ON THE QUANTIZATION OF CONJUGACY CLASSES

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Abstract. Let \( G \) be a compact, simple, simply connected Lie group. A theorem of Freed-Hopkins-Teleman identifies the level \( k \geq 0 \) fusion ring \( R_k(G) \) of \( G \) with the twisted equivariant \( K \)-homology at level \( k + h^V \), where \( h^V \) is the dual Coxeter number of \( G \). In this paper, we will review this result using the language of Dixmier-Douady bundles. We show that the additive generators of the group \( R_k(G) \) are obtained as \( K \)-homology push-forwards of the fundamental classes of pre-quantized conjugacy classes in \( G \).

1. Introduction

A classical result of Dixmier-Douady [10] states that the integral degree three cohomology group \( H^3(X) \) of a space \( X \) classifies bundles of \( C^* \)-algebras \( A \to X \), with typical fiber the compact operators on a Hilbert space. For any such Dixmier-Douady bundle \( A \to X \), one defines the twisted \( K \)-homology and \( K \)-cohomology groups of \( X \) as the \( K \)-groups of the \( C^* \)-algebra of sections of \( A \), vanishing at infinity:

\[
K_q(X, A) := K_q(\Gamma_0(X, A)) \quad \text{and} \quad K^q(X, A) := K^q(\Gamma_0(X, A)).
\]

If a group \( G \) acts by automorphisms of \( A \), one has definitions of \( G \)-equivariant \( K \)-groups.

The twisted \( K \)-groups have attracted a lot of interest in recent years, mainly due to their applications in string theory. For the case of torsion twistings, they were pioneered by Donovan-Karoubi [11] in 1963, while the general case was developed by Rosenberg [36] in 1989. Rosenberg also gave an alternative characterization of \( K^0(X, A) \) as homotopy classes of sections of a bundle of Fredholm operators; this viewpoint was further explored by Atiyah-Segal [4] (see [6, 43] for alternative approaches).

One of the most natural examples of an integral degree three cohomology class comes from Lie theory. Let \( G \) be a compact, simple, simply connected Lie group, acting on itself by conjugation. The generator of \( H^3_G(G) = \mathbb{Z} \) is realized by a \( G \)-Dixmier-Douady bundle \( A \to G \). Let \( h^V \) be the dual Coxeter number of \( G \), and \( k \geq 0 \) a non-negative integer (the level). A beautiful result of Freed-Hopkins-Teleman [13, 14, 15, 16, 17] asserts that the twisted equivariant \( K \)-homology at the shifted level \( k + h^V \) coincides with the level \( k \) fusion ring (Verlinde algebra) of \( G \):

\[
K_0^G(G, A^{k+h^V}) = R_k(G).
\]

Here \( R_k(G) \) may be defined as the ring of positive energy level \( k \) representations of the loop group \( LG \), or equivalently as the quotient \( R_k(G) = R(G)/I_k(G) \) of the usual representation ring by the level \( k \) fusion ideal. The quotient map \( R(G) \to R_k(G) \) is realized on the \( K \)-homology side as push-forward under inclusion \( \{ e \} \hookrightarrow G \), while the product on \( R_k(G) \) is given by push-forward under group multiplication.

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As a \( \mathbb{Z} \)-module, the fusion ring \( R_k(G) \) is freely generated by the set \( \Lambda^*_k \) of *level \( k \) weights* of \( G \). In this paper the isomorphism \( R_k(G) = \mathbb{Z}[\Lambda^*_k] \) is realized as follows. Given \( \mu \in \Lambda^*_k \subset t^* \), (where \( t \) is the Lie algebra of a maximal torus), let \( \mathcal{C} \) be the conjugacy class of the element \( \exp(\mu/k) \in G \), where the basic inner product is used to identify \( t^* \cong t \). We will show that there is a canonical stable isomorphism between the restriction \( A^+_{k \hbar} |_{C} \) and the Clifford algebra bundle \( \text{Cl}(TC) \). This then defines a push-forward map in twisted \( K \)-homology, and the image of the \( K \)-homology fundamental class \( [C] \in K^0_G(C, \text{Cl}(TC)) \) under the push-forward is exactly the generator of \( R_k(G) \) labeled by \( \mu \). This is parallel to the fact that the generators of \( R(G) = \mathbb{Z}[\Lambda^*_\infty] \) are obtained by geometric quantization of the coadjoint orbits through dominant weights. In fact, as shown in [15] the generators of \( R_k(G) \) can also be obtained by geometric quantization of coadjoint orbits of the loop group of \( G \). Hence, our modest observation is that this can also be carried out in finite-dimensional terms. In a forthcoming paper with A. Alekseev, we will discuss more generally the quantization of group-valued moment maps [1] along similar lines.

A second theme in this paper is the construction of a canonical resolution of \( R_k(G) \) in the category of \( R(G) \)-modules,

\[
0 \longrightarrow C_l \overset{\partial}{\longrightarrow} C_{l-1} \overset{\partial}{\longrightarrow} \cdots \overset{\partial}{\longrightarrow} C_0 \overset{\varepsilon}{\longrightarrow} R_k(G) \longrightarrow 0
\]

where \( l = \text{rank}(G) \). In more detail, let \( \{0, \ldots, l\} \) label the vertices of the extended Dynkin diagram of \( G \). For each non-empty subset \( I \subset \{0, \ldots, l\} \), let \( G_I \subset G \) be the maximal rank subgroup whose Dynkin diagram is obtained by deleting the vertices labeled by \( I \). These groups have canonical central extensions \( 1 \to U(1) \to \hat{G}_I \to G_I \to 1 \) (described below). Let \( R(\hat{G}_I)_k \) denote the Grothendieck group of all \( \hat{G}_I \)-representations for which the central circle acts with weight \( k \). Define

\[
C_p = \bigoplus_{|I|=p+1} R(\hat{G}_I)_k.
\]

The differentials \( \partial \) in (2) are given by holomorphic induction maps relative to the inclusions \( \hat{G}_I \hookrightarrow \hat{G}_J \) for \( J \subset I \). As we will explain, the chain complex \( (C_\bullet, \partial) \) arises as the \( E^1 \)-term of a spectral sequence computing \( K^G_0(G, \mathcal{A}^{k+h^\vee}) \), and the exactness of (2) implies that the spectral sequence collapses at the \( E^2 \)-term. Since \( R_k(G) \) is free Abelian, there are no extension problems, and one recovers the equality \( K^G_0(G, \mathcal{A}^{k+h^\vee}) = R_k(G) \) as \( R(G) \)-modules, and hence also as rings.

This article does not make great claims of originality. In particular, I learned that a very similar computation of the twisted equivariant \( K \)-groups of a Lie group had appeared in the article *Thom Prospects for loop group representations* by Kitchloo-Morava [25]. The argument itself may be viewed as a natural generalization of the Mayer-Vietoris calculation for \( G = \text{SU}(2) \), as explained by Dan Freed in [13]. Independently, the chain complex had been obtained by Christopher Douglas (unpublished), who used it to obtain information about the algebraic structure of the fusion ring \( R_k(G) \).
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### 2. Review of twisted equivariant $K$-homology

Throughout this paper, all Hilbert spaces $\mathcal{H}$ will be taken to be separable, but not necessarily infinite-dimensional. All (topological) spaces $X$ will be assumed to allow the structure of a countable CW-complex (respectively $G$-CW complex, in the equivariant case).

#### 2.1. Dixmier-Douady bundles

[10, 35, 36] For any Hilbert space $\mathcal{H}$, we denote by $U(\mathcal{H})$ the unitary group, with the strong operator topology. Let $\mathbb{K}(\mathcal{H})$ be the $C^*$-algebra of compact operators, that is, the norm closure of the finite rank operators. The conjugation action of the unitary group on $\mathbb{K}(\mathcal{H})$ descends to the projective unitary group, and provides an isomorphism, $\text{Aut}(\mathbb{K}(\mathcal{H})) = PU(\mathcal{H})$. A Dixmier-Douady bundle $\mathcal{A} \to X$ is a locally trivial bundle of $C^*$-algebras, with typical fiber $\mathbb{K}(\mathcal{H})$ and structure group $PU(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. That
is,

\[ \mathcal{A} = \mathcal{P} \times_{\text{PU}(\mathcal{H})} \mathbb{K}(\mathcal{H}) \]

for a principal \( \text{PU}(\mathcal{H}) \)-bundle \( \mathcal{P} \to X \). Dixmier-Douady bundles of finite rank are also known as Azumaya bundles [26, 27]. A gauge transformation of \( \mathcal{A} \) is a bundle automorphism inducing the identity on \( X \), and whose restriction to the fibers are \( C^* \)-algebra automorphisms. Equivalently, the group of gauge transformations consists of sections of the associated group bundle, \( \text{Aut}(\mathcal{A}) = \mathcal{P} \times_{\text{PU}(\mathcal{H})} \text{Aut}(\mathbb{K}(\mathcal{H})) \). This group bundle has a central extension

\[ 1 \to X \times \text{U}(1) \to \widetilde{\text{Aut}}(\mathcal{A}) \to \text{Aut}(\mathcal{A}) \to 1, \]

where \( \widetilde{\text{Aut}}(\mathcal{A}) = \mathcal{P} \times_{\text{PU}(\mathcal{H})} \text{U}(\mathcal{H}) \).

If \( \mathcal{A}_1, \mathcal{A}_2 \) are Dixmier-Douady bundles modeled on \( \mathbb{K}(\mathcal{H}_1), \mathbb{K}(\mathcal{H}_2) \), then their (fiberwise) \( C^* \)-tensor product \( \mathcal{A}_1 \otimes \mathcal{A}_2 \) is a Dixmier-Douady bundle modeled on \( \mathbb{K}(\mathcal{H}_1 \otimes \mathcal{H}_2) \). Also, the (fiberwise) opposite \( \mathcal{A}^{\text{opp}} \) of a Dixmier-Douady bundle modeled on \( \mathbb{K}(\mathcal{H}) \) is a Dixmier-Douady bundle modeled on \( \mathbb{K}(\mathcal{H}^{\text{opp}}) \). Here the Hilbert space \( \mathcal{H}^{\text{opp}} \) is equal to \( \mathcal{H} \) as an additive group, but with the new scalar multiplication by \( z \in \mathbb{C} \) equal to the old scalar multiplication by \( \overline{z} \).

A Morita isomorphism between two Dixmier-Douady bundles \( \mathcal{A}_1, \mathcal{A}_2 \to X \) is a lift of the structure group \( \text{PU}(\mathcal{H}_2) \times \text{PU}(\mathcal{H}_1^{\text{opp}}) \) of \( \mathcal{A}_2 \otimes \mathcal{A}_1^{\text{opp}} \) to the group \( \text{P}(\mathcal{H}_2) \times \text{U}(\mathcal{H}_1^{\text{opp}}) \). It is thus given by a bundle \( \mathcal{E} \to X \) of \( \mathcal{A}_2 - \mathcal{A}_1 \)-bimodules, modeled on the \( \mathbb{K}(\mathcal{H}_2) - \mathbb{K}(\mathcal{H}_1) \)-bimodule \( \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \). We will write \( \mathcal{A}_1 \simeq_\mathcal{E} \mathcal{A}_2 \) if \( \mathcal{E} \) defines such a Morita isomorphism, and \( \mathcal{A}_1 \simeq \mathcal{A}_2 \) if \( \mathcal{A}_1, \mathcal{A}_2 \) are Morita isomorphic for some \( \mathcal{E} \). Morita isomorphism is an equivalence relation: In particular, if \( \mathcal{A}_1 \simeq_\mathcal{E} \mathcal{A}_2 \) and \( \mathcal{A}_2 \simeq_\mathcal{F} \mathcal{A}_3 \), then the bundle \( \mathcal{F} \otimes \mathcal{A}_2 \mathcal{E} \) (a completion of the algebraic tensor product over \( \mathcal{A}_2 \)) defines a Morita isomorphism between \( \mathcal{A}_1, \mathcal{A}_3 \). The set of Morita isomorphism classes of Dixmier-Douady bundles over \( X \) is an Abelian group, with sum \( [\mathcal{A}_1] + [\mathcal{A}_2] = [\mathcal{A}_1 \otimes \mathcal{A}_2] \), neutral element \( 0 = [\mathbb{C}] \), and inverse \( -[\mathcal{A}] = [\mathcal{A}^{\text{opp}}] \).

In particular, a Morita trivialization \( \mathbb{C} \simeq_\mathcal{E} \mathcal{A} \) is a Hilbert space bundle \( \mathcal{E} \) together with an isomorphism \( \mathcal{A} \cong \mathbb{K}(\mathcal{E}) \). The obstruction to the existence of a Morita trivialization is given by the Dixmier-Douady class\(^1\) [10, 35]

\[ \text{DD}(\mathcal{A}) \in H^3(X). \]

The Dixmier-Douady class descends to a group isomorphism between Morita isomorphism classes of Dixmier-Douady bundles \( \mathcal{A} \to X \) and \( H^3(X) \).

Example 2.1. Let \( V \to X \) be an oriented Euclidean vector bundle of rank \( k \), and let \( \text{Cl}(V) \to X \) be the complex Clifford algebra bundle. If \( k \) is even, the bundle \( \text{Cl}(V) \) is a bundle of matrix algebras, and hence is a Dixmier-Douady bundle. A Morita trivialization

\[ \mathbb{C} \simeq_\mathcal{S} \text{Cl}(V) \]

is equivalent to the choice of a spinor module \( \mathcal{S} \to X \), which in turn is equivalent to the choice of a \( \text{Spin}_k \) structure on \( V \). For details, see Plymen [34]. The canonical anti-involution of \( \text{Cl}(V) \) defines an isomorphism \( \text{Cl}(V) \cong \text{Cl}(V)^{\text{opp}} \), thus

\[ \text{DD}(\text{Cl}(V)) = \text{DD}(\text{Cl}(V)^{\text{opp}}) = -\text{DD}(\text{Cl}(V)) \]

showing that \( \text{DD}(\text{Cl}(V)) \) is 2-torsion. The Dixmier-Douady class \( \text{DD}(\text{Cl}(V)) \) is the third integral Stiefel-Whitney class \( w^2(\text{Cl}(V)) \in H^2(X, \mathbb{Z}_2) \)

\( ^1 \)We take all cohomology groups with integer coefficients, unless indicated otherwise.
under the Bockstein homomorphism. In the case of $k$ odd, the even part $\text{Cl}^+(V)$ is a Dixmier-
Douady bundle, and a similar discussion applies.

If both $\mathcal{E}, \mathcal{E}' \rightarrow X$ define Morita isomorphisms $A_1 \simeq A_2$, then the bundle of bi-module
homomorphisms $L = \text{Hom}_{A_2 - A_1}(\mathcal{E}, \mathcal{E}')$ is a Hermitian line bundle. We will call $\mathcal{E}, \mathcal{E}'$ equivalent if this line bundle is isomorphic to the trivial line bundle. Conversely, if $\mathcal{E}$ is a Morita
isomorphism then so is $\mathcal{E}' = \mathcal{E} \otimes L$, for any line bundle $L$. Thus, if $A_1, A_2$ have the same
Dixmier-Douady class, then the equivalence classes of Morita isomorphisms $A_1 \cong_{\mathcal{E}} A_2$ are a principal
homogeneous space (torsor) over $H^2(X, \mathbb{Z})$. (In the example $\mathcal{A} = \text{Cl}(V)$, this is the usual twist of Spin$_c$-structures by line bundles.)

Given a compact Lie group $G$ acting on $X$, one may similarly define $G$-equivariant Dixmier-
Douady bundles. All of the above extends to this equivariant setting: In particular, there is a $G$-equivariant
Dixmier-Douady class $DD_G(A) \in H^3_G(X)$, which classifies $G$-Dixmier-Douady bundles up to $G$-equivariant Morita isomorphisms. The extension of the Dixmier-Douady theorem to the $G$-equivariant case was proved by Atiyah-Segal [4].

