THE CUBIC DIRAC OPERATOR FOR INFINITE-DIMENSIONAL LIE ALGEBRAS

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Abstract. Let \( g = \bigoplus_{i \in \mathbb{Z}} g_i \) be an infinite-dimensional graded Lie algebra, with \( \dim g_i < \infty \), equipped with a non-degenerate symmetric bilinear form \( B \) of degree 0. The quantum Weil algebra \( \widehat{W}_g \) is a completion of the tensor product of the enveloping and Clifford algebras of \( g \). Provided that the Kac-Peterson class of \( g \) vanishes, one can construct a cubic Dirac operator \( D \in \widehat{W}(g) \), whose square is a quadratic Casimir element. We show that this condition holds for symmetrizable Kac-Moody algebras. Extending Kostant’s arguments, one obtains generalized Weyl-Kac character formulas for suitable ‘equal rank’ Lie subalgebras of Kac-Moody algebras. These extend the formulas of G. Landweber for affine Lie algebras.

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0. Introduction

Let \( g \) be a finite-dimensional complex Lie algebra, equipped with a non-degenerate invariant symmetric bilinear form \( B \). For \( \xi \in g \), the corresponding generators of the enveloping algebra \( U(g) \) are denoted \( s(\xi) \), while those of the Clifford algebra \( \text{Cl}(g) \) are denoted simply by \( \xi \). The quantum Weil algebra \( W(g) = U(g) \otimes \text{Cl}(g) \), with even generators \( s(\xi) \) and odd generators \( \xi \). Let \( D \in W(g) \) be the odd element, written in terms of a basis \( e_a \) of \( g \) as

\[
D = \sum_a s(e_a)e^a - \frac{1}{12} \sum_{abc} f_{abc} e^a e^b e^c,
\]

where \( e^a \) is the \( B \)-dual basis and \( f_{abc} \) are the structure constants. The key property of this element is that its square lies in the center of \( W(g) \):

\[
D^2 = \text{Cas}_g + \frac{1}{24} \text{tr}_g(\text{Cas}_g),
\]

where \( \text{Cas}_g = \sum_a s(e_a)s(e^a) \in U(g) \) is the quadratic Casimir element. The element \( D \) is called the cubic Dirac operator, following Kostant [10]. More generally, Kostant introduced cubic Dirac operators \( D_{g,u} \) for pairs of a quadratic Lie algebra \( g \) and a quadratic Lie subalgebra \( u \). For \( g \) semi-simple and \( u \) an equal rank subalgebra, he used this to prove, among other things, generalizations of the Bott-Borel-Weil theorem and of the Weyl character formula (see also [2, 11]).
In this article, we will consider generalizations of this theory to infinite-dimensional Lie algebras. We assume that \( g \) is \( \mathbb{Z} \)-graded, with finite dimensional graded pieces \( g_i \), and equipped with a non-degenerate invariant symmetric bilinear form \( B \) of degree 0. A priori, the formal expressions defining \( D \), \( \text{Cas}_g \) are undefined since they involve infinite sums. It is possible to replace these expressions with ‘normal-ordered’ sums, leading to well-defined elements \( D', \text{Cas}'_g \) in suitable completion of \( \mathcal{W}(g) \). However, it is no longer true in general that \( (D')^2 - \text{Cas}'_g \) is a constant, and in any case \( \text{Cas}'_g \) is not a central element. One may attempt to define elements \( D, \text{Cas}_g \) having these properties by adding lower order correction terms to \( D', \text{Cas}'_g \). Our main observation is that this is possible if and only the Kac-Peterson class \( [\psi_{KP}] \in H^2(g) \) is zero. In fact, given \( \rho \in g^* \) with \( \psi_{KP} = d\rho \), the elements \( D = D' + \rho, \text{Cas}_g = \text{Cas}'_g + 2\rho \) have the desired properties. These results are motivated by the work of Kostant-Sternberg [12], who had exhibited the Kac-Peterson class as an obstruction class in their BRST quantization scheme.

For symmetrizable Kac-Moody algebras, the existence of a corrected Casimir element \( \text{Cas}_g \) is a famous result of Kac [4]. In particular, \( [\psi_{KP}] = 0 \) in this case. As we will see, Kostant’s theory carries over to the symmetrizable Kac-Moody case in a fairly straightforward manner. For suitable ‘regular’ Kac-Moody subalgebras \( u \subset g \), we thus obtain generalized Weyl-Kac character formulas as sums over multiplets of \( u \)-representations.

For affine Lie algebras or loop algebras, similar Dirac operators were described in Kac-Todorov [7] and Kazama-Suzuki [8], and more explicitly in Landweber [14] and Wassermann [19]. In fact, Wassermann uses this Dirac operator to give a proof of the Weyl-Kac character formula for affine Lie algebras, while Landweber proves generalized Weyl character formulas for ‘equal rank loop algebras’. The cubic Dirac operator \( D \) for general symmetrizable Kac-Moody algebras is very briefly discussed in Kitchloo [9].

1. Completions

In this Section we will define completions of the exterior and Clifford algebras of a graded quadratic vector space. We recall from [6] how the Kac-Peterson cocycle appears in this context.

1.1. Kac-Peterson cocycle. Let \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) be a \( \mathbb{Z} \)-graded vector space over \( \mathbb{C} \), with finite-dimensional graded components. The (graded) dual space is the direct sum over the duals of \( V_i \), with grading \( (V^*)_i = (V_{-i})^* \). Given another graded vector space \( V' \) with \( \dim V'_i < \infty \), we let \( \text{Hom}(V, V') \) be the direct sum over the spaces \( \text{Hom}(V, V')_i = \bigoplus_r \text{Hom}(V_r, V'_i + r) \) of finite rank maps of degree \( i \). We let

\[
\widehat{\text{Hom}}(V, V')_i = \prod_r \text{Hom}(V_r, V'_{i+r})
\]

be the space of all linear maps \( V \to V' \) of degree \( i \), and \( \widehat{\text{Hom}}(V, V') \) their direct sum. If \( V = V' \) we write \( \text{End}(V) = \widehat{\text{Hom}}(V, V) \) and \( \widehat{\text{End}}(V) = \widehat{\text{Hom}}(V, V) \). Note that \( \widehat{\text{End}}(V) \) is an algebra with unit \( I \).

Define a splitting \( V = V_+ \oplus V_0 \) where \( V_+ = \bigoplus_{i > 0} V_i, V_- = \bigoplus_{i < 0} V_i \). Denote by \( \pi_-, \pi_+ \) the projections to the two summands. The Kac-Peterson cocycle ([6]; see also [5, Exercise
7.28]) on $\hat{\text{End}}(V)$ is a Lie algebra cocycle given by the formula,

$$\psi_{KP}(A_1, A_2) = \frac{1}{2} \text{tr}(A_1 \pi_- A_2 \pi_+) - \frac{1}{2} \text{tr}(A_2 \pi_- A_1 \pi_+).$$

(2)

This is well-defined since the compositions $\pi_- A_i \pi_+: V \to V$ have finite rank. Observe that $\psi_{KP}$ has degree 0, that is, (2) vanishes unless the degrees of $A_1, A_2$ add to zero. On the Lie subalgebra $\text{End}(V) \subset \hat{\text{End}}(V)$, the Kac-Peterson cocycle restricts to a coboundary:

$$\psi_{KP}(A_1, A_2) = \frac{1}{2} \text{tr}(\pi_+[A_1, A_2]).$$

(3)

