A \in \mathfrak{o}(V;B)$, that is $B(Av_1, v_2) + B(v_1, Av_2) = 0$ for all v_1, v_2, then L_A also preserves the ideal $\mathcal{I}(V;B)$ and consequently descends to an even derivation of $\mathrm{Cl}(V;B)$, of filtration degree 0.

One has (on the tensor algebra, and hence also on the exterior and Clifford algebras)

$$[\iota(\alpha_1), \iota(\alpha_2)] = 0,\ [L_{A_1}, L_{A_2}] = L_{[A_1,A_2]},\ [L_A, \iota(\alpha)] = \iota(A.\alpha),$$

where $A.\alpha = -A^*\alpha$ with A^* the dual map. This proves the first part of:

PROPOSITION 2.12. *The map $A \mapsto L_A$, $\alpha \mapsto \iota(\alpha)$ defines an action of the graded Lie algebra $\mathfrak{o}(V;B) \ltimes V^*$ (where elements of V^* have degree -1) on $\mathrm{Cl}(V;B)$ by derivations. The symbol map intertwines this with the corresponding action by derivations of $\wedge(V)$.*

PROOF. It suffices to check on elements $\phi = v_1 \wedge \cdots \wedge v_k \in \wedge(V)$ where $v_1, \ldots, v_k$ are pairwise orthogonal. Then $q(\phi) = v_1 \cdots v_k$, and the quantization of $\iota(\alpha)\phi$ (given by (1)) coincides with $\iota(\alpha)(q(\phi))$ (given by (6)). The argument for the Lie derivatives is similar. □

Any element $v \in V$ defines a derivation of $\mathrm{Cl}(V;B)$ by graded commutator: $x \mapsto [v, x]$. For generators $w \in V$, we have $[v, w] = B(v, w) =$

$\langle B^\flat(v),\ w\rangle$. This shows that this derivation agrees with the contraction by $B^\flat(v)$:

(7) $$[v,\cdot] = \iota(B^\flat(v))$$

As a simple application, we find:

LEMMA 2.13. *The super-center of the $\mathbb{Z}_2$-graded algebra* $\mathrm{Cl}(V;B)$ *is the exterior algebra over* $\mathrm{rad}(B) = \ker B^\flat$.

PROOF. Indeed, suppose x lies in the super-center. Then $0 = [v,x] = \iota(B^\flat(v))x$ for all $v \in V$. Hence $\sigma(x)$ is annihilated by all contractions $B^\flat(v))$, and is therefore an element of the exterior algebra over $\mathrm{ann}(\mathrm{ran}(B^\flat)) = \ker(B^\flat)$. Consequently $x = q(\sigma(x))$ is in $\mathrm{Cl}(\ker(B^\flat)) = \wedge(\ker(B^\flat))$. □

2.9. The homomorphism $\wedge^2 V \to \mathfrak{o}(V;B)$. Consider next the derivations of $\mathrm{Cl}(V;B)$ defined by elements of $q(\wedge^2 V)$. Define a map

(8) $$\wedge^2 V \to \mathfrak{o}(V;B),\ \lambda \mapsto A_\lambda$$

where $A_\lambda(v) = -\iota(B^\flat(v))\lambda$. This does indeed lie in $\mathfrak{o}(V;B)$, since

$$B(A_\lambda(v), w) = -\iota(B^\flat(w))A_\lambda(v) = -\iota(B^\flat(w))\iota(B^\flat(v))\lambda$$

is anti-symmetric in v, w. We have:

(9) $$[q(\lambda), \cdot] = L_{A_\lambda}$$

since both sides are derivations extending the map $v \mapsto A_\lambda(v)$ on generators. Define a bracket $\{\cdot,\cdot\}$ on $\wedge^2(V)$ by

(10) $$\{\lambda, \lambda'\} = L_{A_\lambda}\lambda'.$$

The calculation

$$[q(\lambda), q(\lambda')] = L_{A_\lambda} q(\lambda') = q(L_{A_\lambda}\lambda') = q(\{\lambda,\lambda'\})$$

shows that q intertwines $\{\cdot,\cdot\}$ with the Clifford commutator; in particular $\{\cdot,\cdot\}$ is a Lie bracket. Furthermore, from

$$[q(\lambda), [q(\lambda'), v]] - [q(\lambda'), [q(\lambda), v]] = [[q(\lambda), q(\lambda')], v] = [q(\{\lambda,\lambda'\}), v]$$

we see that $[A_\lambda, A_{\lambda'}] = A_{\{\lambda,\lambda'\}}$, that is, the map $\lambda \mapsto A_\lambda$ is a Lie algebra homomorphism. To summarize:

PROPOSITION 2.14. *The formula* (10) *defines a Lie bracket on* $\wedge^2(V)$*. Relative to this bracket, the map*

$$\wedge^2(V) \rtimes V[1] \to \mathfrak{o}(V;B) \rtimes V^*[1], \quad (\lambda, v) \mapsto (A_\lambda, B^\flat(v))$$

is a homomorphism of graded Lie algebras. (The symbol $[1]$ *indicates a degree shift: We assign degree* -1 *to the elements of* V, V^* *while* $\wedge^2(V)$*,* $\mathfrak{o}(V;B)$ *are assigned degree* 0*.) It intertwines the derivation actions of* $q(\lambda), v$ *on* $\mathrm{Cl}(V;B)$ *by Clifford commutator with the action by Lie derivatives and contractions.*

Note that we can also think of $\wedge^2(V) \rtimes V[1]$ as a graded subspace of $\wedge(V)[2]$, using the standard grading on $\wedge(V)$ shifted down by 2. We will see in the following Section **??** that the graded Lie bracket on this subspace extends to a graded Lie bracket on all of $\wedge(V)[2]$.

2.10. A formula for the Clifford product. It is sometimes useful to express the Clifford multiplication

$$m_{\mathrm{Cl}} \colon \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

in terms of the exterior algebra multiplication,

$$m_\wedge \colon \wedge(V \oplus V) = \wedge(V) \otimes \wedge(V) \to \wedge(V).$$

Recall that by definition of the isomorphism $\wedge(V \oplus V) = \wedge(V) \otimes \wedge(V)$, if $\phi, \psi \in \wedge(V^*)$, the element $\phi \otimes \psi \in \wedge(V^*) \otimes \wedge(V^*)$ is identified with the element $(\phi \oplus 0) \wedge (0 \oplus \psi) \in \wedge(V^* \oplus V^*)$. Similarly for the Clifford algebra.

Let $e_i \in V$ be an orthogonal basis, $e^i \in V^*$ the dual basis, and $e_I \in \wedge(V)$, $e^I \in \wedge(V^*)$ the corresponding dual bases indexed by subsets $I \subset \{1, \ldots, n\}$. Then the element

$$\Psi = \sum_I \frac{1}{(-2)^{|I|}} \, e^I \otimes B^\flat((e_I)^\top) \in \wedge(V^*) \otimes \wedge(V^*)$$

is independent of the choice of bases.

Proposition 2.15. *Under the quantization map, the exterior algebra product and the Clifford product are related as follows:*

$$m_{\mathrm{Cl}} \circ q = q \circ m_\wedge \circ \iota(\Psi)$$

Proof. Let V_i be the 1-dimensional subspace spanned by e_i. Then $\wedge(V)$ is the graded tensor product over all $\wedge(V_i)$, and similarly $\mathrm{Cl}(V)$ is the graded tensor product over all $\mathrm{Cl}(V_i)$. The formula for Ψ factorizes as

$$\Psi = \prod_{i=1}^{n} \big(1 - \tfrac{1}{2} e^i \otimes B^\flat(e_i)\big). \tag{11}$$

It hence suffices to prove the formula for the case $V = V_1$. We have,

$$\begin{aligned} q \circ m_\wedge \circ \iota\big(1 - \tfrac{1}{2} e^1 \otimes B^\flat(e_1)\big)(e_1 \otimes e_1) &= q \circ m_\wedge\big(e_1 \otimes e_1 + \tfrac{1}{2} B(e_1, e_1)\big) \\ &= q\big(\tfrac{1}{2} B(e_1, e_1)\big) \\ &= e_1 e_1. \end{aligned}$$

∎

If $\mathrm{char}(\mathbb{K}) = 0$, we may also write the element Ψ as an exponential:

$$\Psi = \exp\big(-\tfrac{1}{2} \sum_i e^i \otimes B^\flat(e_i)\big).$$

This follows by rewriting (11) as $\prod_i \exp\big(-\tfrac{1}{2} e^i \otimes B^\flat(e_i)\big)$, and then writing the product of exponentials as an exponential of a sum.

Remark 2.16. Consider the addition map

$$\mathrm{Add}\colon V \oplus V \to V,\ v \oplus w \mapsto v + w.$$

This map is linear, and hence to an algebra homomorphism

$$\wedge(\mathrm{Add})\colon \wedge(V \oplus V) \to \wedge(V).$$

In terms of the identification $\wedge(V \oplus V) = \wedge(V) \otimes \wedge(V)$, this is exactly the map $m_\wedge$. The dual map $\mathrm{Add}^*\colon V^* \to V^* \oplus V^*$ is the diagonal inclusion. The composition

$$\tilde{m}_{\mathrm{Cl}} = \sigma \circ m_{\mathrm{Cl}} \circ q\colon \wedge(V \oplus V) \to \wedge(V)$$

has the property,

$$\tilde{m}_{\mathrm{Cl}} \circ \iota(\mathrm{Add}^*(\alpha)) = \iota(\alpha) \circ \tilde{m}_{\mathrm{Cl}}$$

for all $\alpha \in V^*$. Hence, by Lemma **??**, there exists a unique element $\Psi \in \wedge(V^* \oplus V^*)$ such that $\tilde{m}_{\mathrm{Cl}} = m_\wedge \circ \iota(\Psi)$, and this is the element determined in the Proposition.