Still more generally, one can also consider $\mathbb{Z}_2$-graded $G$-equivariant Dixmier-Douady bundles $A \rightarrow X$. Here, isomorphisms and tensor products are understood in the $\mathbb{Z}_2$-graded sense, and the bi-
modules in the definition of Morita isomorphism are $\mathbb{Z}_2$-graded. We continue to denote by $DD_G(A)$ the Dixmier-Douady class of $A$ as an ungraded bundle. If $DD_G(A) = 0$, so that $\mathbb{C} \cong_{\mathcal{E}} A$, there is an obstruction in $H^1(X, \mathbb{Z}_2)$ for the existence of a compatible $\mathbb{Z}_2$-grading on $\mathcal{E}$. Hence, the map from Morita isomorphism classes of $\mathbb{Z}_2$-graded $G$-Dixmier-Douady bundles to those of ungraded $G$-Dixmier-Douady bundles is onto, with kernel $H^1(X, \mathbb{Z}_2)$. See Parker [32] and Atiyah-Segal [4] for details.

2.2. Dixmier-Douady bundles related to central extensions. We assume that $G$ is compact and connected. Then $H^1_G(\text{pt}) = 0$, while $H^2_G(\text{pt})$ is the group of $G$-equivariant line bundles
over a point, or equivalently $H^2_G(\text{pt}) = \text{Hom}(G, U(1))$. The group $H^3_G(\text{pt})$ is realized as the
isomorphism classes of central extensions of $G$ by $U(1)$,

\begin{equation}
1 \rightarrow U(1) \rightarrow \hat{G} \rightarrow G \rightarrow 1.
\end{equation}

For any such extension there is an associated $G$-equivariant line bundle $L = \hat{G} \times_{U(1)} \mathbb{C} \rightarrow G$ from which $\hat{G}$ is recovered as the unit circle bundle. The group structure is encoded in an isomorphism

$$\text{Mult}^* L \cong \text{pr}_1^* L \otimes \text{pr}_2^* L$$

where $\text{Mult}: G \times G \rightarrow G$ is group multiplication, and $\text{pr}_i$ are the two projections. For any $l \in \mathbb{Z}$,
the $l$-th power $\hat{G}^{(l)}$ of the extension is defined in terms of the $l$-th power of the corresponding
line bundle. More generally one defines products of central extensions of $G$ by $U(1)$ in terms of
the tensor products of the corresponding line bundles. The group of gauge transformations of a
given central extension $\hat{G}$ (i.e. group automorphisms covering the identity on $G$) is $H^2_G(\text{pt}) = \text{Hom}(G, U(1))$.

From the interpretation via Dixmier-Douady bundles, the identification of $H^3_G(\text{pt})$ with
isomorphism classes of central extensions may be seen as follows: Given a $G$-equivariant Dixmier-
Douady bundle $A \rightarrow \text{pt}$, the action of $G$ defines a group homomorphism $G \rightarrow \text{Aut}(A)$, and hence a central extension of $G$ by pull-back of (5) (in the case $X = \text{pt}$). Conversely, given a central extension $\hat{G}$, choose a unitary representation $\hat{G} \rightarrow U(\mathcal{E})$ where the central circle $U(1)$
acts by scalar multiplication. Then \( \mathbb{K}(\mathcal{E}) \to \text{pt} \) is a \( G \)-Dixmier-Douady bundle with the prescribed class in \( H^3_G(\text{pt}) \). Note that we may take \( \mathcal{E} \) to be of finite rank, reflecting that \( H^3_G(\text{pt}) \) is torsion. (Recall that \( H^p_G(\text{pt}, \mathbb{R}) = H^p(BG, \mathbb{R}) = 0 \) for \( p \) odd.)

Suppose \( X \) is a connected space, with \( H^1(X) \) torsion-free, and with the trivial action of \( G \). The Kunneth theorem [38, Chapter 5.5] for \( H^*_G(X) = H^*_G(X \times BG) \) gives a direct sum decomposition,

\[
H^3_G(X) = H^3(X) \oplus (H^1(X) \oplus H^2_G(\text{pt})) \oplus H^3_G(\text{pt}).
\]

For any \( G \)-Dixmier-Douady bundle \( \mathcal{A} \to X \), we obtain a corresponding decomposition of \( \text{DD}_G(\mathcal{A}) \). The first component is the non-equivariant class \( \text{DD}(\mathcal{A}) \). The last summand is the class of the central extension of \( G \), defined by the homomorphism \( G \to \text{Aut}(\mathcal{A}_{x_0}) \) at any given base point \( x_0 \in X \). To describe the middle summand, note that the family of actions \( G \to \text{Aut}(\mathcal{A}_x) \) defines a family of central extensions, by pull-back of (5),

\[
1 \to \text{U}(1) \to \widehat{G}_x \to G \to 1.
\]

For any \( x' \in X \), there exists an isomorphism \( \widehat{G}_x \to \widehat{G}_{x'} \) of central extensions, unique up to \( \text{Hom}(G, \text{U}(1)) \cong H^2_G(\text{pt}) \). Since the latter group is discrete, it follows that the family \( \widehat{G}_x \) carries a flat connection: Any path from a base point \( x_0 \) to \( x \) defines an isomorphism \( \widehat{G} := \widehat{G}_{(x_0)} \to \widehat{G}_{(x)} \), depending only on the homotopy class of the path. We therefore obtain the holonomy homomorphism \( \tau: \pi_1(X; x_0) \to H^2_G(\text{pt}) \), hence an element of \( H^1(X) \otimes H^2_G(\text{pt}) \subset H^3_G(X) \). This element is identified with the corresponding component of \( \text{DD}_G(\mathcal{A}) \).

Remark 2.2. Any element of \( H^1(X) \otimes H^2_G(\text{pt}) \) is realized in this way. Indeed, let \( \mathcal{H} = L^2(G) \) with the left-regular representation of \( G \). The homomorphism \( \tau: \pi_1(X) \to H^2_G(\text{pt}) = \text{Hom}(G, \text{U}(1)) \) defines a unitary action of \( \pi_1(X) \) on \( \mathcal{H} \), where \( \lambda \in \pi_1(X) \) acts as pointwise multiplication by the function \( \tau(\lambda) \). The actions of \( G \) and \( \pi_1(X) \) commute up to a scalar. The bundle \( \mathcal{A} = \overline{X} \times_{\pi_1(X)} \mathbb{K}(\mathcal{H}) \) associated to the universal covering \( \overline{X} \to X \) is a \( G \)-equivariant Dixmier-Douady bundle, with \( \text{DD}_G(\mathcal{A}) \) the prescribed class in \( H^1(X) \otimes H^2_G(\text{pt}) \). Note that the component in \( H^3(X) \) is zero, since non-equivariantly \( \mathcal{A} = \mathbb{K}(\mathcal{E}) \) for \( \mathcal{E} = \overline{X} \times_{\pi_1(X)} \mathcal{H} \).

2.3. Twisted \( K \)-homology. The input for the twisted equivariant \( K \)-homology of a \( G \)-space \( X \) is a \( \mathbb{Z}_2 \)-graded \( G \)-Dixmier-Douady bundle \( \mathcal{A} \to X \). From now on, we will usually omit explicit mention of the \( \mathbb{Z}_2 \)-grading (which may be trivial), with the understanding that all tensor products are in the \( \mathbb{Z}_2 \)-graded sense, isomorphisms should preserve the \( \mathbb{Z}_2 \)-grading, and so on.

Given \( \mathcal{A} \to X \), the space \( \mathcal{A} = \Gamma_0(X, \mathcal{A}) \) of continuous sections of \( \mathcal{A} \) vanishing at infinity is a \( (\mathbb{Z}_2 \text{-graded}) \) \( G \)-\( C^* \)-algebra, with norm \( ||s|| = \sup_{x \in X} ||s_x||_{\mathcal{A}_x} \). Following J. Rosenberg [36], we define the twisted equivariant \( K \)-homology and \( K \)-cohomology groups as the equivariant \( C^* \)-algebra \( K \)-homology and \( K \)-cohomology groups of \( \mathcal{A} \):

\[
K^G_q(X, \mathcal{A}) := K^G_q(\Gamma_0(X, \mathcal{A})), \quad K^G_q(X, \mathcal{A}) := K^G_q(\Gamma_0(X, \mathcal{A})).
\]

In this paper, we will mostly work with the \( K \)-homology groups. See the appendix for a quick review of the \( K \)-homology of \( C^* \)-algebras, and some examples. We list some basic properties of the \( K \)-homology groups.
(i) **Morita isomorphisms.** Any Morita isomorphism \( \mathcal{A}_1 \simeq_{\mathcal{E}} \mathcal{A}_2 \) of G-Dixmier-Douady bundles over \( X \) induces an isomorphism in \( K \)-homology,

\[
K_q^G(X, \mathcal{A}_1) \cong K_q^G(X, \mathcal{A}_2).
\]

(ii) **Push-forwards.** The morphisms in the category of G-Dixmier-Douady bundles \((X, \mathcal{A})\) are the equivariant \( C^* \)-algebra bundle maps \( \mathcal{A}_1 \to \mathcal{A}_2 \) for which the induced map on the base \( f: X_1 \to X_2 \) is proper. Any such morphism induces a morphism of \( G-C^* \)-algebras \( f^*: \Gamma_0(X_2, \mathcal{A}_2) \to \Gamma_0(X_1, \mathcal{A}_1) \), hence a push-forward in \( K \)-homology

\[
K_q^G(f): K_q^G(X_1, \mathcal{A}_1) \to K_q^G(X_2, \mathcal{A}_2).
\]

In this way \( K_q^G \) becomes a covariant functor, invariant under proper \( G \)-homotopies.

(iii) **Excision.** For any closed, invariant subset \( Y \subset X \), with complement \( U = X \setminus Y \), there is a long exact sequence

\[
\cdots \to K_q^G(Y, \mathcal{A}|_Y) \to K_q^G(X, \mathcal{A}) \to K_q^G(U, \mathcal{A}|_U) \to K_{q-1}^G(Y, \mathcal{A}|_Y) \to \cdots
\]

Here the restriction map \( K_q^G(X, \mathcal{A}) \to K_q^G(U, \mathcal{A}|_U) \) is induced by the \( C^* \)-algebra morphism \( \Gamma_0(U, \mathcal{A}|_U) \to \Gamma_0(X, \mathcal{A}) \), given as extension by 0. More generally, one obtains a spectral sequence for any filtration of \( X \) by closed, invariant subspaces.

(iv) **Products.** Suppose \( \mathcal{A} \to X \) and \( \mathcal{B} \to Y \) are two G-Dixmier-Douady bundles. Then the exterior tensor product \( \mathcal{A} \boxtimes \mathcal{B} \to X \times Y \) is again a G-Dixmier-Douady bundle. Its space of sections is the \( C^* \)-tensor product of the spaces of sections of \( \mathcal{A}, \mathcal{B} \). As a special case of the Kasparov product in \( K \)-homology, one has a natural associative cross product,

\[
K_q^G(X, \mathcal{A}) \otimes K_q^G(Y, \mathcal{B}) \to K_q^G(X \times Y, \mathcal{A} \boxtimes \mathcal{B}).
\]

(v) **Module structure.** The group \( K_0^G(\text{pt}) \) is canonically identified with the representation ring \( R(G) \). The ring structure on \( K_0^G(\text{pt}) \) is defined by the cross product for \( \mathbb{C} \boxtimes \mathbb{C} \to \text{pt} \times \text{pt} \). Similarly, if \( \mathcal{A} \to X \) is a G-Dixmier-Douady bundle, the cross product for \( \mathbb{C} \boxtimes \mathcal{A} \to \text{pt} \times X \) makes \( K_q^G(X, \mathcal{A}) \) into a module over \( R(G) \). The maps \( K_q^G(f) \) are \( R(G) \)-module homomorphisms.

If \( M \) is a manifold, one has the **Poincaré duality isomorphism** relating twisted \( K \)-homology and \( K \)-cohomology,

\[
K_q^G(M, \mathcal{A}) \cong K_q^G(M, \mathcal{A}^{\text{opp}} \otimes \text{Cl}(TM)).
\]

Here \( \text{Cl}(TM) \) is the Clifford algebra bundle for some choice of invariant metric. For \( \mathcal{A} = \mathbb{C} \) the Poincaré duality was proved by Kasparov in [21, Section 8]; the result in the twisted case was obtained by J.-L. Tu [41, Theorem 3.1]. (See also [9, Section 2]). The image of \( 1 \in K_0^G(M) \) under this isomorphism is Kasparov’s **\( K \)-homology fundamental class** [24]

\[
[M] \in K_0^G(M, \text{Cl}(TM)).
\]

**Remark 2.3.** Note that \( \text{Cl}(TM) \) is a Dixmier-Douady bundle only if \( \dim M \) is even. However, the definition of the twisted \( K \)-groups works for arbitrary bundles of \( C^* \)-algebras, and the isomorphism (7) holds in this sense (but with \( \mathcal{A} \) a Dixmier-Douady bundle). Alternatively, one

\[\text{Note that } K \text{-homology is analogous to Borel-Moore homology (homology with non-compact supports), rather than ordinary homology.}\]
may state the result in terms of Dixmier-Douady bundles, using \( \text{Cl}(TM) = \text{Cl}^+(TM) \otimes \text{Cl}(\mathbb{R}) \) and the isomorphism \( K^G_{q+1}(M, \mathcal{B}) = K^G_q(M, \mathcal{B} \otimes \text{Cl}(\mathbb{R})) \).

The following basic computations in twisted equivariant \( K \)-homology may be deduced from their \( K \)-theory counterparts, using Poincaré duality.

(a) If \( M = \text{pt} \), the twisted \( K \)-homology is

\[ K^G_0(\text{pt}, \mathcal{A}) = R(\hat{G})_{-1}, \]

while \( K^G_1(\text{pt}, \mathcal{A}) = 0 \). Here \( \hat{G} \) is the central extension defined by the action \( G \to \text{Aut}(\mathcal{A}) \), and \( R(\hat{G})_{-1} \) is the Grothendieck group of \( \hat{G} \)-representations where the central \( U(1) \) acts with weight \(-1\).

(b) Suppose \( H \) is a closed subgroup of \( G \). For any \( H \)-Dixmier-Douady bundle \( \mathcal{B} \to Y \), there is a natural isomorphism

\[ \text{ind}^G_H : K^H_q(Y, \mathcal{B} \otimes \text{Cl}(\mathfrak{g} \oplus \mathfrak{h})) \cong K^G_q(G \times_Y Y, G \times_Y \mathcal{B}), \]

which is Poincaré dual to the isomorphism \( K^G_q(G \times_Y Y, G \times_Y \mathcal{B}^{\text{opp}}) \cong K^G_q(Y, \mathcal{B}^{\text{opp}}) \).

If \( Y = \text{pt} \), the left hand side may be evaluated as in (a). If \( H \subset H' \subset G \) are closed subgroups, we have

\[ \text{ind}^G_H = \text{ind}^G_{H'} \circ \text{ind}^G_{H''}. \]

Here we are identifying \( \text{Cl}(\mathfrak{g} \oplus \mathfrak{h}) \cong \text{Cl}(\mathfrak{g}'' \oplus \mathfrak{h}') \), and we are using the canonical isomorphism \( H' \times_H \text{Cl}(\mathfrak{g}'' \oplus \mathfrak{h}') \cong H'' \times_H \text{Cl}(\mathfrak{g}'' \oplus \mathfrak{h}') \).

(c) Let \( \mathcal{A} \to \text{pt} \) be a \( G \)-Dixmier-Douady algebra as in (a), and let \( H \) be a closed subgroup of \( G \). Then \( G \times_Y \mathcal{A} \) is canonically isomorphic to \( \pi^* \mathcal{A} \), the pull-back under the map \( \pi : G/H \to \text{pt} \). By composing the map \( \text{ind}^G_H \) with the push-forward \( K^G_q(\pi) \), we obtain an induction homomorphism,

\[ \text{ind}^G_H : K^H_q(\text{pt}, \mathcal{A} \otimes \text{Cl}(\mathfrak{g} \oplus \mathfrak{h})) \to K^G_q(\text{pt}, \mathcal{A}). \]

An \( H \)-invariant complex structure on \( \mathfrak{g} \oplus \mathfrak{h} \) defines a spinor module \( S \), hence a Morita trivialization \( \mathbb{C} \cong S \text{Cl}(\mathfrak{g} \oplus \mathfrak{h}) \). In this case the induction map simplifies to a map

\[ \text{ind}^G_H : K^H_0(\text{pt}, \mathcal{A}) = R(\hat{H})_{-1} \to K^G_0(\text{pt}, \mathcal{A}) = R(\hat{G})_{-1} \]

known as holomorphic induction.