1.2. Completion of symmetric and exterior algebras. Let $S(V)$ be the symmetric algebra of $V$, with $\mathbb{Z}$-grading defined by assigning degree $i$ to generators in $V_i$. Let $V^*$ be the graded dual as above. The pairing between $S(V)$ and $S(V^*)$ identifies $S(V)_i$ as a subspace of the space of linear maps $S(V^*)^{-i} \to \mathbb{K}$. We define a completion $\hat{S}(V)_i$ as the space of all linear maps $S(V^*)^{-i} \to \mathbb{K}$. Equivalently,

$$\hat{S}(V)_i = \prod_{r \geq 0} S(V_+)^{-i-r} \otimes S(V_-)^r.$$

We let $\hat{S}(V)$ be the direct sum over the $\hat{S}(V)_i$. The multiplication map of $S(V)$ extends to the completion, making $\hat{S}(V)$ into a $\mathbb{Z}$-graded algebra. For each $k \geq 0$ one similarly has a completion $\hat{S}^k(V) \subset \hat{S}(V)$ of each component $S^k(V)$. Then $\hat{S}(V)_i$ is the direct product over all $\hat{S}^k(V)_i$. The space $\hat{S}^2(V^*)_0$ may be identified with the space of symmetric bilinear maps $B: V \times V \to \mathbb{C}$ of degree 0, that is $B(V_i, V_j) = 0$ for $i + j \neq 0$.

In a similar fashion, one defines a completions $\hat{\wedge}(V)_i$ as the spaces of all linear maps $\hat{\wedge}(V^*)^{-i} \to \mathbb{K}$, or equivalently

$$\hat{\wedge}(V)_i = \prod_{r \geq 0} \wedge(V_+)^{-i-r} \otimes \wedge(V_-)^r.$$

We let $\hat{\wedge}(V)$ be the $\mathbb{Z}$-graded super algebra given as the direct sum over all $\hat{\wedge}(V)_i$. Again, one also has completions of the individual $\wedge^k(V)$. The space $\hat{\wedge}^2(V^*)_0$ may be identified with the skew-symmetric bilinear maps $V \times V \to \mathbb{C}$ of degree 0. In particular:

$$\psi_{KP} \in \hat{\wedge}^2(\hat{\text{End}}(V^*))_0.$$

1.3. Clifford algebras. Suppose $B$ is a (possibly degenerate) symmetric bilinear form on $V = \bigoplus V_i$ of degree 0. Let $\text{Cl}(V)$ be the corresponding Clifford algebra, i.e. the super algebra with odd generators $v \in V$ and relations $vw + vw = 2B(v, w)$ for $v, w \in V$. The $\mathbb{Z}$-grading on $V$ defines a $\mathbb{Z}$-grading on $\text{Cl}(V)$, compatible with the algebra structure.

Using the restrictions of the bilinear form to $V_{\pm}$, we may similarly form the Clifford algebras $\text{Cl}(V_{\pm})$. These are $\mathbb{Z}$-graded subalgebras of $\text{Cl}(V)$, and the multiplication map defines an isomorphism of super vector spaces, $\text{Cl}(V) \cong \text{Cl}(V_-) \otimes \text{Cl}(V_+)$. Note that $\text{Cl}(V_+) = \wedge(V_+)$ since $B$ restricts to 0 on $V_+$.

We obtain a $\mathbb{Z}$-graded superalgebra $\hat{\text{Cl}}(V)$ as the direct sum over all

$$\hat{\text{Cl}}(V)_i = \prod_{r \geq 0} \text{Cl}(V_-)^{-i-r} \otimes \text{Cl}(V_+)^r.$$
Let \( q^0 : \wedge (V) \to \text{Cl}(V) \) denote the standard quantization map for the Clifford algebra, defined by super symmetrization:

\[
q^0(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)},
\]

where \( S_k \) is the permutation group on \( k \) elements, and \( \text{sign}(\sigma) = \pm 1 \) is the parity of the permutation \( \sigma \). The map \( q^0 \) is an isomorphism of super spaces, preserving the \( \mathbb{Z} \)-gradings and taking \( \wedge (V_{\pm}) \) to \( \text{Cl}(V_{\pm}) \). While \( q^0 \) itself does not extend to the completions, we obtain a well-defined normal-ordered quantization map

\[
q : \hat{\wedge}(V) \to \hat{\text{Cl}}(V)
\]

by taking the direct sum over \( i \in \mathbb{Z} \) and direct product over \( r \geq 0 \) of

\[
q^0 \otimes q^0 : \wedge (V\wedge)_{i-r} \otimes \wedge (V\vee)_r \to \text{Cl}(V\wedge)_{i-r} \otimes \text{Cl}(V\vee)_r.
\]

The quantization map is an isomorphism of \( \mathbb{Z} \)-graded super vector spaces, with the property that for \( \lambda \in \hat{\wedge}^k(V) \), \( \mu \in \hat{\wedge}^l(V) \),

\[
q^{-1}(q(\lambda)q(\mu)) = \lambda \wedge \mu \mod \hat{\wedge}^{k+l-2}(V).
\]

Any element \( v \in V \) defines an odd derivation \( \iota_v \), called contraction, of the super algebra \( \wedge (V) \), given on generators by \( \iota_v(w) = B(v, w) \). The same formula also defines a derivation of the Clifford algebra, again denoted \( \iota_v \). In both cases, the contractions extend to the completions. The map \( q : \hat{\wedge}(V) \to \hat{\text{Cl}}(V) \) intertwines contractions:

\[
q \circ \iota_v = \iota_v \circ q,
\]

since \( q^0 \circ \iota_v = \iota_v \circ q^0 \) and since contractions preserve \( \wedge (V_{\pm}) \) and \( \text{Cl}(V_{\pm}) \).

Let \( \mathfrak{o}(V) \subset \text{End}(V) \) and \( \hat{\mathfrak{o}}(V) \subset \hat{\text{End}}(V) \) denote the \( B \)-skew-symmetric endomorphisms. Let

\[
\hat{\wedge}^2(V) \to \hat{\mathfrak{o}}(V), \quad \lambda \mapsto A_\lambda
\]

be the map defined by \( A_\lambda(v) = -2\iota_v \lambda \). The map \( (4) \) is \( \hat{\mathfrak{o}}(V) \)-equivariant, that is,

\[
A_{[X,A_\lambda]} = [X,A_\lambda]
\]

for \( X \in \hat{\mathfrak{o}}(V) \).