For other examples of calculations of twisted \( K \)-groups, see [6, Section 8].

3. The Dixmier-Douady bundle over \( G \)

For the rest of this paper, \( G \) will denote a compact, simple, simply connected Lie group, acting on itself by conjugation. Then \( H^2_G(G) \) is canonically isomorphic to \( \mathbb{Z} \). Hence there exists a \( G \)-Dixmier-Douady bundle \( \mathcal{A} \to G \), unique up to Morita isomorphism, such that \( \text{DD}_G(G, \mathcal{A}) \) corresponds to the generator \( 1 \in \mathbb{Z} \). Any two bundles \( \mathcal{A}, \mathcal{A}' \to G \) representing the generator are related by a \( G \)-equivariant Morita isomorphism, unique up to equivalence (since \( H^2_G(G) = 0 \)).

The quickest construction of \( \mathcal{A} \) is as an associated bundle

\[ \mathcal{A} = P_e G \times_{L_e G} \mathbb{K}(\mathcal{H}), \]

where \( P_e G \) is the space of based paths in \( G \), \( L_e G = LG \cap P_e G \) the based loop group, and \( \mathcal{H} \) a representation of the standard central extension \( \widehat{LG} \) of \( LG \) where the central circle acts with
weight $-1$. The construction given in this Section is essentially just a slow-paced version of this model for $\mathcal{A}$, avoiding some infinite-dimensional technicalities. Our strategy is to first give a direct construction of the family of central extensions of the centralizers $G_g \subset G$, corresponding to their action on $\mathcal{A}$.

### 3.1. Pull-back to the maximal torus.

Let $T \subset G$ be a maximal torus of $G$, with Lie algebra $\mathfrak{t}$. Consider the map

$$H^3_G(G) \to H^3_T(T)$$

obtained by first restricting the action to $T$ and then pulling back to $T$. We will compute the image of the generator of $H^3_G(G)$ under this map. Denote by $\Lambda \subset \mathfrak{t}$ the integral lattice (i.e. the kernel of $\exp : \mathfrak{t} \to T$). Recall that the basic inner product $B$ on the Lie algebra $\mathfrak{g}$ is the unique invariant inner product, with the property that the smallest length of a non-zero element $\lambda \in \Lambda$ equals $\sqrt{2}$. One of the key properties of $B$ is that it restricts to an integer-valued bilinear form on $\Lambda$. That is, $B|_t \in \Lambda^* \otimes \Lambda^*$ where $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$ is the (real) weight lattice.

**Proposition 3.1.** The map (8) is injective, and takes the generator of $H^3_G(G)$ to the element

$$-B|_t \in \Lambda^* \otimes \Lambda^* \cong H^3_T(pt) \otimes H^1(T) \subset H^3_T(T)$$

given by minus the basic inner product.

**Proof.** Since $H_G(G)$ and $H_T(T)$ have no torsion in degree $\leq 3$, we may pass to real coefficients, and hence work with Cartan’s equivariant de Rham model $\Omega^*_G(M) = \bigoplus_{2i+j=m} (S^i \mathfrak{g}^* \otimes \Omega^j(M))^G$ for the equivariant cohomology $H_G(M, \mathbb{R})$ of a $G$-manifold, with differential $(d_G \alpha)(\xi) = d\alpha(\xi) - \iota(\xi_M)\alpha(\xi)$ where $\xi_M$ is the vector field defined by $\xi \in \mathfrak{g}$. Note $H^3_G(T, \mathbb{R}) = \mathfrak{t}^* \otimes H^1(T) \oplus H^3(T, \mathbb{R})$ since the $T$-action on $T$ is trivial. Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left-, right- invariant Maurer Cartan forms. The generator of $H^3_G(G)$ is represented by an equivariant de Rham form,

$$\eta_G(\xi) = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) - \frac{1}{2} B(\theta^L + \theta^R, \xi).$$

Its pull-back to $T$ is $\iota^*_T \eta_G(\xi) = -B(\theta_T, \xi)$, where $\theta_T \in \Omega^1(T, \mathfrak{t})$ the Maurer-Cartan form for $T$. Thus

$$\iota^*_T[\eta_G] = [B^p(\theta_T)] \in \mathfrak{t}^* \otimes H^1(T, \mathbb{R}) \subset H^3_T(T, \mathbb{R}).$$

The identification $H^1(T, \mathbb{R}) \cong \mathfrak{t}^*$ takes $[B^p(\theta_T)]$ to $B|_t \in \mathfrak{t}^* \otimes \mathfrak{t}^*$. \hfill $\Box$

### 3.2. The family of central extensions $\widehat{T}(\mathfrak{t})$.

As discussed in Section 2.2, any element of $H^3_T(pt) \otimes H^1(T)$ is realized as the holonomy of a family of central extensions. For any $\mu \in \Lambda^*$ let $T \to U(1)$, $t \mapsto t^u$ be the corresponding homomorphism. Let the lattice $\Lambda$ act on $\widehat{T} = T \times U(1)$ as

$$\Lambda \times \widehat{T} \to \widehat{T}, \quad \lambda.(h, z) = (h, h^{-B^p(\lambda)}z).$$

Then the holonomy of the family

$$t \times_\Lambda \widehat{T} \to \mathfrak{t}/\Lambda = T.$$ 

is the element $B|_t$. The action of the Weyl group $W = N(T)/T$ on $T$ lifts to an action on this family, by

$$w.[(\xi; h, z)] = [(w\xi; wh, z)].$$
Let $\hat{T}(t)$ be the fiber of (11) over $t \in T$. The choice of $\xi$ with $\exp \xi = t$ defines a trivialization
\[(13)\quad T \rightarrow \hat{T}(t), \quad h \mapsto [(\xi; h, 1)] \in t \times \Lambda \hat{T}.
\]
Shifting $\xi$ by $\lambda \in \Lambda$ changes the trivialization by the homomorphism $T \rightarrow U(1), h \mapsto h^{-B^0(\lambda)}$.

3.3. Simplicial description. It will be useful to have the following equivalent description of the bundle (11). Let $t_\pm \subset t$ be the choice of a closed Weyl chamber, and let $\Delta \subset t_+$ be the corresponding closed Weyl alcove. Recall that $\Delta$ labels the $W$-orbits in $T$, in the sense that every orbit contains a unique point in $\exp(\Delta)$. Label the vertices of $\Delta$ by $0, \ldots, l = \mathrm{rank}(G)$, in such a way that the label $0$ corresponds to the origin. For every non-empty subset $I \subset \{0, \ldots, l\}$ let $\Delta_I$ denote the closed simplex spanned by the vertices in $I$, and let $W_I \subset W$ denote the subgroup fixing $\exp(\Delta_I) \subset T$. Then the maps $W/W_I \times \Delta_I \rightarrow T$, $(wW_I, \xi) \mapsto \exp \xi$ define an isomorphism
\[(14)\quad T \cong \coprod_I W/W_I \times \Delta_I / \sim
\]
using the identifications,
\[(15)\quad (x, \iota_J^I(\xi)) \sim (\phi_J^I(x), \xi), \quad J \subset I.
\]
Here $\iota_J^I: \Delta_I \hookrightarrow \Delta_I$ is the natural inclusion, giving rise to an inclusion $W_I \hookrightarrow W_J$ of Lie groups and hence to projection $\phi_J^I: W/W_I \rightarrow W/W_J$.

Let $\lambda_I: W_I \rightarrow \Lambda$ be defined by $w\Delta_I = \Delta_I - \lambda_I(w)$. It is a group cocycle, $\lambda_I(wv) = \lambda_I(u) + u \cdot \lambda_I(v)$, and $\lambda_J|_{W_I} = \lambda_I$ for $J \subset I$. We thus obtains compatible actions of $wI$ on $\hat{T} = T \times U(1)$:
\[(16)\quad w_I(h, z) = (wh, h^{-B^0(\lambda_I(w^{-1}))}z).
\]

Lemma 3.2. The isomorphism (14) extends to an isomorphism of the family (11) of central extensions,
\[(17)\quad \bigcup_{t \in T} \hat{T}_t = t \times \Lambda \hat{T} \cong \coprod_I (W \times W_I \hat{T}) \times \Delta_I / \sim.
\]

Proof. The maps $\hat{T} \times \Delta_I \rightarrow t \times \Lambda \hat{T}$, $(h, z; \xi) \mapsto [(\xi; h, z)]$ are $W_I$-equivariant, by the calculation (for $\xi \in \Delta_I$, $w \in W_I$)
\[
w_I[(\xi; h, z)] = [(w\xi; wh, z)] = [(\xi - \lambda_I(w); wh, z)] = [(\xi; wh, (wh)^{B^0(\lambda_I(w^{-1}))}z)] = [(\xi; wh, h^{-B^0(\lambda_I(w^{-1}))}z)].
\]
They hence extend to $W$-equivariant maps $(W \times W_I \hat{T}) \times \Delta_I \rightarrow t \times \Lambda \hat{T}$, which glue to the desired isomorphism. \hfill $\square$

3.4. The centralizers $G_I$ and their central extensions. For any $g \in G$, we denote by $G_g$ its centralizer. For any given $I$, the centralizer $G_{\exp \xi}$ for $\xi$ in the interior of $\Delta_I$ is independent of the choice of $\xi$, and will be denote $G_I$. Equivalently, $G_I$ is the closed subgroup of $G$ fixing
\[ G \cong \coprod_I G/G_I \times \Delta_I / \sim \]

using the equivalence relations (15) for the natural maps \( \phi_I^J : G/G_I \to G/G_J \) for \( J \subset I \). In this Section, we generalize (17) to define a \( G \)-equivariant collection of central extensions,

\[ \bigcup_{g \in G} \hat{G}_g \cong \coprod_I (G \times_{G_I} \hat{G}_I) \times \Delta_I / \sim . \]

(Of course, this is no longer a fiber bundle.) Our construction of \( A \to G \) will realize \( \hat{G}_g \) as the opposite of the central extension, defined by action of \( G_g \) on the fiber \( A_g \).

**Lemma 3.3.** There are distinguished central extensions

\[ 1 \to U(1) \to \hat{G}_I \to G_I \to 1, \]

together with lifts \( \hat{i}_I^J : \hat{G}_I \hookrightarrow \hat{G}_J \) of the inclusions \( i_I^J : G_I \hookrightarrow G_J \) for \( J \subset I \), such that

\begin{enumerate}[(a)]
  \item \( \hat{G}_{\{0, \ldots, I\}} = \hat{T} \),
  \item the lifted inclusions satisfy the coherence condition \( \hat{i}_I^K = \hat{i}_J^K \circ \hat{i}_I^J \) for \( K \subset J \subset I \),
  \item the \( W_I \)-action on \( \hat{T} \subset \hat{G}_I \) (cf. (16)) is induced by the conjugation action of \( N_{G_I}(T) \).
\end{enumerate}

**Proof.** Recall \( \pi_1(G_I) = \Lambda / \Lambda_I \), where \( \Lambda_I \) is the co-root lattice of \( G_I \) [8, Theorem (7.1)]. But

\[ \lambda \in \Lambda_I, \ t \in \exp(\Delta_I) \Rightarrow t^{B^\flat(\lambda)} = 1 \]

(see [28, Proposition 5.4]). Hence, for any given \( t \in \exp(\Delta_I) \), there is a homomorphism

\[ g_{t,I} : \pi_1(G_I) = \Lambda / \Lambda_I \to U(1), \quad \lambda + \Lambda_I \mapsto t^{B^\flat(\lambda)}. \]

We therefore obtain a family of central extensions \( \hat{G}_{I,(t)} = \hat{G}_I \times \pi_1(G_I) U(1) \) parametrized by the points of \( \exp(\Delta_I) \). Since \( \exp(\Delta_I) \cong \Delta_I \) is contractible, we may use the flat connection on the family of central extensions (cf. Section 2.2) to identify all \( \hat{G}_{I,(t)} \). The resulting \( \hat{G}_I \) has the desired properties. In particular, if \( J \subset I \) and \( t \in \exp(\Delta_J) \subset \exp(\Delta_I) \), the homomorphism \( g_{t,I} \) is given by the inclusion \( \pi_1(G_I) \to \pi_1(G_J) \) followed by \( g_{t,J} \). This defines an inclusion \( \hat{G}_{I,(t)} \hookrightarrow \hat{G}_{J,(t)} \), compatible with the flat connection and (hence) satisfying the coherence condition. Fix \( \xi \in \Delta \) with \( \exp_T \xi = t \). The inclusion of \( \hat{T} = T \times U(1) \) into \( \hat{G}_I \cong \hat{G}_{I,(t)} \) is explicitly given as

\[ i^I : (\exp_T \zeta, z) \mapsto [(\exp_{\hat{G}_I} \zeta, e^{-2\pi \sqrt{-1}B(\xi, \zeta)} z)], \]

for \( \zeta \in t, \ z \in U(1) \). If \( g \in N_{G_I}(T) \) lifts \( w \in W_I \), we have

\[ g.\left[(\exp_{\hat{G}_I} \zeta, e^{-2\pi \sqrt{-1}B(\xi, \zeta)} z)\right] = \left[(\exp_{\hat{G}_I} (w, \zeta), e^{-2\pi \sqrt{-1}B(\xi, \zeta)} z)\right] \]
\[ = i^I(\left[(\exp_T (w, \zeta), e^{-2\pi \sqrt{-1}B(\xi, \zeta)} z)\right]) \]
\[ = i^I(\left[(w. (\exp_T \zeta, \ z))\right]) \]

proving that \( i^I \) is equivariant for the actions of \( W_I \) and \( N_{G_I}(T) \). \( \square \)
Remarks 3.4. (i) The central extension \( \hat{G}_I \) admits a trivialization if and only if the affine span of \( B^\perp(\Delta_I) \subset \mathfrak{t}^* \) contains a point in the weight lattice, \( \Lambda^* \). In particular, this is the case whenever \( 0 \in I \). If \( G \) is of type \( A_n \) or \( C_n \), then all \( \hat{G}_I \) are isomorphic to trivial extensions. (ii) The choice of any \( t \in \exp(\Delta_I) \) gives a trivialization \( \hat{g}_I \cong \hat{g}_{I,(t)} = \mathfrak{g}_I \times \mathbb{R} \), by the definition of \( \hat{G}_{I,(t)} \) as a quotient of \( \hat{G}_I \times \text{U}(1) \).

3.5. Construction of the Dixmier-Douady bundle \( A \to G \). Our construction of the Dixmier-Douady bundle \( A \to G \) involves a suitable Hilbert space \( \mathcal{H} \).

Lemma 3.5. There exists a Hilbert space \( \mathcal{H} \), equipped with unitary representations of the central extensions \( \hat{G}_I \) such that (i) the central \( \text{U}(1) \) acts with weight \(-1\), and (ii) for \( J \subset I \) the action of \( \hat{G}_J \) restricts to the action of \( \hat{G}_I \).

One may construct such an \( \mathcal{H} \) using the theory of affine Lie algebras. Let \( \mathcal{L}(\mathfrak{g}) = \mathfrak{g}^\mathbb{C} \otimes \mathbb{C}[z, z^{-1}] \) be the loop algebra associated to \( \mathfrak{g} \). For all roots \( \alpha \) of \( G \), let \( e_\alpha \in \mathfrak{g}^\mathbb{C} \) be the corresponding root vector. Then \( \mathfrak{g}_I^\mathbb{C} \) is spanned by \( \mathfrak{t}^\mathbb{C} \) together with the root vectors \( e_\alpha \) such that \( \langle \alpha, \xi \rangle \in \mathbb{Z} \) for \( \xi \in \Delta_I \). The map \( j_I : \mathfrak{g}_I^\mathbb{C} \to \mathcal{L}(\mathfrak{g}) \) given by \( \zeta \mapsto \zeta \otimes 1 \) for \( \zeta \in \mathfrak{t}^\mathbb{C} \) and
\[
e_\alpha \mapsto e_\alpha \otimes z^{\langle \alpha, \xi \rangle},\]
for \( \langle \alpha, \xi \rangle \in \mathbb{Z} \) is an injective Lie algebra homomorphism (independent of \( \xi \)). Consider the standard central extension \( \hat{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C} \), with bracket
\[
[\zeta_1 \otimes f_1 + s_1 \zeta, \zeta_2 \otimes f_2 + s_2 \zeta'] = ([\zeta_1, \zeta_2] \otimes f_1 f_2) + B(\zeta_1, \zeta_2) \text{Res}(f_1 d f_2) \zeta.
\]
Its restriction to constant loops is canonically trivial, thus \( \hat{\mathfrak{t}}^\mathbb{C} \) is embedded in \( \mathcal{L}(\mathfrak{g}^\mathbb{C}) \) by the map \( (\zeta, s) \mapsto \zeta + sc \). The inclusions \( j_I \) lift to inclusions \( \hat{j}_I : \hat{\mathfrak{g}}_I \to \hat{\mathcal{L}}(\mathfrak{g}) \) extending the given inclusion of \( \mathfrak{t}^\mathbb{C} \). To see this, take \( \zeta \in \Delta_I \) (defining a trivialization \( \mathfrak{g}_I \cong \mathfrak{g}_{I,(\exp \xi)} = \mathfrak{g}_I \times \mathbb{R} \)). Then the desired lift reads,
\[
\hat{j}_I : \hat{\mathfrak{g}}_{I,(\exp \xi)} \to \hat{\mathcal{L}}(\mathfrak{g}), \quad \hat{j}_I(\zeta, s) = j_I(\zeta) + (s + B(\zeta, \xi)) \zeta.
\]
By the theory of affine Lie algebras [20], there exists a unitarizable \( \hat{\mathfrak{g}} \)-module where the central element \( \xi \) acts as \(-1\). Unitarizability means in particular that the \( \mathfrak{t} \)-action exponentiates to a unitary \( \hat{T} \)-action, and hence all \( \hat{\mathfrak{g}}_I \)-actions exponentiate to unitary \( \hat{G}_I \)-actions.

With \( \mathcal{H} \) as in the Lemma, put \( A_I = G \times G_I \mathbb{K}(\mathcal{H}) \). For \( J \subset I \), the map \( \phi_J^I : G/G_I \to G/G_J \) is covered by a homomorphism of Dixmier-Douady bundles, \( A_I \to A_J \). Hence we may define a \( G \)-Dixmier-Douady bundle,
\[
A = \prod_I (A_I \times \Delta_I)/\sim \tag{20}
\]
with identifications similar to those in (18). By construction, the central extension of \( G_I \) defined by the restriction \( A|_{\exp(\Delta_I)} \) coincides with the opposite of \( \hat{G}_I \). Hence, the family of central extensions defined by the action of \( T \) on \( A|_T \) is the opposite of the family \( \hat{T} \). We had seen that the class in \( H^2_T(\text{pt}) \otimes H^1(T) \subset H^2_T(T) \) is the class defined by \(-B_I\), and hence coincides with the image of the generator of \( H^2_G(G) \cong \mathbb{Z} \). It follows that \( DD_G(A) \) is a generator of \( H^2_G(G) \).
4. Conjuncty classes

As is well-known, coadjoint orbits \( O \subset \mathfrak{g}^* \) carry a distinguished invariant complex structure, hence a Spin\(_c\)-structure. If \( O \) admits a pre-quantum line bundle \( L \rightarrow O \) (i.e. a line bundle with curvature equal to the symplectic form), one may twist the original Spin\(_c\)-structure by this line bundle. The resulting equivariant index is the irreducible representation parametrized by \( O \). In this Section, we will describe a similar picture for conjugacy classes \( \mathcal{C} \subset G \).

4.1. Pull-back to conjugacy classes. Given \( \xi \in \Delta \), define a \( G \)-equivariant map \( \Psi : G/T \rightarrow G \), \( gT \mapsto \text{Ad}_g(\exp \xi) \). The pull-back \( \Psi^*A \) admits a canonical Morita trivialization, defined by the Hilbert space bundle \( G \times_T \mathcal{H} \). More generally, for any \( l \in \mathbb{Z} \) and any weight \( \mu \in \Lambda^* \) there is a Morita trivialization,

\[
\mathbb{C} \simeq_{\xi} \Psi^*A_l, \quad \mathcal{E} = G \times_T (\mathcal{H}^l \otimes \mathbb{C}_\mu)
\]

where \( \mathbb{C}_\mu \) is the 1-dimensional 1-dimensional \( T \)-representation of weight \( \mu \). Equivariant Dixmier-Douady bundles over \( G \), together with Morita trivializations of their pull-backs by \( \Psi \), are classified by the relative cohomology group \( H^3_{G/\mathbb{C}}(\Psi) \). (See Appendix A.) The map \( \Psi =: \Psi_1 \) is equivariantly homotopic to the constant map \( \Psi_0 : gT \mapsto e \), by the homotopy \( \Psi_t(gT) = \exp(t \text{Ad}_g(\xi)) \). Hence \( H^3_{G}(\Psi) = H^3_{G}(\Psi_0) = H^3_{G}(G/T) \oplus H^3_{G}(G) \). Identifying \( H^3_{G}(G/T) = H^3_{T}(\text{pt}) = \Lambda^* \) and \( H^3_{G}(G) = \mathbb{Z} \), we obtain an isomorphism

\[
H^3_{G}(\Psi) = \Lambda^* \oplus \mathbb{Z},
\]

The element \( (\mu, l) \in H^3_{G}(\Psi) \) is realized by the Morita trivialization \( (21) \).

Now let \( \mathcal{C} \) be the conjugacy class of \( \exp(\xi) \), and \( \Phi : \mathcal{C} \rightarrow G \) the inclusion. Let \( \pi : G/T \rightarrow \mathcal{C} \) be the \( G \)-invariant projection such that \( \Psi = \Phi \circ \pi \). We obtain a map of long exact sequences in relative cohomology,

\[
\cdots \rightarrow 0 \rightarrow H^2_{G}(\mathcal{C}) \rightarrow H^2_{G}(\Phi) \rightarrow H^2_{G}(G) \rightarrow H^2_{G}(\mathcal{C}) \rightarrow \cdots
\]

From the identifications \( H^2_{G}(\mathcal{C}) = \text{Hom}(G_{\exp \xi}, U(1)) \) and \( H^2_{G}(G/T) = \text{Hom}(T, U(1)) \), it is evident that the second vertical map is injective. Hence the 5-Lemma implies that the map \( H^2_{G}(\Phi) \rightarrow H^2_{G}(\Psi) \) is injective. Hence we obtain an injective map,

\[
H^2_{G}(\Phi) \rightarrow H^2_{G}(\Psi) = \Lambda^* \oplus \mathbb{Z}.
\]

By a parallel discussion with real coefficients, there is an injective map \( H^3_{G}(\Phi, \mathbb{R}) \rightarrow H^3_{G}(\Psi, \mathbb{R}) = t^* \oplus \mathbb{R} \).

4.2. Pre-quantization of conjugacy classes. We return to Cartan’s de Rham model for \( H^*_{G}(M, \mathbb{R}) \) (cf. the proof of Proposition 3.1) with \( \eta_G \in \Omega^3_{G}(G) \) representing the generator of \( H^3_{G}(G) \). The conjugacy class \( \mathcal{C} \) carries a unique invariant 2-form \( \omega \in \Omega^2(\mathcal{C})^G \subset \Omega^2_{G}(\mathcal{C}) \) with the property \( [1, 18] \),

\[
d_G \omega = \Phi^* \eta_G.
\]
The triple \((\mathcal{C}, \omega, \Phi)\) is an example of a quasi-Hamiltonian \(G\)-space in the terminology of [1]. Equation (22) together with \(d_G\eta_G = 0\) say that \((\omega, \eta_G) \in \Omega^3_G(\Phi)\) is a relative equivariant cocycle. Let \([(\omega, \eta_G)]\) be its class in \(H^3_G(\Phi, \mathbb{R})\).

**Lemma 4.1.** The inclusion \(H^3_G(\Phi, \mathbb{R}) \to \mathfrak{t}^* \oplus \mathbb{R}\) takes the class \([(\omega, \eta_G)]\) to the element \((B^3(\xi), 1)\).

**Proof.** Let \(h_t: \Omega^*_G(G) \to \Omega^*_{G^{-1}}(G/T)\) be the homotopy operator defined by homotopy \(\Psi_t\). Thus \(d \circ h_t + h_t \circ d = \Psi^*_t - \Psi^*_0\). Then

\[
\Omega^*_G(\Psi_t) \to \Omega^*_G(\Psi_0), \quad (\alpha, \beta) \mapsto (\alpha - h_t(\beta), \beta)
\]

is an isomorphism of chain complexes, inducing the isomorphism \(H^*_G(\Psi_t, \mathbb{R}) \to H^*_G(\Psi_0, \mathbb{R})\). In particular, the isomorphism \(H^3_G(\Psi_t, \mathbb{R}) \to H^3_G(\Psi_0, \mathbb{R})\) takes \([(\omega, \eta_G)]\) to \([(\omega - h^*_t \eta_G; \eta_G)]\).

The family of maps \(\Psi_t\) is a composition of the map \(f: G/T \to \mathfrak{g}, \ gT \mapsto \text{Ad}_g(\xi)\) with the family of maps \(g \to G, \ \zeta \mapsto \exp(t\zeta)\). Let \(j_1: \Omega^*_G(G) \to \Omega^*_{G^{-1}}(\mathfrak{g})\) be the homotopy operator for the second family of maps. Then \(h_t = f^* \circ j_1\). By [28], we have \(j_1 \eta_G = \Psi_G\), where \(\Psi_G \in \Omega^2_G(\mathfrak{g})\) is of the form \(\Psi_G(\xi) = \|\xi\| - B(\xi, \cdot)\). It follows that the image of \([(\omega, \eta_G)]\) under the map to \(\mathfrak{t}^* \oplus \mathbb{R}\) is \((B^3(\xi), 1)\).

As a special case of pre-quantization of group-valued moment maps [2], we define:

**Definition 4.2.** A level \(k \in \mathbb{Z}\) pre-quantization of a conjugacy class \(\mathcal{C}\) is a lift of the class \(k [(\omega, \eta_G)] \in H^3_G(\Phi, \mathbb{R})\) to an integral class.

By the long exact sequence in relative cohomology, if \(\mathcal{C}\) admits a level \(k\) pre-quantization, then the latter is unique (since \(H^3_G(\mathcal{C})\) has no torsion).

**Proposition 4.3.** The conjugacy class \(\mathcal{C}\) of the element \(\exp \xi\) with \(\xi \in \Delta\) admits a pre-quantization at level \(k\) if and only if \((B^k(\xi), k) \in \Lambda^* \times \mathbb{Z}\).

**Proof.** According to the Lemma, \(k [(\omega, \eta_G)]\) maps to \((B^k(\xi), k) \in \mathfrak{t}^* \oplus \mathbb{R}\). Since all maps in the commutative diagram

\[
\begin{array}{ccc}
H^3_G(\Phi) & \longrightarrow & \Lambda^* \oplus \mathbb{Z} \\
\downarrow & & \downarrow \\
H^3_G(\Phi, \mathbb{R}) & \longrightarrow & \mathfrak{t}^* \oplus \mathbb{R}
\end{array}
\]

are injective, it follows that \(k [(\omega, \eta_G)]\) is integral if and only if \((B^k(\xi), k) \in \Lambda^* \times \mathbb{Z}\). \(\square\)

Geometrically, a level \(k\) pre-quantization is given by a \(G\)-equivariant Morita trivialization of \(\Phi^* A^k\). This can be seen explicitly, as follows.

**Lemma 4.4.** Let \(\xi \in \Delta_I\), and suppose that \((B^k(\xi)) \in \Lambda^*\). Then the \(k\)-th power of the central extension of \(G_I\) admits a unique trivialization \(G_I \to \tilde{G}^{(k)}_I\) extending the map

\[
T \to \tilde{T}^{(k)} = T \times U(1), \ h \mapsto (h, h^{B^k(\xi)}).
\]

**Proof.** By \(G_I\)-equivariance, a trivialization \(G_I \to \tilde{G}^{(k)}_I\) is uniquely determined by its restriction to \(T\). For existence, recall that \(t = \exp \xi\) determines an identification \(\tilde{G}_I \cong \tilde{G}_{I,(t)} = \tilde{G}_I \times \pi_1(G_I) U(1)\), using the homomorphism \(g_{t,I}: \pi_1(G_I) = \Lambda/\Lambda_I \to U(1), \ \lambda + \Lambda_I \mapsto t^{-B^k(\lambda)}\). The powers
are obtained similarly, using the $l$-th powers of the homomorphism $\varrho_{t,I}$. Since $B^\psi(k\xi)$ is a weight, we have

$$(\varrho_{t,I})^k(\lambda + \Lambda_I) = e^{-2\pi \sqrt{-1}B^\psi(k\xi,\lambda)} = 1.$$  

This defines a trivialization,

$$\hat{G}^{(k)}_I \cong \hat{G}^{(k)}_{I,(t)} = G_I \times U(1).$$  

By (19), this trivialization intertwines the standard inclusion $\hat{T}^{(k)}_I \to \hat{G}^{(k)}_I$ with the map

$$\hat{T} = T \times U(1) \to G_I \times U(1), \ (h,z) \mapsto (h,h^{-B^\psi(k\xi)}z).$$  

The composition of this map with (23) is $h \mapsto (h,1)$, as required. \hfill $\Box$

Let $\Phi : C \to G$ be the conjugacy class of $t = \exp \xi$, and let $I$ be the unique index set such that $\xi$ lies in the relative interior of $\Delta_I$. If $C$ is pre-quantizable at level $k$, so that $B^\psi(k\xi) \in \Lambda^*$, the Lemma defines a trivialization of $G^{(k)}_I$. Hence, its action on $H^k$ descends to an action of $G_I$, and the Hilbert bundle $\mathcal{E} = G \times_{G_I} \mathcal{H}^k$ defines a Morita trivialization of $\Phi^*A^k$.

**Proposition 4.5.** The relative Dixmier-Douady class $DD_G(A^k,\mathcal{E}) \in H^3_G(C)$ (cf. Appendix A) is an integral lift of the class $k[(\omega,\eta_G)] \in H^3_G(G,C,\mathbb{R})$.

**Proof.** We have to show that the image of $DD_G(A,\mathcal{E})$ in $H^3_G(C) = \Lambda^* \oplus \mathbb{Z}$ is $(B^\psi(k\xi),k)$. But this follows from the discussion in the last Section, since the pull-back of $\mathcal{E}$ under the map $\pi : G/T \to C$ is

$$\pi^*\mathcal{E} = G \times_T (\mathcal{H}^k \otimes \mathbb{C}_{B^\psi(k\xi)}).$$ \hfill $\Box$

4.3. The $h^\vee$-th power of the Dixmier-Douady bundle. For any coadjoint orbit $O \subset g^*$, the compatible complex structure defines a $G$-invariant Spin$^c$-structure, i.e. Morita trivialization of $Cl(TO)$. We show that similarly, for all conjugacy classes $C \subset G$, there is a distinguished Morita isomorphism between $Cl(TC)$ and $A^{h^\vee}|_C$, where $h^\vee$ is the dual Coxeter number. That is, conjugacy classes carry a canonical ‘twisted Spin$^c$-structure’. There are examples of conjugacy classes that do not admit Spin$^c$-structures let alone almost complex structures.