**Lemma 1.1.** For all \( \lambda \in \hat{\wedge}^2(V) \),

\[
q(\lambda) = q^0(\lambda) - \frac{1}{2} \text{tr}(\pi_+ A_\lambda).
\]

**Proof.** It suffices to check for elements of the form \( \lambda = u \wedge v \) for \( u, v \in V \). We have \( A_{u \wedge v}(w) = 2(B(v, w)u - B(u, w)v) \), hence \( \text{tr}(\pi_+ A_{u \wedge v}) = 2(B(\pi_+ u, v) - B(\pi_+ v, u)) \). On the other hand, by considering the special cases that \( u, v \) are both in \( V_- \), both in \( V_+ \), or \( u \in V_-, v \in V_+ \) we find

\[
q(u \wedge v) = q^0(u \wedge v) + B(\pi_+ v, u) - B(\pi_+ u, v). \quad \square
\]

The map \( q^0 \) is \( \mathfrak{o}(V) \)-equivariant. For the normal-ordered quantization map this is no longer the case.
Proposition 1.2 (Kac-Peterson). [6] For all \( \lambda \in \hat{\Lambda}^2(V) \) and \( X \in \hat{\mathfrak{o}}(V) \), one has
\[
L_X q(\lambda) = q(L_X \lambda) + \psi_{KP}(X, A_\lambda).
\]

Proof. It is enough to prove this for \( X \in \mathfrak{o}(V) \) and \( \lambda \in \Lambda^2(V) \). Since \( q^0 \) intertwines Lie derivatives, Lemma 1.1 together with (3) give
\[
L_X q(\lambda) - q(L_X \lambda) = \frac{1}{2} \text{tr}(\pi_+ A_{LX} \lambda) = \frac{1}{2} \text{tr}(\pi_+ [X, A_\lambda]) = \psi_{KP}(X, A_\lambda). \quad \square
\]

If \( B \) is non-degenerate, the map \( \lambda \mapsto A_\lambda \) defines an isomorphism \( \Lambda^2(V) \to \mathfrak{o}(V) \). Let
\[
\lambda: \mathfrak{o}(V) \to \Lambda^2(V), \ A \mapsto \lambda(A)
\]
be the inverse map. It extends to a map \( \hat{\mathfrak{o}}(V) \to \hat{\Lambda}^2(V) \) of the completions. In a basis \( e_a \) of \( V \), with \( B \)-dual basis \( e^a \) (i.e. \( B(e_a, e^b) = \delta^b_a \)), one has
\[
\lambda(A) = \frac{1}{4} \sum_a A(e_a) \wedge e^a.
\]

If \( A \in \mathfrak{o}(V) \), the elements \( \gamma^0(A) = q^0(\lambda(A)) \) are defined. As is well-known, \( [\gamma^0(A_1), \gamma^0(A_2)] = \gamma^0([A_1, A_2]) \) for \( A_i \in \mathfrak{o}(V) \), and
\[
L_A = [\gamma^0(A), \cdot].
\]
If \( A \in \hat{\mathfrak{o}}(V) \), one still has \( L_A = [\gamma'(A), \cdot] \) with
\[
\gamma'(A) = q(\lambda(A)),
\]
but the map \( \gamma' \) is no longer a Lie algebra homomorphism. Instead, Proposition 1.2 shows [6]
\[
[\gamma'(A_1), \gamma'(A_2)] = \gamma'([A_1, A_2]) + \psi_{KP}(A_1, A_2)
\]
for \( A_1, A_2 \in \hat{\mathfrak{o}}(V) \).

2. Graded Lie algebras

We will now specialize to the case that \( V = \mathfrak{g} \) is a \( \mathbb{Z} \)-graded Lie algebra. We show that in the quadratic case, the obstruction to defining a reasonable ‘Casimir operator’ is precisely the Kac-Peterson class of \( \mathfrak{g} \).

2.1. Kac-Peterson cocycle of \( \mathfrak{g} \). Let \( \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \) be a graded Lie algebra, with \( \dim \mathfrak{g}_i < \infty \). That is, we assume that the grading is compatible with the bracket: \([\mathfrak{g}_i, \mathfrak{g}_j]_\mathfrak{g} \subset \mathfrak{g}_{i+j}\). The map \( \text{ad}_\xi: \mathfrak{g} \to \mathfrak{g} \) defines a homomorphism of graded Lie algebras
\[
\text{ad}: \mathfrak{g} \to \hat{\text{End}}(\mathfrak{g}).
\]
Recall that \( \mathfrak{g}^* = \bigoplus_i (\mathfrak{g}^*)_i \) denotes the restricted dual where \( (\mathfrak{g}^*)_i = (\mathfrak{g}_{-i})^* \). The algebra \( \Lambda(\mathfrak{g}^*) \) carries contraction operators and Lie derivatives \( \iota_\xi \), \( L_\xi \) for \( \xi \in \mathfrak{g} \), given on generators by \( \iota_\xi \mu = \langle \mu, \xi \rangle \) and \( L_\xi \mu = (-\text{ad}_\xi)^* \mu \). If \( \dim \mathfrak{g} < \infty \) it also carries a differential \( d \), given on generators by
\[
d\mu = 2\lambda(\mu)
\]
where \( \lambda(\mu) \) is defined by \( \iota_\xi \lambda(\mu) = \frac{1}{2} L_\xi \mu \). On generators,
\[
(d\mu)(\xi_1, \xi_2) = -\langle \mu, [\xi_1, \xi_2]_\mathfrak{g} \rangle.
\]
In the infinite-dimensional case, $\lambda(\mu)$ and hence $d$ are well-defined on the completion $\widehat{\mathfrak{g}}^*(\mu)$. The operators $i\xi, L\xi, d$ make $\widehat{\mathfrak{g}}^*(\mu)$ into a $\mathfrak{g}$-differential algebra. Define

$$\psi_{KP}(\xi_1, \xi_2) := \psi_{KP}(\text{ad}_{\xi_1}, \text{ad}_{\xi_2})$$

for $\xi_i \in \mathfrak{g}$. Thus $\psi_{KP} \in \widehat{\mathfrak{g}}^2(\mathfrak{g}^*)_0$ is a degree 2 Lie algebra cocycle of $\mathfrak{g}$, called the Kac-Peterson cocycle of $\mathfrak{g}$. Its class $[\psi_{KP}] \in H^2(\mathfrak{g})$ will be called the Kac-Peterson class of the graded Lie algebra $\mathfrak{g}$. Note that $d$ has $\mathbb{Z}$-degree 0, so that it restricts to a differential on each $\widehat{\mathfrak{g}}^*(\mu)_i$. Hence, if $\psi_{KP}$ admits a primitive in $\mathfrak{g}^*$, then it admits a primitive in $\mathfrak{g}^*_0$.

**Example 2.1.** [6] Suppose $\mathfrak{t}$ is a finite-dimensional Lie algebra, and let $\mathfrak{g} = \mathfrak{t}[z, z^{-1}]$ the loop algebra with its usual $\mathbb{Z}$-grading. Let $B^{Kil}(x, y) = \text{tr}_\mathfrak{t}(\text{ad}_x \text{ad}_y)$ for $x, y \in \mathfrak{t}$ be the Killing form on $\mathfrak{t}$. One finds

$$\psi_{KP}(\xi, \zeta) = \text{Res} \ B^{Kil}\left(\frac{\partial \xi}{\partial z}, \zeta\right)$$

for $\xi, \zeta \in \mathfrak{t}[z, z^{-1}]$, where $\text{Res}$ picks out the coefficient of $z^{-1}$. One may check that unless $B\text{Kil} = 0$, the Kac-Peterson class $[\psi_{KP}]$ is non-zero.

**Example 2.2** (Heisenberg algebra). Let $\mathfrak{g}$ be the Lie algebra with basis $K, e_1, f_1, e_2, f_2, \ldots$, where $K$ is a central element and $[e_i, f_j] = \delta_{i,j}K$. Define a grading on $\mathfrak{g}$ such that $e_i$ has degree $i$ and $f_i$ has degree $-i$, while $K$ has degree 0. One finds $\psi_{KP} = 0$.