**Example 4.6.** The simplest example of a conjugacy class not admitting an almost complex structure is the conjugacy class $C \cong \text{Spin}(5)/\text{Spin}(4) \cong S^4$ of the group $\text{Spin}(5)$. (Its image in $\text{SO}(5)$ is the conjugacy class of the matrix with entries $(-1,-1,-1,1,1)$ down the diagonal.) Similarly, the group $G = \text{Spin}(9)$ has a conjugacy class $G/H$ with $H = (\text{SU}(2) \times \text{Spin}(6))/\mathbb{Z}_2$ that does not admit a Spin$^c$-structure. Indeed, if such a Spin$^c$-structure existed it could be made $G$-equivariant (since $G$ is simply connected), hence it would give an $H$-invariant Spin$^c$-structure on $g/\mathfrak{h}$. Since $H$ is semi-simple, this is equivalent to the condition that the half-sum of positive roots of $H$, is a weight of $H$. But by explicit calculation, one checks that this is not the case. I thank Reyer Sjamaar for discussion of these and similar examples.

We will need some additional notation. Let $\mathfrak{S}_0 = \{\alpha_1,\ldots,\alpha_l\}$, $l = \text{rank}(G)$, be a set of simple roots for $g$, relative to our choice of fundamental Weyl chamber. We denote by $\alpha_0 = -\alpha_{\text{max}}$ minus the highest root, and let

$$\mathfrak{S} = \mathfrak{S}_0 \cup \{\alpha_0\} = \{\alpha_0,\ldots,\alpha_l\}.$$
Thus $\Delta \subset t_+$ is the $l$-simplex cut out by the inequalities $\langle \alpha_i, \cdot \rangle + \delta_{i,0} \geq 0$ for $i = 0, \ldots, l$, and $t_+$ is cut out by the inequalities with $i > 0$. The roots of $G_I$ are those roots $\alpha$ of $G$ for which $\langle \alpha, \xi \rangle \in \mathbb{Z}$ for $\xi \in \Delta_I$, and a set of simple roots is

$$\mathcal{G}_I = \{ \alpha_i \in \mathcal{G} | i \not\in I \}$$

That is, the Dynkin diagram of $G_I$ is obtained from the extended Dynkin diagram of $G$ by removing the vertices labeled by $i \in I$. Let $\rho$ be the half-sum of positive roots of $G$, let $\rho^\sharp = B^\sharp(\rho)$ with $B^\sharp = (B^\flat)^{-1}$, and let

$$h^\vee = 1 + \langle \alpha_{\max}, \rho^\sharp \rangle$$

be the dual Coxeter number.

**Theorem 4.7.** For any conjugacy class $\Phi: \mathcal{C} \rightarrow G$, there is a distinguished $G$-equivariant Morita isomorphism $\text{Cl}(\mathcal{C}) \simeq \Phi^* \mathcal{A}^{h^\vee}$.

**Proof.** Let $\xi \in \Delta$ be the unique point of the alcove corresponding with $\exp \xi \in \mathcal{C}$, and $I$ the index set such that $\xi \in \text{int}(\Delta_I)$. Thus $\mathcal{C} = G/G_I$ and $\text{Cl}(\mathcal{C}) = G \times_{G_I} \text{Cl}(\mathfrak{g}_I^\perp)$, where $\mathfrak{g}_I^\perp$ is the orthogonal complement of $\mathfrak{g}_I$ in $\mathfrak{g}$. By construction, $\Phi^* \mathcal{A}^{h^\vee} = G \times_{G_I} \mathbb{K}(\mathcal{H}^{h^\vee})$. Hence it is our task to construct a $G_I$-equivariant Morita isomorphism

$$\text{Cl}(\mathfrak{g}_I^\perp) \simeq \mathbb{K}(\mathcal{H}^{h^\vee})$$

Let $\tilde{G}_I'$ be the central extension of $G_I$ defined by its action on $\text{Cl}(\mathfrak{g}_I^\perp)$. It fits into a pull-back diagram,

$$\begin{array}{ccc}
\tilde{G}_I' & \longrightarrow & \text{Spin}_v(\mathfrak{g}_I^\perp) \\
\downarrow & & \downarrow \\
G_I & \longrightarrow & \text{SO}(\mathfrak{g}_I^\perp).
\end{array}$$

Equivalently, $\tilde{G}_I' = \tilde{G}_I \times_{\pi_1(G_I)} \text{U}(1)$ where $\tilde{G}_I$ is the universal covering group, and the homomorphism $\pi_1(G_I) \rightarrow \text{U}(1)$ is defined by the commutative diagram,

$$\begin{array}{cccc}
1 & \longrightarrow & \pi_1(G_I) & \longrightarrow & \tilde{G}_I & \longrightarrow & G_I & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \text{U}(1) & \longrightarrow & \text{Spin}_v(\mathfrak{g}_I^\perp) & \longrightarrow & \text{SO}(\mathfrak{g}_I^\perp) & \longrightarrow & 1
\end{array}$$

Let $\Lambda_I$ be the co-root lattice of $G_I$, so that $\pi_1(G_I) = \Lambda / \Lambda_I$. By a direct calculation (cf. Sternberg [40, Section 9.2]), the homomorphism $\pi_1(G_I) \rightarrow \text{U}(1)$ is

$$\pi_1(G_I) = \Lambda / \Lambda_I \rightarrow \text{U}(1), \; \lambda \mapsto e^{2\pi \sqrt{-1}(\rho - \rho_I, \lambda)} = \pm 1$$

where $\rho$ is the half-sum of positive roots of $G$, and $\rho_I$ is the half-sum of positive roots of $G_I$, relative to the given system $\mathcal{G}_I$ of simple roots. Let

$$\nu_I = \frac{1}{h^\vee} (\rho - \rho_I), \quad \nu_I^\sharp = B^\sharp(\nu_I).$$

The element $\nu_I^\sharp$ is contained in the the interior of the face $\Delta_I$ (see e.g. [30]). Hence, the homomorphism (24) is just the $-h^\vee$-th power of the homomorphism $\varrho_{t,I}$, $t = \exp \nu_I^\sharp$ in the
Recall that $\hat{G}_I$ acts with weight $-h^\vee$ on $\mathcal{H}^{h^\vee}$, or equivalently $\hat{G}_I^{-h^\vee}$ acts with weight 1. Hence, if $S_I$ is any spinor module over $\text{Cl}(g_I^h)$, the $\text{Cl}(g_I^h)$-$\mathbb{K}(\mathcal{H}^{h^\vee})$-bimodule

$$\text{Hom}(\mathcal{H}^{h^\vee}, S_I)$$

is $G_I$-equivariant, and gives the desired Morita isomorphism $\text{Cl}(g_I^h) \simeq \mathbb{K}(\mathcal{H}^{h^\vee})$. An explicit spinor module $S_I$ for $\text{Cl}(g_I^h)$, constructed as follows. Let $n_+ \subset g^C$ and $n_{I,+} \subset g_I^h$ be the sum of root spaces for positive roots of $G$ and $G_I$, respectively. (Here positivity is defined by the respective sets $\mathcal{S}_0, \mathcal{S}_I$ of simple roots.) Then $S = \bigwedge n_+$ is a spinor module for $\text{Cl}(t^\perp)$, and $S' = \bigwedge n_{I,+}$ is a spinor module for $\text{Cl}(g_I \cap t^\perp)$. (Cf. [40, Section 9.2].) We define

$$S_I = \text{Hom}_{\text{Cl}(g_I \cap t^\perp)}(S', S).$$

The spinor modules $S, S'$ are $T$-equivariant, since they are constructed using $T$-invariant complex structures on $t^\perp, g_I \cap t^\perp$. Hence $S_I$ is $T$-equivariant as well.

**Proposition 4.8.** Let $\mathcal{C}$ be the conjugacy class of $\exp \xi, \xi \in \Delta$. The pull-back of $\text{Cl}(TC)$ under the projection map

$$\pi: G/T \to \mathcal{C}, gT \mapsto \text{Ad}_g(\exp(\xi)).$$

admits a canonical $G$-equivariant Morita trivialization

$$\mathcal{C} \simeq \pi^* \text{Cl}(TC).$$

**Proof.** Let $I$ be the index set such that $G_I$ is the stabilizer of $\exp \xi$. We have $\pi^* \text{Cl}(TC) = \text{Cl}(\pi^*TC) = G \times_T \text{Cl}(g_I^h)$. Hence we need a $T$-equivariant Morita trivialization of $\text{Cl}(g_I^h)$, and this is provided by $S_I$. $\square$

If the conjugacy class $\mathcal{C}$ is pre-quantized at level $k$, the Morita equivalences $\text{Cl}(TC) \simeq \Phi^* \mathcal{A}^{h^\vee}$ and $\mathcal{C} \simeq \Phi^* \mathcal{A}^k$, combine to a Morita isomorphism

$$\text{Cl}(TC) \simeq \Phi^* \mathcal{A}^{k+h^\vee}$$

Recall $\Psi = \Phi \circ \pi: G/T \to G$. The composition of the Morita isomorphisms (27) and $\text{Cl}(TC) \simeq \Phi^* \mathcal{A}^{h^\vee}$ is the Morita trivialization $\mathcal{C} \simeq \Psi^* \mathcal{A}^{h^\vee}$ defined by the bundle $G \times_T \mathcal{H}^{h^\vee}$. It is thus labeled by $(0, h^\vee) \in \Lambda^* \oplus \mathbb{Z}$. Hence, in the pre-quantized case, the composition of (27) and (28) is the Morita trivialization of $\Psi^* \mathcal{A}^{k+h^\vee}$ parametrized by $(B^\Psi(k\xi), k + h^\vee) \in \Lambda^* \oplus \mathbb{Z}$.

4.4. **Freed-Hopkins-Teleman.** The twisted equivariant $K$-homology group

$$K_F^G(G, \mathcal{A}^{k+h^\vee})$$

carries a ring structure, with product given by the cross-product for $G \times G$, followed by push-forward under group multiplication $\text{Mult}: G \times G \to G$. Indeed, since $\text{Mult}^* x = \text{pr}_1^* x + \text{pr}_2^* x$ for all $x \in H_G^3(G, \mathbb{Z})$, there is a Morita isomorphism

$$\text{pr}_1^* \mathcal{A}^{k+h^\vee} \otimes \text{pr}_2^* \mathcal{A}^{k+h^\vee} \simeq \text{Mult}^* \mathcal{A}^{k+h^\vee}.$$
The Morita bimodule is unique up to equivalence since \( H^2_G(G \times G) = 0 \). It defines a product structure

\[
K^G_* (\text{Mult}) : K^G_* (G, A^{k+h^\vee}) \otimes K^G_* (G, A^{k+h^\vee}) \to K^G_* (G, A^{k+h^\vee}),
\]
given by the cross product \( K^G_* (G, A^{k+h^\vee}) \otimes K^G_* (G, A^{k+h^\vee}) \to K^G_* (G \times G, pr_1^* A^{k+h^\vee} \otimes pr_2^* A^{k+h^\vee}) \)
followed by \( K^G_* (\text{Mult}) \). The product is commutative and associative, again since the relevant Morita bimodules are unique up to equivalence. (For non-simply connected groups, the existence of a ring structures on the twisted K-homology is a much more subtle matter [42].)

The inclusion \( i : \{ e \} \rightarrow G \) of the group unit induces a ring homomorphism

\[
K^G_* (i) : R(G) = K^G_* (pt) \to K^G_* (G, A^{k+h^\vee}).
\]

**Theorem 4.9** (Freed-Hopkins-Teleman). For all non-negative integers \( k \geq 0 \) the ring homomorphism (29) is onto, with kernel the level \( k \) fusion ideal \( I_k(G) \subset R(G) \). That is, \( K^G_1 (G, A^{k+h^\vee}) = 0 \), while \( K^G_0 (G, A^{k+h^\vee}) \) is canonically isomorphic to the level \( k \) fusion ring, \( R_k(G) = R(G)/I_k(G) \).

We will explain a proof of this Theorem in Section 5. The ring \( R_k(G) \) may be defined as the ring of level \( k \) projective representations of the loop group \( LG \), or in finite-dimensional terms (cf. [3]): Let

\[
\Lambda^*_k = \Lambda^* \cap B^\vee (k\Delta)
\]
be the set of *level \( k \) weights*. Identify \( R(G) \) with ring of characters of \( G \). Then \( R_k(G) = R(G)/I_k(G) \), where \( I_k(G) \) is the vanishing ideal of the set of elements \( \{ t_\nu \in T, \nu \in \Lambda^*_k \} \) where

\[
t_\nu = \exp (B^\vee (\frac{\nu}{k+h^\vee})).
\]

It turns out that as an additive group, \( R_k(G) \) is freely generated by the images of irreducible characters \( \chi_\mu \) for \( \mu \in \Lambda^*_k \). Thus \( R_k(G) = \mathbb{Z}[\Lambda^*_k] \) additively.

**Remark 4.10.** If \( G \) has type ADE (so that all roots have equal length), the lattice \( B^\vee (\Lambda^*) \subset t \) is identified with the set of elements \( \xi \in t \) with \( \exp \xi \in Z(G) \), the center of \( G \). Hence the ideal \( I_k(G) \) may be characterized, in this case, as the vanishing ideal of the set of all \( g \in G_{\text{reg}} \) such that \( g^{k+h^\vee} \in Z(G) \).

**Remark 4.11.** Freed-Hopkins-Teleman compute twisted K-homology groups of \( G \) for arbitrary compact groups, not necessarily simply connected. The case of simple, simply connected groups considered here is considerably easier than the general case.

**Remark 4.12.** It is also very interesting to consider the non-equivariant twisted K-homology rings \( K_* (G, A^{k+h^\vee}) \). These are studied in the work of V. Braun [7] and C. Douglas [12].

### 4.5. Quantization of conjugacy classes.

Suppose \( \Phi : C \hookrightarrow G \) is the conjugacy class of \( \exp \xi, \xi \in \Delta \), pre-quantized at level \( k \geq 0 \). Thus \( \mu := B^\vee (k\xi) \) is a weight. The Morita isomorphism (28) defines a push-forward map in K-homology,

\[
K^G_* (\Phi) : K^G_0 (C, \text{Cl}(T C)) \rightarrow K^G_0 (G, A^{k+h^\vee})
\]
where \( \Phi : C \hookrightarrow G \) is the inclusion.