**Example 2.3.** Suppose $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$ and a system $\Delta^+ \subset \mathfrak{h}^*$ of positive roots. Let $\mathfrak{g}$ carry the principal grading, i.e. $\mathfrak{g}_0 = \mathfrak{h}$ while $\mathfrak{g}_i, i \neq 0$ is the direct sum of root spaces for roots of height $i$. Using (3) one finds that $\psi_{KP} = d\rho$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

**2.2. Enveloping algebras.** The $\mathbb{Z}$-grading on $\mathfrak{g}$ defines a $\mathbb{Z}$-grading on the enveloping algebra $U(\mathfrak{g})$. Both $\mathfrak{g}_+ = \bigoplus_{i > 0} \mathfrak{g}_i$ and $\mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i$ are graded Lie subalgebras, thus $U(\mathfrak{g}_\pm)$ are graded subalgebras of $U(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt theorem, the multiplication map defines an isomorphism of vector spaces, $U(\mathfrak{g}) = U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+)$. We define a completion $\widehat{U}(\mathfrak{g})$ as a direct sum over

$$\widehat{U}(\mathfrak{g})_i = \prod_{r \geq 0} U(\mathfrak{g}_-)_i \otimes U(\mathfrak{g}_+)_r.$$ 

The multiplication map extends to the completion, making $\widehat{U}(\mathfrak{g})$ into a graded algebra. Let $q^0: S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ be the isomorphism given by the standard (PBW) symmetrization map,

$$q^0(\xi_1 \cdots \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}.$$ 

This preserves $\mathbb{Z}$-degrees and takes $S(\mathfrak{g}_\pm)$ to $U(\mathfrak{g}_\pm)$. While the map itself does not extend to the completions, we define a normal-ordered symmetrization (quantization) map $q: \widehat{S}(\mathfrak{g}) \rightarrow \widehat{U}(\mathfrak{g})$ by taking the direct sum over $i$ and direct product over $r$ of the maps

$$q^0 \otimes q^0: S(\mathfrak{g}_-)_i \otimes S(\mathfrak{g}_+)_r \rightarrow U(\mathfrak{g}_-)_i \otimes U(\mathfrak{g}_+)_r.$$
Then \( q \) is an isomorphism of \( \mathbb{Z} \)-graded vector spaces. Let 
\[
S^2(\mathfrak{g}) \to \text{Hom}(\mathfrak{g}^*, \mathfrak{g}), \ p \mapsto A_p
\]
be the linear map given for \( p = uv, \ u, v \in \mathfrak{g} \) by 
\[
A_p(\mu) = (\mu, u)v + (\mu, v)u.
\]
It extends to a \( \mathfrak{g} \)-equivariant linear map \( \hat{S}^2(\mathfrak{g}) \to \hat{\text{Hom}}(\mathfrak{g}^*, \mathfrak{g}) \). Let 
\[
\text{br}: \text{Hom}(\mathfrak{g}^*, \mathfrak{g}) \to \mathfrak{g}
\]
be the linear map, given by the identification \( \text{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g} \) followed by the Lie bracket. In a basis \( e_a \) of \( \mathfrak{g} \) with dual basis \( e^a \in \mathfrak{g}^*, \ \text{br}(A) = \sum_a [A(e^a), e_a]_\mathfrak{g} \). The counterpart to Lemma 1.1 reads:

**Lemma 2.4.** For \( p \in S^2(\mathfrak{g}) \),
\[
q(p) = q^0(p) - \frac{1}{2} \text{br}(\pi_+ A_p).
\]

*Proof.* It suffices to check for \( p = uv \), where the formula reduces to (cf. (6))
\[
q(uv) = q^0(uv) + \frac{1}{2}[u, \pi_+ v]_\mathfrak{g} + \frac{1}{2}[v, \pi_+ u]_\mathfrak{g},
\]
but this is straightforward in each of the cases that \( u, v \) are both in \( \mathfrak{g}_+ \), both in \( \mathfrak{g}_- \), or \( u \in \mathfrak{g}_+, v \in \mathfrak{g}_- \).

In contrast to \( q^0 \), the map \( q \) is not \( \mathfrak{g} \)-equivariant. Similar to Proposition 1.2 we have:

**Proposition 2.5.** On \( \hat{S}^2(\mathfrak{g}) \),
\[
L_\xi(q(p)) - q(L_\xi(p)) = \frac{1}{2} \text{br} \left( (\pi_+ \text{ad}_\xi \pi_- - \pi_- \text{ad}_\xi \pi_+) A_p \right).
\]

The right hand side is well-defined, since \( \pi_+ \text{ad}_\xi \pi_+ \) and \( \pi_- \text{ad}_\xi \pi_- \) are in \( \text{Hom}(\mathfrak{g}, \mathfrak{g}) \), hence \( (\pi_+ \text{ad}_\xi \pi_- - \pi_- \text{ad}_\xi \pi_+) A_p \in \text{Hom}(\mathfrak{g}^*, \mathfrak{g}) \).

*Proof.* It suffices to verify this for \( p \in S^2(\mathfrak{g}) \), so that \( A_p \) has finite rank. Since \( L_\xi q^0(p) - q^0(L_\xi p) = 0 \), Lemma 2.4 gives
\[
L_\xi q(p) - q(L_\xi p) = -\frac{1}{2} \left( L_\xi \text{br}(\pi_+ A_p) - \text{br}(\pi_+ L_\xi A_p) \right)
\]
\[
= -\frac{1}{2} \left( [L_\xi, \pi_+ A_p] - \pi_+ [L_\xi, A_p] \right)
\]
\[
= -\frac{1}{2} \text{br} \left( L_\xi \pi_+ A_p - \pi_+ L_\xi A_p \right)
\]
\[
= \frac{1}{2} \text{br} \left( (\pi_+ L_\xi \pi_- - \pi_- L_\xi \pi_+) A_p \right).
\]

\[ \square \]

### 2.3. Quadratic Lie algebras

We assume that \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \) comes equipped with a non-degenerate ad-invariant symmetric bilinear form \( B \) of degree 0. Thus, \( B(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \) for \( i + j \neq 0 \), while \( B \) defines a non-degenerate pairing between \( \mathfrak{g}_i, \mathfrak{g}_{-i} \). We will often use \( B \) to identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \). The examples we have in mind are the following:

1. Let \( \mathfrak{k} \) be a finite-dimensional Lie algebra, with an invariant symmetric bilinear form \( B_\mathfrak{k} \). Then \( B \) extends to an inner product on the loop algebra \( \mathfrak{g} = \mathfrak{k}[z, z^{-1}] \).
(b) Let \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \) be a graded Lie algebra, with finite-dimensional homogeneous components, and \( \mathfrak{g}^* = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^*_i \) its restricted dual, with grading \( (\mathfrak{g}^*)_i = \mathfrak{g}_i^* \). The semi-direct product \( \mathfrak{g} = \mathfrak{g} \ltimes \mathfrak{g}^* \), with \( B \) given by the pairing, satisfies our assumptions. This case was studied by Kostant and Sternberg in [12].

(c) Let \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \) be a symmetrizable Kac-Moody Lie algebra, with grading the principal grading (defined by the height of roots). Then \( \mathfrak{g} \) carries a 'standard' non-degenerate invariant symmetric bilinear form, see [5]. We will return to the Kac-Moody case in Section 6.

Under the identification \( \hat{\wedge}^2(\mathfrak{g}) \cong \hat{\mathcal{s}}(\mathfrak{g}) \), the Kac-Peterson cocycle \( \psi_{KP} \) corresponds to an element

\[
\Psi_{KP} \in \hat{\mathcal{s}}(\mathfrak{g}), \quad \psi_{KP}(\xi, \zeta) = B(\Psi_{KP}(\xi), \zeta).
\]

Since \( \psi_{KP} \) has \( \mathbb{Z} \)-degree 0, the transformation \( \Psi_{KP} \) preserves each \( \mathfrak{g}_i \). Since \( \psi_{KP} \) is a cocycle, \( \Psi_{KP} \) is a derivation of the Lie bracket on \( \mathfrak{g} \). Moreover, \( \psi_{KP} \) is a coboundary if and only if the derivation \( \Psi_{KP} \) is inner:

\[
\psi_{KP} = d\rho \iff \Psi_{KP} = [\rho^\sharp, \cdot]_{\mathfrak{g}},
\]

where \( \rho^\sharp \) is the image of \( \rho \in \mathfrak{g}_0^* \) under the isomorphism \( B^\sharp : \mathfrak{g}^* \to \mathfrak{g} \).

**Example 2.6.** Let \( \mathfrak{g} = \mathfrak{k}[z, z^{-1}] \), with \( \mathfrak{k} \) semi-simple, and with bilinear form defined in terms of the Killing form on \( \mathfrak{k} \) as \( B(\xi, \zeta) = \text{Res}(z^{-1}B^\text{Kil}(\xi, \zeta)) \), for \( \xi, \zeta \in \mathfrak{k}[z, z^{-1}] \). Then \( \Psi_{KP} \) is the degree operator:

\[
\Psi_{KP}(\xi) = z \frac{\partial \xi}{\partial z}.
\]

2.4. **Casimir elements.** Let \( p \in \hat{\wedge}^2(\mathfrak{g}) \) be the element

\[
p = \sum a e^a \in \hat{\wedge}^2(\mathfrak{g}),
\]

where \( e_a \) is a homogeneous basis of \( \mathfrak{g} \), with \( B \)-dual basis \( e^a \). The corresponding transformation \( A_p \in \text{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \text{End}(\mathfrak{g}) \) is \( 2 \text{Id}_{\mathfrak{g}} \). We refer to

\[
\text{Cas}^\prime_\mathfrak{g} = q(p) \in \hat{U}(\mathfrak{g})
\]

as the normal-ordered Casimir element. It is not an element of the center, in general:

**Theorem 2.7.** The normal-ordered Casimir element satisfies

\[
L_\xi \text{Cas}^\prime_\mathfrak{g} = 2\Psi_{KP}(\xi),
\]

for all \( \xi \in \mathfrak{g} \).

**Proof.** From the definition of \( \text{br} \), one finds

\[
B(\text{br}(A), \zeta) = \text{tr}(\text{ad}_\zeta A)
\]

for all \( A \in \text{End}(\mathfrak{g}) \) and \( \zeta \in \mathfrak{g} \). Since \( A_p = 2 \text{Id}_\mathfrak{g} \) and \( L_\xi p = 0 \), Proposition 2.5 therefore gives

\[
B(L_\xi \text{Cas}^\prime_\mathfrak{g}, \zeta) = B(\text{br}(\pi_+ \text{ad}_\xi \pi_+ - \pi_\xi \text{ad}_\xi \pi_+), \zeta)
\]

\[
= \text{tr}(\text{ad}_\xi \pi_+ \text{ad}_\xi \pi_+ - \pi_\xi \pi_\xi \text{ad}_\xi \pi_+)
\]

\[
= 2\Psi_{KP}(\xi, \zeta)
\]

\[
= 2B(\Psi_{KP}(\xi), \zeta).
\]

\( \square \)
The normal-ordered Casimir element \( \text{Cas}'_g \) admits a linear correction to a central element if and only if the Kac-Peterson class is zero. More precisely:

**Corollary 2.8.** For \( \rho \in g^0 \),

\[
\text{Cas}_g := \text{Cas}'_g + 2\rho^\sharp
\]

lies in the center of \( \hat{U}(g) \) if and only if \( \psi_{KP} = d\rho \).

**Proof.** This is a direct consequence of Theorem 2.7, since \( \psi_{KP} = d\rho \) if and only if \( L_\xi \rho^\sharp = -\Psi_{KP}(\xi) \), see Equation (9). \( \square \)

**Example 2.9.** For a loop algebra \( g = \mathfrak{k}[z, z^{-1}] \), with \( \mathfrak{k} \) a semi-simple Lie algebra, the Kac-Peterson cocycle of \( g \) defines a non-trivial cohomology class. Hence it is impossible to make \( \text{Cas}'_g \) invariant by adding linear terms. On the other hand, for a symmetrizable Kac-Moody algebra \( g \), a classical result of Kac shows that \( \text{Cas}'_g \) becomes invariant after a \( \rho \)-shift. Hence the Kac-Peterson class of such a \( g \) is trivial. See Section 6 below.

### 2.5. The structure constants tensor and its quantization.

Recall the definition of \( \lambda: \hat{\mathfrak{o}}(g) \to \hat{\wedge}^2(g) \). We will write

\[
\lambda(\xi) = \lambda(\text{ad}_\xi),
\]

that is \( \iota_\xi \lambda(\xi) = \frac{1}{2}[[\xi, \xi]]_g \). In a basis \( e_a \) of \( g \), with \( B \)-dual basis \( e^a \), we have \( \lambda(\xi) = \frac{1}{4} \sum_a [[\xi, e_a]]_g \wedge e^a \).

**Lemma 2.10.** There is a unique element \( \phi \in \hat{\wedge}^3(g)_0 \) with the property

\[
\iota_\xi \iota_\eta \iota_\zeta \phi = \frac{1}{2} B([\xi_1, [\xi_2, \xi_3]]_g, \xi_3), \quad \xi_1, \xi_2, \xi_3 \in g.
\]

**Proof.** The right-hand side is a skew-symmetric trilinear form of degree 0 on \( g \). Hence it defines an element of \( \hat{\wedge}^3(g) \). \( \square \)

Equivalently, \( \iota_\xi \phi = 2\lambda(\xi), \quad \xi \in g \). In a basis,

\[
\phi = -\frac{1}{12} \sum_{abc} f_{abc} e^a \wedge e^b \wedge e^c,
\]

where \( f_{abc} = B([e_a, e_b], e_c) \) are the structure constants. From the definition, it is clear that \( \phi \) is \( g \)-invariant. This need no longer be true of its normal-ordered quantization. Write

\[
\gamma'(\xi) = q(\lambda(\xi)), \quad \phi'_{Cl} = q(\phi),
\]

so that \( L_\xi = [\gamma'(\xi), \cdot] \). Denote by \( \psi^\sharp_{KP} \in \hat{\wedge}^2(g^\ast) \) the image of \( \psi_{KP} \in \hat{\wedge}^2(g^\ast) \) under the isomorphism \( B^\sharp: \hat{\wedge}(g^\ast) \to \hat{\wedge}(g) \).