**Theorem 4.13.** The push-forward map (30) takes the fundamental class \( [C] \in K^G_0 (C, \text{Cl}(T C)) \) to the equivalence class of the character \( \chi_\mu \) in \( R_k(G) = R(G)/I_k(G) \).
Proof. Let \( \pi : G/T \to \mathcal{C} \) and \( \Psi = \Phi \circ \pi : G/T \to G \) be as in Section 4.1. The Morita trivializations
\[
\mathcal{C} \simeq \text{Cl}(T(G/T)), \quad \mathcal{C} \simeq \pi^* \text{Cl}(TC)
\]
defined by \( G \times_T S \) resp. \( G \times_T S_I \) (cf. Proposition 4.8) define a push-forward map
\[
K_0^G(\pi) : K_0^G(G/T, \text{Cl}(T(G/T))) \cong K_0^G(G/T) \to K_0^G(\mathcal{C}, \text{Cl}(TC))
\]
with \( K_0^G(\pi)([G/T]) = [\mathcal{C}] \). Hence
\[
K_0^G(\Phi)([\mathcal{C}]) = K_0^G(\Psi)([G/T]).
\]
Recall now that \( \Psi = \Psi_1 \) is equivariantly homotopic to the constant map \( \Psi_0 \) onto \( e \in G \). That is, the diagram
\[
\begin{array}{ccc}
G/T & \xrightarrow{\pi} & \mathcal{C} \\
\downarrow p & & \downarrow \Phi \\
\text{pt} & \xleftarrow{\iota} & G,
\end{array}
\]
commutes up to a \( G \)-equivariant homotopy. As discussed at the end of Section 4.3, the composition of the Morita isomorphisms \( \mathcal{C} \simeq \pi^* \text{Cl}(TC) \) and \( \text{Cl}(TC) \simeq \Phi^* \mathcal{A}^{k+h^\vee} \) (see Equations (27) and (28)) is the Morita trivialization,
\[
\Psi^* \mathcal{A}^{k+h^\vee} \cong \mathbb{K}(G \times_T (\mathbb{C}_\mu \otimes \mathcal{H}^{k+h^\vee})).
\]
On the other hand, \( \iota^* \mathcal{A}^{k+h^\vee} = \mathbb{K}(\mathcal{H}^{k+h^\vee}) \) by construction of \( \mathcal{A} \), hence
\[
\Psi_0^* \mathcal{A}^{k+h^\vee} \cong p^* \mathbb{K}(\mathcal{H}^{k+h^\vee}) = \mathbb{K}(G \times_T \mathcal{H}^{k+h^\vee}).
\]
The two Morita isomorphisms are thus related by a twist by the line bundle \( G \times_T \mathbb{C}_\mu \). It follows that \( K_0^G(\Psi) \) is the automorphism of \( K_0^G(G/T) \) defined by the class of the line bundle \( G \times_T \mathbb{C}_\mu \), followed by \( K_0^G(\Psi_0) = K_0^G(\iota) \circ K_0^G(p) \). But \( K_0^G(p) \) is just the equivariant index map for \( G/T \). As is well-known, it takes \( [G/T] \), twisted by \( G \times_T \mathbb{C}_\mu \), to the class \([V_\mu] \in K_0^G(\text{pt})\) of the irreducible \( G \)-representation labeled by \( \mu \). We conclude that
\[
K_0^G(\Psi)([G/T]) = K_0^G(\iota)([V_\mu]).
\]
The identification \( K_0^G(\text{pt}) \cong R(G) \) takes \([V_\mu]\) to the character \( \chi_\mu \). \( \square \)

4.6. **Twisted \( K \)-homology of the conjugacy classes.** Suppose \( \Phi : \mathcal{C} \hookrightarrow G \) is an arbitrary conjugacy class (not necessarily pre-quantized) corresponding to \( \xi \in \Delta \). Let \( I \) be the index set such that \( \xi \in \text{int}(\Delta_I) \), thus \( \mathcal{C} = G \times G/I \). Write \( \mathcal{B} = \mathbb{K}(\mathcal{H}) \) so that \( \mathcal{A}_I = G \times_{G/I} \mathcal{B} \). In 4.3 we had constructed a \( G_I \)-equivariant Morita isomorphism \( \text{Cl}(\mathfrak{g}_I^\vee) \simeq \mathcal{B}^{h^\vee} \), or equivalently (since \( \text{Cl}(\mathfrak{g}_I^\vee) \cong \text{Cl}(\mathfrak{g}_I^\vee)^{\text{opp}} \)) \( \mathcal{C} \simeq \mathcal{B}^{h^\vee} \otimes \text{Cl}(\mathfrak{g}_I^\vee) \). We have, by 2.3(a)-(c),
\[
K_q^G(\mathcal{C}, \Phi^* \mathcal{A}^{k+h^\vee}) = K_q^G(G/G_I, \ G \times_{G/I} \mathcal{B}^{k+h^\vee})
\]
\[
= K_q^G(\text{pt}, \mathcal{B}^{k+h^\vee} \otimes \text{Cl}(\mathfrak{g}_I^\vee))
\]
\[
= K_q^G(\text{pt}, \mathcal{B}^k).
\]
This vanishes for \( q = 1 \), and is equal to \( R(\hat{G}_{I}^{-k})_{-1} \) for \( q = 0 \). But a representation of \( \hat{G}_{I}^{-k} \), where the central circle acts with weight \(-1\), is the same as a representation of \( \hat{G}_{I} \) where the central circle acts with weight \( k \). Thus

\[
K_{0}^{G}(C, \Phi^{*}A^{k+h^{H}}) = K_{0}^{G}(G/G_{I}, G \times G_{I} B^{k+h^{H}}) \cong R(\hat{G}_{I})_{k}
\]
as \( R(G) \)-modules. (The module structure is given by the restriction homomorphism \( R(G) \rightarrow R(G_{I}) = R(\hat{G}_{I})_{0} \), which acts on \( R(\hat{G}_{I}) \) by multiplication.) If \( J \subset I \), we have a natural map \( \phi_{I}^{J}: G/G_{I} \rightarrow G/G_{J} \) covered by a map of Dixmier-Douady bundles \( G \times G_{I} B \rightarrow G \times G_{J} B \). Hence we obtain a push-forward map,

\[
K_{0}^{G}(\phi_{I}^{J}): K_{0}^{G}(G/G_{I}, G \times G_{I} B^{k+h^{H}}) \rightarrow K_{0}^{G}(G/G_{J}, G \times G_{J} B^{k+h^{H}})
\]
The naturality of the maps \( \mathbf{I}_{H}^{G} \) (cf. 2.3(b)) and the definition of \( \text{ind}_{I}^{J} \equiv \text{ind}_{G_{I}}^{G_{J}} \) (cf. 2.3(c)) gives a commutative diagram,

\[
\begin{array}{ccc}
K_{0}^{G}(\text{pt}, B^{k+h^{H}} \otimes \text{Cl}(g/g_{I})) & \xrightarrow{\text{ind}_{I}^{J}} & K_{0}^{G}(\text{pt}, B^{k+h^{H}} \otimes \text{Cl}(g/g_{J})) \\
\downarrow \mathbf{I}_{G_{I}}^{G_{J}} & & \downarrow = \\
K_{0}^{G}(G/G_{J}, (G \times G_{J} B^{k+h^{H}}) \otimes \text{Cl}(g/g_{J})) & \xrightarrow{} & K_{0}^{G}(\text{pt}, B^{k+h^{H}} \otimes \text{Cl}(g/g_{J})) \\
\downarrow \mathbf{I}_{G_{J}}^{G} & & \downarrow \mathbf{I}_{G_{J}}^{G} \\
K_{0}^{G}(G/G_{I}, G \times G_{I} B^{k+h^{H}}) & \xrightarrow{K_{0}^{G}(\phi_{I}^{J})} & K_{0}^{G}(G/G_{J}, G \times G_{J} B^{k+h^{H}})
\end{array}
\]

That is, \( K_{0}^{G}(\phi_{I}^{J}) \circ \mathbf{I}_{G_{I}}^{G_{J}} = \mathbf{I}_{G_{J}}^{G_{I}} \circ \text{ind}_{I}^{J} \). The entries on the top row are identified with \( R(\hat{G}_{I})_{k} \) and \( R(\hat{G}_{J})_{k} \), and (cf. 2.3(c)) the map \( \text{ind}_{I}^{J} \) is the holomorphic induction map

\[
\text{ind}_{I}^{J}: R(\hat{G}_{I})_{k} \rightarrow R(\hat{G}_{J})_{k},
\]
relative to the complex structure on \( G_{J}/G_{I} = \hat{G}_{J}/\hat{G}_{I} \) defined by the collections of simple roots \( \mathcal{G}_{J} \subset \mathcal{G}_{I} \). To summarize,

**Proposition 4.14.** The identifications \( K_{0}^{G}(G/G_{I}, G \times G_{I} B^{k+h^{H}}) \cong R(\hat{G}_{I})_{k} \) intertwine the push-forward maps \( K_{0}^{G}(\phi_{I}^{J}) \) with the holomorphic induction maps \( \text{ind}_{I}^{J} \).

### 5. Computation of \( K_{•}^{G}(G, A^{k+h^{H}}) \)

The Dixmier-Douady bundle \( A \rightarrow G \), as described in (20), may be viewed as the geometric realization of a co-simplicial Dixmier-Douady bundle, with non-degenerate \( p \)-simplices the bundle \( \coprod_{|I|=p+1} A_{I} \) over \( \coprod_{|I|=p+1} G/G_{I} \). This defines a spectral sequence computing the \( K \)-homology group \( K_{•}^{G}(A^{k+h^{H}}) \), in terms of the known \( K \)-homology groups \( K_{•}^{G}(G/G_{I}, A_{I}^{k+h^{H}}) = R(\hat{G}_{I})_{k} \) and the holomorphic induction maps between these groups. As it turns out, the spectral sequence collapses at the \( E_{2} \)-stage, and computes the level \( k \) fusion ring.
5.1. The spectral sequence for $K^G_\bullet(G, \mathcal{A}^{k+h'})$. The construction (20) of $A \to G$ as a quotient of $\coprod J \mathcal{A}^l \times \Delta_l \to \coprod J G/G_I \times \Delta_I$ may be thought of as the geometric realization of a ‘co-simplicial Dixmier-Douady bundle’. See [37] and [31] for background on co-simplicial (semi-simplicial) techniques. Here the $G$-Dixmier-Douady bundles

$$\coprod_{|I|=p+1} \mathcal{A}_I \to \coprod_{|I|=p+1} G/G_I$$

are the non-degenerate $p$-simplices; the full set of $p$-simplices is a union $\coprod_J \mathcal{A}_{f([p])} \to \coprod_J G/G_{f([p])}$ over all non-decreasing maps $f: [p] = \{0, \ldots, p\} \to \{0, \ldots, l\}$. By the theory of co-simplicial spaces (see [37, Section 5]), one obtains a spectral sequence $E^1_{p,q} = K^G_{p+q}(G, \mathcal{A}^{k+h'})$ where

$$(34) \quad E^1_{p,q} = \bigoplus_{|I|=p+1} K^G_q(G/G_I, \mathcal{A}^{k+h'}_I).$$

The differential $d^1: E^1_{p,q} \to E^1_{p-1,q}$ is given on $K^G_q(G/G_I, \mathcal{A}^{k+h'}_I)$ as an alternating sum,

$$d^1 = \sum_{r=0}^p (-1)^r K^G_{\delta(I)}(\phi^r_I).$$

Here $\delta_r I$ is obtained from $I$ by omitting the $r$-th entry: $\delta_r I = \{i_0, \ldots, i_r, \ldots, i_p\}$ for $I = \{i_0, \ldots, i_p\}$ with $i_0 < \cdots < i_p$. Recall that $\phi^r_I: G/G_I \to G/G_J$ are the natural maps for $J \subset I$.

By mod 2 periodicity of the $K$-homology, we have $E^1_{p,q} = E^1_{p,q+2}$. Since the groups $G_I$ are connected, and since $\dim G/G_I$ is even, one has $K^G_1(G/G_I, \mathcal{A}^{k+h'}_I) = 0$, thus $E^1_{\bullet,1} = 0$. Hence, the $E^1$-term is described by a single chain complex $(C_\bullet, \partial)$, where

$$C_p = E^1_{p,0}, \quad \partial = d^1.$$

The map $R(G) \to K^G_\bullet(G, \mathcal{A}^{k+h'})$ defined by the inclusion $\iota: e \hookrightarrow G$ may be also be described by the spectral sequence. Think of $\iota$ as the geometric realization of a map of co-simplicial manifolds, given as the inclusion of $\{e\} = G/G_{\{0\}}$ into $\coprod_{i=0}^l G/G_{\{i\}}$. The co-simplicial map gives rises to a morphism of spectral sequences, $E^\bullet \to E^\bullet$, where

$$\tilde{E}^1_{p,q} = \begin{cases} K^G_q(pt, \mathbb{C}) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

At the $E^1$-stage, this boils down to a chain map

$$(35) \quad R(G) \to C_\bullet$$

where $R(G) = \tilde{E}^1_{0,0}$ carries the zero differential. Our goal is to show that the homology of $(C_\bullet, \partial)$ vanishes in positive degrees, while the induced map in homology $R(G) \to H_0(C, \partial)$ is onto, with kernel $I_k(G)$.

5.2. The induction maps in terms of weights. To get started, we express the chain complex in terms of weights of representations. Recall that $R(T)$ is isomorphic to the group ring $\mathbb{Z}[\Lambda^*]$. The restriction map $R(G) \to R(T)$ is injective, and identifies

$$R(G) \cong \mathbb{Z}[\Lambda^*]^W.$$
Let us next describe $R(\widehat{G}_I)_k$ in terms of weights. Each $\widehat{G}_I$ has maximal torus $\widehat{T} = T \times U(1)$, hence the weight lattice is
\[ \Lambda^* = \Lambda^* \times \mathbb{Z} \subset \mathfrak{t}^* = \mathfrak{t}^* \times \mathbb{R}. \]
The simple roots for $\widehat{G}_I$ are $(\alpha_i, 0)$ with $\alpha_i \in \mathfrak{g}_I$, the corresponding co-roots are
\[ (\alpha^*_i, \delta_{i,0}) \in \mathfrak{t} = \mathfrak{t} \times \mathbb{R}, \quad \alpha_i \in \mathfrak{g}_I. \]
These define a fundamental Weyl chamber
\[ \mathfrak{t}^*_{I,+} = \{ (\nu, s) \mid \langle \nu, \alpha^*_i \rangle + s \delta_{i,0} \geq 0, \quad \alpha_i \in \mathfrak{g}_I \} \]
The elements $\nu_I$ satisfy $\langle \nu_I, \alpha^*_I \rangle + \delta_{i,0} = 0$. Hence, $(\nu, s) \in \mathfrak{t}^*_{I,+}$ is and only if $\nu - s \nu_I \in \mathfrak{t}^*_I$.

Let $\Lambda^*_{I,k} \subset \Lambda^*$ be the intersection of (37) with $\Lambda^*_\times \{ k \} \cong \Lambda^*$. Thus
\[ \Lambda^*_{I,k} = \{ \nu \in \Lambda^* \mid \langle \nu, \alpha^*_I \rangle + k \delta_{i,0} \geq 0, \quad i \not\in I \} \]
labels the irreducible $\widehat{G}_I$-representations for which the central circle acts with weight $k$. The Weyl group $W_I$ of $G_I$ is also the Weyl group of $\widehat{G}_I$. Its action on $\widehat{\Lambda}^*$ preserves the levels $\Lambda^*_\times \{ k \}$, hence it takes the form $w.(\nu, k) = (w \cdot_k \nu, k)$ for a level $k$-action $\nu \mapsto w \cdot_k \nu$ on $\Lambda^*$. Explicitly,
\[ w \cdot_k \nu = w(\nu - k \nu_I) + k \nu_I. \]
Fix $k$, and denote by $\mathbb{Z}[\Lambda^*]^{W_I-\text{as}}$ the anti-invariant part for the $W_I$-action $\nu \mapsto w \cdot_k \nu$ at the shifted level $k + h^\vee$. Observe that this space is invariant under the action of $\mathbb{Z}[\Lambda^*]^W$. Let
\[ \text{Sk}^I : \mathbb{Z}[\Lambda^*] \to \mathbb{Z}[\Lambda^*]^{W_I-\text{as}}, \quad \nu \mapsto \sum_{w \in W_I} (-1)^{\text{length}(w)} w \cdot_k \nu \]
denote skew-symmetrization relative to the action at level $k + h^\vee$. For $\mu \in \Lambda^*_k$, let $\chi^I_k(\mu) \in R(\widehat{G}_I)_k$ be the character of the irreducible $\widehat{G}_I$-representation of weight $(\mu, k)$.

**Lemma 5.1.** The map $\chi^I_k \mapsto \text{Sk}^I(\mu + \rho)$ extends to an isomorphism
\[ R(\widehat{G}_I)_k \to \mathbb{Z}[\Lambda^*]^{W_I-\text{as}}. \]
Under this isomorphism, the $R(G) \cong \mathbb{Z}[\Lambda^*]^W$-module structure is given by multiplication in the group ring. Furthermore, the identification (39) intertwines the holomorphic induction maps $\text{ind}^I_J : R(\widehat{G}_I)_k \to R(\widehat{G}_J)_k$ for $J \subset I$ with skew-symmetrizations
\[ \text{Sk}^I = \frac{1}{|W_I|} \text{Sk}^J : \mathbb{Z}[\Lambda^*]^{W_I-\text{as}} \to \mathbb{Z}[\Lambda^*]^{W_J-\text{as}}. \]

Note that the statement involves a shift by $\rho$, rather than $\rho_I$. Thus, even in the case $I = \{ 0, \ldots, I \}$ where $G_I = T$ and $W_I = \{ 1 \}$, $\rho_I = 0$, the identification $R(\widehat{T})_k \to \mathbb{Z}[\Lambda^*]$ involves a $\rho$-shift.

**Proof.** Let $\Lambda^*_{I,k+h^\vee,\text{reg}}$ be the intersection of $\Lambda^* \times \{ k+h^\vee \}$ with $\text{int}(\mathfrak{t}^*_{I,+})$. Since obviously $R(\widehat{G}_I)_k = \mathbb{Z}[\Lambda^*_{I,k}]$, the first part of the Lemma amounts to the assertion that
\[ \mu \in \Lambda^*_{I,k} \iff \mu + \rho \in \Lambda^*_{I,k+h^\vee,\text{reg}}. \]
We have $\mu \in \Lambda^*_{I,k}$ if and only if $\langle \mu, \alpha^*_I \rangle + k \delta_{i,0} \geq 0$ for $i \not\in I$. Since $\langle \rho, \alpha^*_I \rangle + h^\vee \delta_{i,0} = 1$ this is equivalent to $\langle \mu + \rho, \alpha^*_I \rangle + (k+h^\vee) \delta_{i,0} \geq 0$, $i \not\in I$, i.e. $\mu + \rho \in \Lambda^*_{I,k+h^\vee,\text{reg}}$ as claimed. The assertion
about the $R(G)$-module structure is obvious. Finally, for $J \subset I$ the holomorphic induction map $\text{ind}^J_I$ is given by
\[
\text{ind}^J_I(\chi^J_k) = (-1)^{\text{length}(w)} \chi^J_{w \cdot k(w + \rho_J) - \rho_J}
\]
if there exists $w \in W_J$ with $w \cdot k(w + \rho_J) - \rho_J \in \Lambda^*_J, k$; while $\text{ind}^J_I(\chi^J_k) = 0$ if there is no such $w$. Using (38) together with $\rho_I - k\nu_I = \rho - (k + h^\vee)\nu_I$ (by the definition of $\nu_I$), this may be re-written in terms of the action at level $k + h^\vee$:
\[
w \cdot k(w + \rho_J) - \rho_J = w \cdot k + h^\vee (w + \rho) - \rho.
\]

By combining this discussion with Proposition 4.14, we have established a commutative diagram
\[
\begin{array}{ccc}
K_0^G(G/G_J, A^k_{J+h^\vee}) & \xrightarrow{\cong} & R(\hat{G}_J)_k & \xrightarrow{\cong} & Z[\Lambda^*_J]^{W_{J-as}}
\end{array}
\]
(40)

We can thus re-express the chain complex $(C_\bullet, \partial)$ in terms of weights:
\[
C_p = \bigoplus_{|I| = p+1} Z[\Lambda^*_I]^{W_{I-as}}, \quad \partial \phi^I = \sum_{r=0}^p (-1)^r \text{Sk}_r^\delta I(\phi^I),
\]
(41)

for $\phi^I \in Z[\Lambda^*_I]^{W_{I-as}}$. The map $R(G) \to C_0 \subset C_\bullet$ given by (35) is expressed as the inclusion of $Z[\Lambda^*_I]^{W_{I-as}}$, as the summand corresponding to $I = \emptyset$. By construction, $C_\bullet$ is a complex of $R(G)$-modules, and the map (35) is an $R(G)$-module homomorphism.

5.3. **Fusion ring.** Let us also describe the fusion ring in terms of weights. The subset $B^\delta(k\Delta) \subset t^*$ defining the set $\Lambda^*_k = \Lambda^* \cap B^\delta(k\Delta)$ of level $k$ weights is cut out by the inequalities
\[
\langle \nu, \alpha^\vee_i \rangle + k\delta_i,0 \geq 0.
\]

It is a fundamental domain for the level $k$ action $\nu \mapsto w \cdot_k \nu$ of the affine Weyl group, generated by the simple affine reflections
\[
\nu \mapsto \nu - \langle \nu, \alpha^\vee_i \rangle + s\delta_i,0 \alpha_i, \quad i = 0, \ldots, l.
\]

This is consistent with our earlier notation: The level $k$ action of $W_{\text{aff}}$ restricts to the level $k$ action of the subgroup $W_I$, generated by the affine reflections with $i \notin I$.

Let $Z[\Lambda^*_k]$ be the $Z[\Lambda^*_k]$-module consisting of all functions $\Lambda^*_k \to Z$, not necessarily of finite support. Let
\[
\text{Sk}_{\text{aff}} : Z[\Lambda^*] \to Z[[\Lambda^*]]^{W_{\text{aff-as}}}, \quad \nu \mapsto \sum_{w \in W_{\text{aff}}} (-1)^{\text{length}(w)} w \cdot_{k + h^\vee} \nu
\]
be skew-symmetrization, using the action at the shifted level $k + h^\vee$. The map $\mu \mapsto \text{Sk}_{\text{aff}}(\mu + \rho)$ extends to an isomorphism, $Z[\Lambda^*_k] \to Z[[\Lambda^*_k]]^{W_{\text{aff-as}}}$. This identifies
\[
R_k(G) \cong Z[[\Lambda^*_k]]^{W_{\text{aff-as}}}
\]
(42)
as an Abelian group. For any $I$ we have $R(G) = \mathbb{Z}[[\Lambda^*]]^{W}$-module homomorphisms $R(\hat{G}_I)_k \to R_k(G)$,

$$Z[\Lambda^*]^{W-\text{as}} \to \mathbb{Z}[\Lambda^*]^{W_{\text{aff},-\text{as}}}, \quad \phi_I \mapsto \frac{1}{|W_I|} \text{Sk}_{\text{aff}} \phi_I.$$  

For $I = \{0\}$ we may use the obvious trivialization $\hat{G} = G \times U(1)$ to identify $R(G) = R(\hat{G}_0)_k$.

The following is clear from the description of the quotient map $R(G) \to R_k(G)$ (see e.g. [3]):

**Lemma 5.2.** The identifications $R(G) = \mathbb{Z}[[\Lambda^*]]^{W-\text{as}}$ and (42) intertwine the quotient map $R(G) \to R_k(G)$ with the skew-symmetrization map,

$$\frac{1}{|W|} \text{Sk}_{\text{aff}} : Z[\Lambda]^{W-\text{as}} \to Z[[\Lambda^*]]^{W_{\text{aff},-\text{as}}}.$$  

In particular, (42) is an isomorphism of $R(G) \cong Z[\Lambda^*]^{W}$-modules.

In fact, we could define the ideal $I_k(G) \subset R(G)$ as the kernel of the map (44). Let $\epsilon : C_0 \to R_k(G)$ be the direct sum of the morphisms (43) for $|I| = 1$.

5.4. **A resolution of the $R(G)$-module $R_k(G)$.**

**Theorem 5.3.** For all $k \geq 0$ the chain complex $(C_\bullet, \partial)$ defines a resolution

$$0 \to C_l \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} R_k(G) \to 0$$

of $R_k(G)$ as an $R(G)$-module.

The proof will be given below. As mentioned in the introduction, Theorem 5.3 is implicit in the work of Kitchloo-Morava [25].

**Remark 5.4.** It turns out that the twisted representations $R(\hat{G}_I)_k$ are projective modules over $R(G)$, hence (by the Quillen-Suslin theorem) free modules over $R(G)$. That is, $(C_\bullet, \partial)$ is a free resolution of the $R(G)$-module $R_k(G)$. If $\hat{G}_I^{(k)} \cong G_I \times U(1)$, the $R(G)$ module $R(\hat{G}_I)_k$ is isomorphic to $R(G_I)$, and the claim follows from the Pittie-Steinberg theorem [33, 39]. The general case requires a mild generalization of the Pittie-Steinberg theorem [29].

**Remark 5.5.** Theorem 5.3 implies the Freed-Hopkins-Teleman theorem (1): By acyclicity of the chain complex $C_\bullet$ the spectral sequence $E^2$ collapses at the $E^2$-term, with

$$E^2_{p,q} = E^\infty_{p,q} = \begin{cases} R_k(G) & \text{if } p = 0 \text{ and } q \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Since $R_k(G)$ is free Abelian as a $\mathbb{Z}$-module, there are no extension problems and we conclude $K^G_1(G, \mathcal{A}^{k+h'}) = 0$, while

$$K^G_0(G, \mathcal{A}^{k+h'}) = R_k(G)$$

as modules over $R(G)$. This isomorphism takes the ring homomorphism $R(G) \to K^G_0(G, \mathcal{A}^{k+h'})$ to the quotient map $R(G) \to R_k(G)$, hence (45) is an isomorphism of rings.
The statement of Theorem 5.3 can be simplified. Indeed, the chain complex $C_\bullet$ breaks up as a direct sum of sub-complexes $C_\bullet(\mu)$, $\mu \in \Lambda^*_k$, given as

$$C_p(\mu) = \bigoplus_{|I|=p+1} Z[W_{\text{aff}} \bullet_{k+h^\vee} \mu]^W_{I-as}.$$ 

Similarly the map $\epsilon: C_0 \rightarrow R_k(G)$ splits into a direct sum of maps

$$\epsilon: C_0(\mu) \rightarrow Z[W_{\text{aff}} \bullet_{k+h^\vee} \mu]^W_{-as} = \begin{cases} Z & \text{for } \mu \in \Lambda^*_{k+h^\vee} \\ 0 & \text{otherwise.} \end{cases}$$

Finally the chain map $R(G) \hookrightarrow C_\bullet$ splits into inclusions of $Z[W_{\text{aff}} \bullet_{k+h^\vee} \mu]^W_{-as}$ as the term corresponding to $I = \{0\}$. Clearly, $(C_\bullet(\mu), \partial)$ depends only on the open face $B^\bullet((k + h^\vee)\Delta_f)$ of $B^\bullet((k + h^\vee)\Delta)$ containing $\mu$. Indeed, since $Z[W_{\text{aff}} \bullet_{k+h^\vee} \mu] = Z[W_{\text{aff}}/W_J]$ we have

$$C_p(J) = \bigoplus_{|I|=p+1} Z[W_{\text{aff}}/W_J]^W_{I-as}.$$ 

The differential $\partial$ is again given by anti-symmetrization as in (41), but with $\phi^J$ now an element of $Z[W_{\text{aff}}/W_J]^W_{I-as}$. The map $\epsilon: C_0 \rightarrow R_k(G)$ translates into the zero map $C_0(J) \rightarrow 0$ unless $J = \{0, \ldots, l\}$, in which case it becomes a map $\epsilon: C_0(J) \rightarrow Z$, given as the direct sum for $i = 0, \ldots, l$ of the maps,

$$Z[W_{\text{aff}}]_{I-as} \rightarrow Z, \quad \sum_w n_w w \mapsto \sum_W n_w (-1)^{\text{length}(w)}.$$ 

The map $R(G) \rightarrow C_\bullet$ is again the inclusion of the summand of $C_0(J)$ corresponding to $I = \{0\}$.

Theorem 5.3 is now reduced to the following simpler statement:

**Theorem 5.6.** The homology $H_\bullet(J)$ of the chain complex $C_\bullet(J)$ vanishes in degree $p > 0$, while

$$H_0(J) = \begin{cases} 0 & \text{if } J \neq \{0, \ldots, l\} \\ Z & \text{if } J = \{0, \ldots, l\} \end{cases}$$

In the second case, the isomorphism is induced by the augmentation map $\epsilon: C_0(J) \rightarrow Z$.

5.5. **Proof of Theorem 5.6.** Throughout this Section, we consider a given face $\Delta_f$ of the alcove. We may think of $W_{\text{aff}}/W_J$ as the $W_{\text{aff}}$-orbit of a point in the interior of the face $\Delta_f$, under the standard action of $W_{\text{aff}}$ on $t$. To be concrete, let us take the point $\nu^+_f$. Denote its orbit by

$$V = W_{\text{aff}}.\nu^+_f \subset t.$$ 

We introduce a length function $\text{length}: V \rightarrow Z$, defined in terms of the function on $W_{\text{aff}}$ as

$$\text{length}(x) = \min\{\text{length}(w)|w \in W_{\text{aff}}, \ x = w.\nu^+_f\}, \quad x \in V.$$ 

Geometrically, $\text{length}(x)$ is the number of affine root hyperplanes in the Stiefel diagram, crossed by a line segment from the a point in the interior of $\Delta$ to the point $x$.

For any $I$ let $t_{I,+}$ be defined by the inequalities $\langle \alpha_i, \cdot \rangle + \delta_{i,0} \geq 0$ for $\alpha_i \in \mathcal{S}_I$. (Equivalently, it is the affine cone over $\Delta$ at $\nu^+_I$.) Then $t_{I,+}$ is a fundamental domain for the $W_I$-action. Let
$V^I \subset \nabla^I \subset V$ be the subsets,

$$V^I = V \cap \text{int}(t_{I,+}), \quad \nabla^I = V \cap t_{I,+}.$$ 

Every $W_I \subset W_{\text{aff}}$-orbit contains a unique point in $\nabla^I$. Thus, if $x \in V$, we may choose $u \in W_I$ with $u.x \in \nabla^I$. Then

$$\text{length}(u.x) \leq \text{length}(x),$$

with equality if and only if $x \in \nabla^I$ and hence $u.x = x$.

The elements

$$(46) \quad \beta_I(x) = \text{Sk}^I(x), \quad x \in V^I$$

form a basis of the $\mathbb{Z}$-module $\mathbb{Z}[V]^{W_1 - \text{as}}$. (Note that if $x \in \nabla^I \setminus V^I$ then $\text{Sk}^I(x) = 0$.) Let us describe the differential in terms of this basis. For $|I| = p + 1$ and $x \in V^I$ we have,

$$\partial \beta_I(x) = \sum_{r=0}^p (-1)^r \text{Sk}^I_r(x).$$

In general, the terms $\text{Sk}^I_r(x)$ are not standard basis elements, since $x$ need not lie in $V^\delta_r I$. Letting $u_r \in W_{\delta' I}$ be the unique element such that $u_r.x \in V^{\delta' I}$, we have

$$(47) \quad \partial \beta_I(x) = \sum_{r=0}^n (-1)^{r+\text{length}(u_r)} \beta_{\delta' I}(u_r.x).$$

5.5.1. Computation of $H_0(J)$. Consider $C_0(J) = \bigoplus_{j=0}^p \mathbb{Z}[V]^{W_j - \text{as}}$. For all $i, j$ and all $x$, the elements $\text{Sk}^I_i(x), \text{Sk}^I_j(x)$ are homologous since they differ by the boundary of $\text{Sk}^I_j(x) \in C_1(J)$. Together with $\text{Sk}^I_j(x) = (-1)^{\text{length}(w)} \text{Sk}^I_j(wx)$ for $w \in W_j$, this implies

$$\text{Sk}^I_i(x) \sim (-1)^{\text{length}(w)} \text{Sk}^I_j(wx)$$

for $w \in W_j$. Since the subgroups $W_j$ generate $W_{\text{aff}}$, this holds in fact for all $w \in W_{\text{aff}}$. Thus

$$\text{Sk}^I_i(w.v^s_j) \sim \text{Sk}^I_i(w.v^s_j) \sim (-1)^{\text{length}(w)} \text{Sk}^I_i(v^s_j)$$

for all $i, j$, and all $w \in W_{\text{aff}}$. If $J \neq \{0, \ldots, l\}$, the choice of any $i \notin J$ gives $\text{Sk}^I_i(v^s_j) = 0$. This proves $H_0(J) = 0$. Suppose now $J = \{0, \ldots, l\}$. The augmentation map $C_0(J) \to \mathbb{Z}$ is described in terms of the basis by $\beta_I(x) \mapsto (-1)^{\text{length}(x)}$. It has a right inverse $\mathbb{Z} \to C_0(J)$, $1 \mapsto \beta_0(v^s_\delta)$. Hence the induced map in homology $\mathbb{Z} \to H_0(J)$ is injective, but also surjective since $\text{Sk}^I_i(x) \sim (-1)^{\text{length}(x)} \beta_0(v^s_\delta)$. Thus $H_0(J) = \mathbb{Z}$ in this case.

5.5.2. Computation of $H_l(J)$. Suppose $\phi \in C_l(J) = \mathbb{Z}[V]$. Then $\partial \phi = 0$ if and only if $\text{Sk}^{0 \cdots l}_i \phi = 0$ for all $i$. That is, $\phi$ is invariant under every reflection $\sigma_i \in W_{\text{aff}}$, hence under the full affine Weyl group $W_{\text{aff}}$. But since $\phi$ has finite length this is impossible unless $\phi = 0$. This shows $H_l(J) = 0$. 