**Proposition 2.11.** The element \( \phi'_{Cl} \in \hat{\text{Cl}}(g) \) satisfies

\[
L_\xi \phi'_{Cl} = \Psi_{KP}(\xi),
\]

and its square is given by the formula

\[
(\phi'_{Cl})^2 = q(\psi^\sharp_{KP}) + \frac{1}{24} \text{tr}_{g^0}(\text{Cas}_{g^0}).
\]

Here \( \text{Cas}_{g^0} \in U(g^0) \) is the quadratic Casimir element for \( g^0 \), and \( \text{tr}_{g^0}(\text{Cas}_{g^0}) \) is its trace in the adjoint representation.
Proof. The first formula follows from the second, since

$$L_\xi \phi'_\mathcal{C} = [\gamma'(\xi), \phi'_\mathcal{C}] = \iota_\xi (\phi'_\mathcal{C})^2.$$

Since

$$\iota_\xi (\phi'_\mathcal{C})^2 = [\gamma'(\xi), \phi'_\mathcal{C}] = L_\xi \phi'_\mathcal{C} = \Psi_{KP}(\xi) = \iota_\xi q(\psi_{KP}),$$

the difference $(\phi'_\mathcal{C})^2 - q(\psi_{KP})$ is a constant. Let $\phi_r$ be the component of $\phi$ in $(\wedge g_+)_r \otimes (\wedge g_-)_r$. The commutator of $\phi'_\mathcal{C}$ with a term $q(\phi_r)$ for $r > 0$ is contained in the right ideal generated by $g_+$, and hence does not contribute to the constant. Hence the constant equals $q(\phi_0)^2$, where $\phi_0 \in \wedge^3 g_0$ is the structure constants tensor of $g_0 \subset g$. By [1, 10] this constant is given by $\frac{1}{24} \text{tr}_{g_0}(\text{Cas}_{g_0})$. □

Corollary 2.12. Suppose $\psi_{KP} = d\rho$ for some $\rho \in g_0$. Define elements of $\hat{\mathcal{C}}(g)$ by

$$\phi_{\mathcal{C}} := \phi'_{\mathcal{C}} + \rho^\sharp, \quad \gamma(\xi) = \gamma'(\xi) + \langle \rho, \xi \rangle,$$

for $\xi \in g$. The following commutator relations hold in $\hat{\mathcal{C}}(g)$:

$$[\xi, \zeta] = 2B(\xi, \zeta),$$

$$[\gamma(\xi), \phi_{\mathcal{C}}] = 0,$$

$$[\xi, \phi_{\mathcal{C}}] = 2\gamma(\xi),$$

$$[\gamma(\xi), \gamma(\zeta)] = \gamma([\xi, \zeta]_g),$$

$$[\gamma(\xi), \zeta] = [\xi, \zeta]_g,$$

$$[\phi_{\mathcal{C}}, \phi_{\mathcal{C}}] = 2B(\rho^\sharp, \rho^\sharp) + \frac{1}{12} \text{tr}_{g_0}(\text{Cas}_{g_0}).$$

Thus $\hat{\mathcal{C}}(g)$ becomes a $g$-differential algebra (see e.g. [16]) with differential $d = [\phi_{\mathcal{C}}, \cdot]$, contractions $\iota_\xi = \frac{1}{2} [\xi, \cdot]$, and Lie derivatives $L_\xi = [\gamma(\xi), \cdot]$.

Proof. Observe first that $\lambda(\rho^\sharp) = -\psi_{KP}$, since

$$\iota_\xi \iota_\xi \lambda(\rho^\sharp) = \iota_\xi [\xi, \rho^\sharp]_g = 2B(\xi, [\xi, \rho^\sharp]_g) = -\langle \rho, [\xi, \zeta]_g \rangle.$$

Consequently $[\rho^\sharp, \phi'_{\mathcal{C}}] = -q(\psi_{KP})$, which implies the formula for $[\phi_{\mathcal{C}}, \phi_{\mathcal{C}}]$. The other assertions are verified similarly. □

Still assuming $\psi_{KP} = d\rho$, consider the algebra morphism

$$\gamma : U(g) \to \hat{\mathcal{C}}(g)$$

extending the Lie algebra homomorphism $\xi \mapsto \gamma(\xi)$.

Proposition 2.13. The map (13) extends to an algebra morphism

$$\gamma : \hat{U}(g) \to \hat{\mathcal{C}}(g).$$

Proof. We claim that for all $i > 0$, $\gamma(g_i)$ is contained in

$$\prod_{r \geq 0} \mathcal{C}(g_-)_r \mathcal{C}(g_+)_i \subset \hat{\mathcal{C}}(g)_i$$

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$$\iota_\xi \iota_\xi \lambda(\rho^\sharp) = \iota_\xi [\xi, \rho^\sharp]_g = 2B(\xi, [\xi, \rho^\sharp]_g) = -\langle \rho, [\xi, \zeta]_g \rangle.$$

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Proof. Observe first that $\lambda(\rho^\sharp) = -\psi_{KP}$, since

$$\iota_\xi \iota_\xi \lambda(\rho^\sharp) = \iota_\xi [\xi, \rho^\sharp]_g = 2B(\xi, [\xi, \rho^\sharp]_g) = -\langle \rho, [\xi, \zeta]_g \rangle.$$

Consequently $[\rho^\sharp, \phi'_{\mathcal{C}}] = -q(\psi_{KP})$, which implies the formula for $[\phi_{\mathcal{C}}, \phi_{\mathcal{C}}]$. The other assertions are verified similarly. □
(i.e. the components in Cl($g_+$) have degree $\geq i$). Indeed, suppose $\xi \in g_i$ with $i > 0$. In particular, $\langle \rho, \xi \rangle = 0$. Let $e_a \in g$ be a basis consisting of homogeneous elements, and $e^a$ the dual basis. Since $\langle \rho, \xi \rangle = 0$, and since $[\xi, e_a]_g$ Clifford commutes with $e^a$, we have
\[
\gamma(\xi) = \frac{1}{2} \sum_+ (\xi, e^a)[e_a - e^a\xi, e_a] + \frac{1}{4} \sum_0 \xi, e^a\]
where $\sum_+$ is a summation over indices with $e_a \in g_+$, and $\sum_0$ is a summation over indices with $e_a \in g_0$. The second and third term in this expression are in (14), as are the summands $[\xi, e^a] e_a$ from the first sum for $e_a \in g_s$ with $s \geq i$. In the remaining case $s < i$ we have $[\xi, e^a] e_a \in g_{i-s} \subset g_+$, and hence $[\xi, e^a] e_a \in Cl(g_+)$. This proves the claim. By induction, one deduces that
\[
\gamma(U(g_+)_i) \subset \prod_{r \geq 0} Cl(g_-)_r Cl(g_+)_i + r.
\]
Similarly, if $j \leq 0$,
\[
\gamma(U(g_-)_j) \subset \prod_{r \geq 0} Cl(g_-)_j Cl(g_+)_r.
\]
It follows that
\[
\gamma(U(g_-)_r U(g_+)_i + r) \subset \prod_{m \geq 0} Cl(g_-)_r Cl(g_+)_i + r + m.
\]
Summing over all $r \geq 0$, one obtains a well-defined map $\hat{U}(g)_i \to \hat{Cl}(g)_i$.

3. Double extension

For the loop algebra $g = \mathfrak{t}[z, z^{-1}]$ of a semisimple Lie algebra $\mathfrak{t}$, the Kac-Peterson class is non-trivial. On the other hand, the usual double extension $\tilde{g}$ of $g$ is a symmetrizable Kac-Moody algebra, hence its Kac-Peterson class is zero. In fact, one has a similar double extension in the general case, as we now explain.

We continue to work with the assumptions from the last sections; in particular $g$ carries an invariant non-degenerate symmetric bilinear form $B$ of degree 0. As noted above, the Kac-Peterson cocycle $\psi_{KP}$ gives rise to a skew-symmetric derivation $\Psi_{KP} \in \hat{\mathfrak{g}}(g)$. By a general construction of Medina-Revoy [15], such a derivation can be used to define a double extension
\[
\tilde{g} = g \oplus \mathbb{C}\delta \oplus \mathbb{C}K,
\]
with the following bracket: For $\xi, \xi_1, \xi_2 \in g$,
\[
[\xi_1, \xi_2]_\tilde{g} = [\xi_1, \xi_2]_g + \psi_{KP}(\xi_1, \xi_2)K,
\]
\[
[\delta, \xi]_\tilde{g} = \Psi_{KP}(\xi),
\]
\[
[\delta, K]_\tilde{g} = 0,
\]
\[
[\xi, K]_\tilde{g} = 0
\]
The bilinear form $B$ on $g$ extends to a non-degenerate invariant bilinear form on $\tilde{g}$, in such a way that $g$ and $\mathbb{C}\delta \oplus \mathbb{C}K$ are orthogonal and
\[
\tilde{B}(\delta, K) = 1, \quad \tilde{B}(\delta, \delta) = \tilde{B}(K, K) = 0.
\]
Introduce the grading \( \tilde{g}_i = g_i \) for \( i \neq 0 \) and \( \tilde{g}_0 = g_0 \oplus \mathbb{C} \delta \oplus \mathbb{C} K \). The resulting splitting is
\[
\tilde{g}_- = g_- \oplus \mathbb{C} \delta \oplus \mathbb{C} K, \quad \tilde{g}_+ = g_+.
\]
Let \( \tilde{\psi}_{KP} \) be the Kac-Peterson cocycle for this splitting, \( \tilde{\psi}_{KP} \) the associated derivation, and denote by \( \tilde{\pi} : \tilde{g} \to \tilde{g}_\pm \) the projections along \( \tilde{g}_\mp \). The adjoint representation for \( \tilde{g} \) will be denoted \( \text{ad} \).

**Proposition 3.1.** The derivation \( \tilde{\psi}_{KP} \) is inner:
\[
\tilde{\psi}_{KP} = [\delta, \cdot]_{\tilde{g}}.
\]
Equivalently \( \tilde{\psi}_{KP} = d\rho \) where \( \rho = \tilde{B}(\delta, \cdot) \).

**Proof.** The desired equation \( \tilde{\psi}_{KP} = [\delta, \cdot]_{\tilde{g}} \) means that \( \tilde{\psi}_{KP}(\xi) = \tilde{\psi}_{KP}(\delta) = 0 \), \( \tilde{\psi}_{KP}(K) = 0 \). Equivalently, we have to show that \( \tilde{\psi}_{KP}(\xi_1, \xi_2) = \tilde{\psi}_{KP}(\xi_1, \xi_2) \) for \( \xi_1, \xi_2 \in g \), while both \( K, \delta \) are in the kernel of \( \tilde{\psi}_{KP} \). The last claim follows from
\[
\pi_+ \text{ad} \xi_1 \pi_+ \text{ad} \xi_2 \pi_+ : g_+ \to g_+
\]
of operators on \( g \) coincides with the composition
\[
\pi_+ \text{ad} \xi_1 \pi_- \text{ad} \xi_2 \pi_+ : g_+ \to g_+
\]
of operators on \( \tilde{g} \). Hence the Kac-Peterson cocycles agree on elements of \( g \subseteq \tilde{g} \). \( \square \)

4. The cubic Dirac operator

We will define the cubic Dirac operator as an element of a completion of the quantum Weil algebra \( W(g) = U(g) \otimes \text{Cl}(g) \). Following [1], we take the viewpoint that the commutator with \( D \) defines a differential, making \( \tilde{W}(g) \) into a \( g \)-differential algebra.

4.1. Weil algebra. We begin with an arbitrary \( \mathbb{Z} \)-graded Lie algebra \( g \) with \( \dim g_i < \infty \). As usual \( g^* \) denotes the restricted dual. Consider the tensor product \( W(g^*) = S(g^*) \otimes \wedge(g^*) \) with grading
\[
W^k(g^*) = \bigoplus_{2r+s=k} S^r(g^*) \otimes \wedge^s(g^*).
\]
For \( \mu \in g^* \) we denote by \( s(\mu) = \mu \otimes 1 \) the degree 2 generators and by \( \mu = 1 \otimes \mu \) the degree 1 generators. Any \( \xi \in g \) defines contraction operators \( \iota_\xi \); these are derivations of degree \(-1\) given on generators by \( \iota_\xi \mu = \mu(\xi) \), \( \iota_\xi s(\mu) = 0 \). The co-adjoint action on \( g^* \) defines Lie derivatives \( L_\xi = L_\xi^S \otimes 1 + 1 \otimes L_\xi^\wedge \). If \( \dim(g) < \infty \), the algebra \( W(g) \) carries a Weil differential \( d^W \), given on generators by\(^1\)
\[
d^W \mu = 2(s(\mu) + \lambda(\mu)), \quad d^W s(\mu) = \sum_a s(L_{e_a} \mu) e^a.
\]
\(^1\)The conventions for the differential follow [16, §6.11]. They are arranged to make the relation with the quantum Weil algebra appear most natural. One recovers the more standard conventions used in e.g. [3] and [1] by a simple rescaling of variables.
Here $e_a$ is a basis of $\mathfrak{g}$ with dual basis $e^a \in \mathfrak{g}^*$.

In the general case, we need to pass to a completion in order for the differential to be defined. Define a second $\mathbb{Z}$-grading on $W(\mathfrak{g}^*)$, in such a way that the generators $s(\mu), \mu$ for $\mu \in (\mathfrak{g}^*)_i = (\mathfrak{g}^-)_i^*$ have degree $i$. Letting $\mathfrak{g}^+_i = \bigoplus_{i>0}(\mathfrak{g}^*)_i$ and $\mathfrak{g}^+ = \bigoplus_{i \leq 0}(\mathfrak{g}^*)_i$ we define a completion $\widehat{W}(\mathfrak{g}^*)$ as the graded algebra with

$$\widehat{W}(\mathfrak{g}^*)_i = \prod_{r \geq 0} W(\mathfrak{g}^*)_{i-r} \otimes W(\mathfrak{g}^*)_r.$$  

(Equivalently, $\widehat{W}(\mathfrak{g}^*)_i$ is the space of all linear maps $(S(\mathfrak{g}) \otimes \wedge(\mathfrak{g}))_{-i} \to \mathbb{K}$.) The Weil differential $d^W$ is defined on generators by the formulas (15). Together with the natural extensions of $\iota_\xi$, $L_\xi$ this makes $\widehat{W}(\mathfrak{g}^*)$ into a $\mathfrak{g}$-differential algebra.