5.5.3. **Computation of** $H_p(J)$, $0 < p < l$. To simplify notation, we will write $C_\bullet$ instead of $C_\bullet(J)$. (This should of course not be confused with the chain complex $C_\bullet$ considered in previous sections.) Introduce a $\mathbb{Z}$-filtration

$$0 = F_{-1}C_\bullet \subset F_0C_\bullet \subset F_1C_\bullet \subset \cdots$$

where $F_N C_p$ is spanned by basis elements (46) with $|I| = p + 1$ and length($x$) $\leq N$. Formula (47) shows that for any basis element $\beta_I(x) \in F_N C_p$,

$$\partial \beta_I(x) = \sum_r^{|I|} (-1)^r \beta_{h_i I}(x) \mod F_{N-1} C_{p-1}$$

where the sum is only over those $r$ for which $x \in V^{\delta^r I} \subset V^I$, i.e. $u_r = 1$ (other terms lower the filtration degree since length($u_r x$) $< \text{length}(x)$ unless $x = u_r x$). In particular, $\partial$ preserves the filtration. Define operators $h_i : C_p \to C_{p+1}$ on basis elements, as follows:

$$h_i \beta_I(x) = \begin{cases} (-1)^r \beta_{I \cup \{i\}}(x) & \text{if } i_{r-1} < i < i_r, \\ 0 & \text{if } i = i_r, \text{ some } r. \end{cases}$$

Note that $h_i$ preserves the filtration: $h_i(F_N C_p) \subset F_{N-1} C_{p+1}$. Let

$$A_i = \text{id} - h_i \partial - \partial h_i.$$ 

Then $A_i$ is a chain map, which is homotopic to the identity map.

**Lemma 5.7.** Let $p > 0$. For any basis element $\beta_I(x) \in F_N C_p$ we have $A_i \beta_I(x) \in F_{N-1} C_p$ unless $i \in I$ and $x \notin V^{I-\{i\}}$. In the latter case,

$$A_i \beta_I(x) = \beta_I(x) \mod F_{N-1} C_p.$$ 

**Proof.** Write $I = \{i_0, \ldots, i_p\}$ where $i_0 < \cdots < i_p$. Using (48) we obtain

$$h_i \partial \beta_I(x) = \sum_r^{|I|} (-1)^r h_i \beta_{\delta^r I}(x) \mod F_{N-1} C_p,$$

summing over indices with $x \in V^{\delta^r I} \subset V^I$. The calculation of $A_i \beta_I(x)$ divides into two cases:

**Case 1:** $i \in I$. Thus $i = i_s$ for some index $s$, and $(-1)^r h_i \beta_{\delta^r I}(x) = 0$ unless $r = s$, in which case one obtains $\beta_I(x)$. Hence all terms in the sum (49) vanish, except possibly for the term $r = s$ which appears if and only if $x \in V^{\delta^r I} = V^{I-\{i\}}$. That is,

$$h_i \partial \beta_I(x) = \begin{cases} \beta_I(x) \mod F_{N-1} C_p & \text{if } x \in V^{I-\{i\}} \\ 0 \mod F_{N-1} C_p & \text{if } x \notin V^{I-\{i\}} \end{cases}$$

(\text{using the assumption } p > 0). Since $h_i \beta_I(x) = 0$ this shows $A_i \beta_I(x) \in F_{N-1} C_p$ unless $x \notin V^{I-\{i\}}$, in which case $A_i \beta_I(x) = \beta_I(x) \mod F_{N-1} C_p$.

**Case 2:** $i \notin I$. Exactly one of the terms in $\partial h_i \beta_I(x)$ reproduces $\beta_I(x)$. The remaining terms are organized in a sum similar to (47):

$$\partial h_i \beta_I(x) = \beta_I(x) - \sum_r^{|I|} (-1)^r h_i \beta_{\delta^r I}(x) \mod F_{N-1} C_p,$$

where the sum is over all $r$ such that $x \in V^{I \cup \{i_r\}}$. But $x \in V^{\delta^r I} \iff x \in V^{I \cup \{i_r\}} - \{i_r\}$, since

$$V^{\delta^r I} = V^{I \cup \{i_r\}} - \{i_r\} \cap V^I.$$
Hence the sum \( \sum_r' \) and \( \sum_r'' \) are just the same. This proves \( A_i \beta_I(x) \in F_{N-1}C_p. \) \( \square \)

Consider now the product \( A := A_0 \cdots A_l. \) By iterated application of the Lemma, we find that if \( 0 < p < l, \) then \( A\beta_I(x) \in F_{N-1}C_p \) (because at least one index \( i \) is not in \( I \)). Thus

\[
A: F_NC_p \to F_{N-1}C_p
\]

for \( 0 < p < l. \) The chain map \( A \) is chain homotopic to the identity, since each of its factors are. Thus, if \( \phi \in F_NC_p \) is a cycle,

\[
\phi \sim A\phi \sim \cdots A^{N+1}\phi = 0.
\]

This proves \( H_p(J) = 0 \) for \( 0 < p < l, \) and concludes the proof of Theorem 5.6.

**Remark 5.8.** N. Kitchloo pointed out a more elegant proof of Theorem 5.6, along the lines of Kitchloo-Morava [25]. His argument produces an inclusion of \( K \) of Kitchloo-Morava [25]. This proves

\[
H_p(J) = 0 \text{ for } 0 < p < l,
\]

and concludes the proof of Theorem 5.6.

**Appendix A. Relative Dixmier-Douady Bundles**

For any map \( f: Y \to X, \) and \( \text{cone}(f) \) its mapping cone, obtained by gluing \( \text{cone}(Y) = Y \times I/Y \times \{0\} \) with \( X \) by the identification \( (y,1) \sim f(y). \) Let \( H^\bullet(f) = H^\bullet(\text{cone}(f)) \) denote the relative cohomology of \( f. \) Equivalently \( H^\bullet(f) \) is the cohomology of the algebraic mapping cone \( C^\bullet(f) \) of the cochain map \( C^\bullet(Y) \to C^\bullet(X), \) i.e. \( C^p(f) = C^{p-1}(Y) \oplus C^p(X) \) with differential \( d(a,b) = (da-f^*b, db). \) If \( f \) is a smooth map of manifolds, the cohomology \( H^\bullet(f, \mathbb{R}) \) may be computed using differential forms, replacing the singular cochains in the above.

The group \( H^2(f) \) has a geometric interpretation as isomorphism classes of relative line bundles, i.e. pairs \( (L, \psi_Y), \) where \( L \) is a Hermitian line bundle over \( X, \) and \( \psi_Y: Y \times \mathbb{C} \to f^*L \) is a unitary trivialization of its pull-back to \( Y. \) The class of a relative line bundle is the Chern class of the line bundle \( \tilde{L} \to \text{cone}(f), \) obtained by gluing cone\((Y) \times \mathbb{C} \) with \( L \) via \( \psi_Y. \)

Similarly, \( H^3(f) \) is interpreted in terms of relative Dixmier-Douady bundles, i.e. pairs \( (\mathcal{A}, \mathcal{E}_Y), \) where \( \mathcal{A} \to X \) is a Dixmier-Douady bundle, and \( \mathcal{E}_Y \to Y \) is a Morita trivialization of the pull-back \( f^*\mathcal{A}. \)

Given such a triple, one may construct a Dixmier-Douady bundle \( \tilde{\mathcal{A}} \to \text{cone}(f). \) First stabilize: Let \( \mathbb{H} \) be a fixed infinite-dimensional Hilbert space, and \( \mathbb{K} = \mathbb{K}(\mathbb{H}) = \) the compact operators. Then \( \mathcal{E}_Y^\mathbb{K} = \mathcal{E}_Y \otimes \mathbb{K} \) defines a Morita trivialization of the pull-back of \( \mathcal{A}^\mathbb{K} = \mathcal{A} \otimes \mathbb{K}. \) Since the Hilbert space bundle \( \mathcal{E}_Y^\mathbb{K} \) is stable, it is equivariantly isomorphic to the trivial bundle \( Y \times \mathbb{H}. \) Define \( \tilde{\mathcal{A}} \) by gluing the trivial bundle cone\((Y) \times \mathbb{K} \) with \( f^*\mathcal{A}^\mathbb{K}, \) using this identification.

We define the relative Dixmier-Douady class \( \text{DD}((\mathcal{A}, \mathcal{E}_Y)) := \text{DD}(\tilde{\mathcal{A}}) \in H^3(f). \)

Tensor products and opposites of relative Dixmier-Douady bundles are defined in the obvious way. A Morita trivialization \( (\mathcal{A}, \mathcal{E}_Y) \) is a Morita trivialization \( \mathbb{C} \simeq_{\mathcal{E}_Y} \mathcal{A} \) together with an isomorphism \( \mathcal{E}_Y \cong f^*\mathcal{E}_X \) intertwining the module structures. From the usual Dixmier-Douady theorem, one deduces that \( \text{DD}((\mathcal{A}, \mathcal{E}_Y)) \) is the obstruction to the existence of a relative Morita trivialization.

More generally, one may define relative equivariant Dixmier-Douady bundles; these are classified by an equivariant class \( \text{DD}_G((\mathcal{A}, \mathcal{E}_Y, \psi_Y)) \in H^3_G(f) := H^3(f_G), \) where \( f_G: Y_G \to X_G \) is the
induced map of Borel constructions. (For the stabilization procedure, one replaces $H$ with the stable $G$-Hilbert space $H_G$ containing all $G$-representations with infinite multiplicity.)

## Appendix B. Review of Kasparov K-homology

In this Section we review Kasparov's definition of K-homology [23, 22] for $C^*$-algebras. Excellent references for this material are the books by Higson-Roe [19] and Blackadar [5].

Suppose $A$ is a $\mathbb{Z}_2$-graded $C^*$-algebra, equipped with an action of a compact Lie group $G$ by automorphisms. An equivariant Fredholm module over $A$ is a triple $x = (\mathcal{H}, \rho, F)$, where $\mathcal{H}$ is a $G$-equivariant $\mathbb{Z}_2$-graded Hilbert space, $\rho: A \to L(\mathcal{H})$ is a morphism of $\mathbb{Z}_2$-graded $G$-$C^*$-algebras, and $F \in L(\mathcal{H})$ is a $G$-invariant odd operator such that for all $a \in A$,

$$(F^2 - I)\rho(a) \sim 0, \ (F^* - F)\rho(a) \sim 0, \ [F, \rho(a)] \sim 0.$$  

Here $\sim$ denotes equality modulo compact operators. There is an obvious notion of direct sum of Fredholm modules over $A$. One defines a semi-group $K^G_0(A)$, with generators $[x]$ for each Fredholm module over $A$, and equivalence relations

$$[x] + [x'] = [x \oplus x'],$$

and

$$[x_0] = [x_1]$$

provided $x_0, x_1$ are related by an ‘operator homotopy’ $x_t = (\mathcal{H}, \rho, F_t)$ (cf. [5, 19]). One then proves that every element in this semi-group has an additive inverse, so that $K^G_0(A)$ is actually a group. More generally, for $q \leq 0$ one defines $K^G_q(A) = K^G_q(A \otimes \text{Cl}(\mathbb{R}^q))$. This has the mod 2 periodicity property $K^G_{q+2}(A) = K^G_q(A)$, which is then used to extend the definition to all $q \in \mathbb{Z}$. The assignment $A \to K^G_q(A)$ is homotopy invariant, contravariant functor, depending only on the Morita isomorphism class of $A$. It has the stability property, $K^G_q(A \otimes \mathbb{K}_G) = K^G_q(A)$, where $\mathbb{K}_G$ are the compact operators on a $G$-Hilbert-space $\mathcal{H}_G$ containing all $G$-representations with infinite multiplicity. With this definition, let us now review some basic examples of twisted $K$-homology groups

$$K^G_q(X, A) = K^G_q(\Gamma_0(X, A))$$

for Dixmier-Douady bundles $A \to X$.

**Example B.1.** Let $A \to pt$ be a $G$-equivariant Dixmier-Douady bundle over a point. Disregarding the $G$-action, we have $A \cong \mathbb{K}(\mathcal{E})$ for some Hilbert space $\mathcal{E}$. As in Section 2.2 the action $G \to \text{Aut}(A)$ defines a central extension $\hat{G}$ of $G$ by $\text{U}(1)$.

The group $\hat{G}$ acts on $\mathcal{E}$, in such a way that the central circle acts with weight 1. Let $V$ be a $\hat{G}$-module where the central circle acts with weight $-1$. Then the Hilbert space $\mathcal{H} = V \otimes \mathcal{E}$ is a $G$-module. Letting $\rho: \mathbb{C} \to L(\mathcal{H})$ be the action by scalar multiplication, the triple $(\mathcal{H}, \rho, 0)$ is a $G$-equivariant Fredholm module over $C(pt) = \mathbb{C}$. This construction realizes the isomorphism $R(\hat{G})_{-1} \to K^G_0(pt, A)$.

**Example B.2.** Let $M$ be a compact Riemannian $G$-manifold, and $D$ an invariant first order elliptic operator acting on a $G$-equivariant $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Suppose also that a finite rank $\mathbb{Z}_2$-graded $G$-Dixmier-Douady bundle $A \to M$ acts on $\mathcal{E}$, where the action is equivariant and compatible with the grading. Let $\mathcal{H}$ be the space of $L^2$-sections of $\mathcal{E}$, with the natural representation $\rho$ of $\Gamma(M, A)$, and $F = D(1 + D^2)^{-1/2} \in L(\mathcal{H})$.

The commutator of $F$ with elements $\rho(a)$ for $a \in \Gamma(M, A)$ are pseudo-differential operators of degree $-1$, hence are compact. Thus $(\mathcal{H}, \rho, F)$ is an equivariant Fredholm module over $\Gamma(M, A)$, defining a class in $K^G_0(M, A)$. 
Example B.3. [24, page 114] Let $M$ be a compact Riemannian $G$-manifold, and $A = \text{Cl}(TM)$ its Clifford bundle. Take $\mathcal{E} = \wedge T^* M$, $\mathcal{H}$ its space of $L^2$-sections, and $\varrho$ the usual action of sections of $\Gamma(M, \text{Cl}(TM))$. Let $D = d + d^*$ be the de-Rham Dirac operator. By B.2 above, we obtain a Fredholm module $(\mathcal{H}, \varrho, F)$ over $\Gamma(M, \text{Cl}(TM))$, defining a class $[M] \in K^G_0(M, \text{Cl}(TM))$. This is the Kasparov fundamental class of $M$. (Actually, $\text{Cl}(TM)$ is a Dixmier-Douady bundle only if $\dim M$ is even. If $\dim M$ is odd, one can use the isomorphism $K^G_0(M, \text{Cl}(TM)) = K^{\text{red}}_1(M, \text{Cl}^+(TM))$ if needed.)

Example B.4. Let $H$ be a closed subgroup of $G$, and $B \to \text{pt}$ an $H$-Dixmier-Douady bundle of finite rank. As explained in B.1, any class in $K^H_0(\text{pt}, \text{Cl}(g/h) \otimes B)$ is realized by a Fredholm module of the form $(\mathcal{E}, \varrho, 0)$ where $\mathcal{E}$ if a Hilbert space of finite dimension. Let $\mathcal{E} = G \times_H \mathcal{E}$. The action of $\text{Cl}(T(G/H))$ defines a Dirac operator, which together with the action of $\mathcal{I}^G_H(B)$ yields a Fredholm module and hence an element of $K^G_0(G/H, \mathcal{I}^G_H(B))$. This construction realizes the isomorphism $K^H_0(\text{pt}, B \otimes \text{Cl}(g/h)) \to K^G_0(G/H, \text{ind}^G_H(B))$ if $B$ has finite rank. As remarked in Section 2.1, all $H$-Dixmier-Douady bundles over pt are Morita isomorphic to finite rank ones.

References


[31] M. Mostow and J. Perchik, Notes on Gelfand-Fuks cohomology and characteristic classes (lectures delivered by R. Bott), Proceedings of the eleventh annual holiday symposium, New Mexico State University, 1973, pp. 1–126.


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