4.2. Quantum Weil algebra. Suppose now that $\mathfrak{g}$ carries an invariant symmetric bilinear form $B$ of degree 0. We use $B$ to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and will thus write $W(\mathfrak{g})$, $\widehat{W}(\mathfrak{g})$ and so on. The non-commutative quantum Weil algebra is the tensor product

$$W(\mathfrak{g}) = U(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{g}).$$

It is a super algebra, with even generators $s(\zeta) = \zeta \otimes 1$ and odd generators $\zeta = 1 \otimes \zeta$. Any $\xi \in \mathfrak{g}$ defines Lie derivatives $L_\xi = L_\xi^U \otimes 1 + 1 \otimes L_\xi^{\text{Cl}}$ and contraction operators $\iota_\xi$, given as odd derivations with $\iota_\xi \zeta = B(\xi, \zeta)$, $\iota_\xi s(\zeta) = 0$. Super symmetrization defines an isomorphism

(16) $$q^0: W(\mathfrak{g}) \to W(\mathfrak{g}),$$

given simply as the tensor product of $q^0: S(\mathfrak{g}) \to U(\mathfrak{g})$ and $q^0: \wedge(\mathfrak{g}) \to \text{Cl}(\mathfrak{g})$. Note that (16) intertwines the contractions and Lie derivatives. We define a completion $\widehat{W}(\mathfrak{g})$ as the graded super algebra with

$$\widehat{W}(\mathfrak{g})_i = \prod_{r \geq 0} W(\mathfrak{g}^*)_{i-r} \otimes W(\mathfrak{g}^*)_r.$$  

The ‘normal-ordered’ quantization map $q: \widehat{W}(\mathfrak{g}) \to \widehat{W}(\mathfrak{g})$ is defined by summing over all

$$q^0 \otimes q^0: W(\mathfrak{g}^*)_{i-r} \otimes W(\mathfrak{g}^*)_r \to W(\mathfrak{g}_i)_{i-r} \otimes W(\mathfrak{g}_i)_r.$$  

It extends the quantization maps $q: S(\mathfrak{g}) \to \widehat{S}(\mathfrak{g})$ and $q: \wedge(\mathfrak{g}) \to \widehat{\wedge}(\mathfrak{g})$.

4.3. The element $q(D)$. If $\dim \mathfrak{g} < \infty$, one obtains a differential $d^W$ on $W(\mathfrak{g})$, as a derivation given on generators by formulas similar to (15),

$$d^W s(\zeta) = 2(s(\zeta) + g_0(\lambda(\zeta))), \quad d^W s(\zeta) = \sum_a s(L_{e_a} \zeta) e^a,$$

see [1]. In fact, $d^W = [q^0(D), \cdot]$, where $D \in W^3(\mathfrak{g})$ is the element

$$D = \sum_a s(e_a) e^a + \phi,$$

with $\phi \in \wedge^3 \mathfrak{g} \subset W^3(\mathfrak{g})$ the structure constants tensor. The fact that $d^W$ squares to zero means that $q^0(D)$ squares to a central element, and indeed one finds

$$q^0(D)^2 = \text{Cas}_g + \frac{1}{24} \text{tr}_g(\text{Cas}_g).$$
If dim \( \mathfrak{g} = \infty \), the element \( D \) is well-defined as an element of the completion \( \hat{W}^3(\mathfrak{g}) \), but \( q^0(D) \) is ill-defined. On the other hand,

\[
D' = q(D) = \sum_a s(e_a) e^a + \phi'_{C_1}
\]

is defined but does not square to a central element.

**Proposition 4.1.** The square of \( D' = q(D) \) is given by

\[
(D')^2 = \text{Cas}'_{g'} + q(\psi_{KP}^*) + \frac{1}{24} \text{tr}_{g_0}(\text{Cas}_{g_0}).
\]

**Proof.** We have

\[
L_\xi D' = L_\xi \phi'_{C_1} = \Psi_{KP}(\xi) = \iota_\xi q(\psi_{KP}^*)
\]

because \( \sum_a s(e_a) e^a \in \hat{W}(\mathfrak{g}) \) is \( \mathfrak{g} \)-invariant. Using that

\[
\iota_\xi (D') = s(\xi) + \iota_\xi (q(\phi)) = s(\xi) + \gamma'(\xi)
\]

are generators for the \( \mathfrak{g} \)-action on \( \hat{W}(\mathfrak{g}) \), we have

\[
\iota_\xi ((D')^2 - q(\psi_{KP}^*)) = [\iota_\xi D', D'] - q(\psi_{KP}^*) = 0.
\]

This shows \( (D')^2 - q(\psi_{KP}^*) \in \hat{U}(\mathfrak{g}) \subset \hat{W}(\mathfrak{g}) \). To find this element we calculate, denoting by \( \ldots \) terms in the kernel of the projection \( \hat{W}(\mathfrak{g}) \to \hat{U}(\mathfrak{g}) \),

\[
(D')^2 = \sum_{ab} s(e_a) s(e_b) e^a e^b + (\phi'_{C_1})^2 + \ldots
\]

\[
= \frac{1}{2} \sum_{ab} s(e_a) s(e_b) [e^a, e^b] + \frac{1}{24} \text{tr}_{g_0}(\text{Cas}_{g_0}) + \ldots
\]

\[
= \text{Cas}'_{g'} + \frac{1}{24} \text{tr}_{g_0}(\text{Cas}_{g_0}) + \ldots
\]

If the Kac-Peterson class is trivial, one obtains an element \( D \) with better properties.

**Corollary 4.2.** Suppose that \( \psi_{KP} = d\rho \) for some \( \rho \in \mathfrak{g}_0^* \). Define

\[
\mathcal{D} = D' + \rho^* \mathcal{W}(\xi) = s(\xi) + \gamma'(\xi) + \langle \rho, \xi \rangle,
\]

and put \( \text{Cas}_g = \text{Cas}'_g + 2\rho^* \) as before. Then

\[
\mathcal{D}^2 = \text{Cas}_g \otimes 1 + \frac{1}{24} \text{tr}_{g_0}(\text{Cas}_{g_0}) + B(\rho^*, \rho^*).
\]

One has the following commutator relations in \( \hat{W}(\mathfrak{g}) \),

\[
[D, \mathcal{D}] = 2 \text{Cas}_g \otimes 1 + \frac{1}{12} \text{tr}_{g_0}(\text{Cas}_{g_0}) + 2B(\rho^*, \rho^*),
\]

\[
[\gamma_W(\xi), \mathcal{D}] = 0,
\]

\[
[\xi, \mathcal{D}] = 2\gamma_W(\xi),
\]

\[
[\gamma_W(\xi), \gamma_W(\zeta)] = \gamma_W(\langle \xi, \zeta \rangle),
\]

\[
[\gamma_W(\xi), \zeta] = [\xi, \zeta]_g,
\]

\[
[\xi, \zeta] = 2B(\xi, \zeta).
\]
Thus $\hat{\mathcal{W}}(\mathfrak{g})$ becomes a $\mathfrak{g}$-differential algebra, with differential, Lie derivatives and contractions given by

$$d^\mathcal{W} = [D, \cdot], \quad L^\mathcal{W}_\xi = [\gamma^\mathcal{W}(\xi), \cdot], \quad \iota^\mathcal{W}_\xi = \frac{1}{2}[\xi, \cdot].$$

We will refer to $D \in \hat{\mathcal{W}}(\mathfrak{g})$ as the cubic Dirac operator, following Kostant [10].

5. **Relative Dirac operators**

In his paper [10], Kostant introduced more generally Dirac operators for any pair of a quadratic Lie algebra $\mathfrak{g}$ and a quadratic Lie subalgebra $\mathfrak{u}$. We consider now an extension of his results to infinite-dimensional graded Lie algebras.

Let $\mathfrak{g}, B$ be as in the last Section, and suppose $\mathfrak{u} \subseteq \mathfrak{g}$ is a graded quadratic subalgebra. That is, $\mathfrak{u}_i \subseteq \mathfrak{g}_i$ for all $i$, and the non-degenerate symmetric bilinear form $B$ on $\mathfrak{g}$ restricts to a non-degenerate bilinear form on $\mathfrak{u}$. We have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$$

where $\mathfrak{p} = \mathfrak{u}^\perp$. For any $\xi \in \mathfrak{u}$, the operator $\text{ad}_\xi \in \hat{\mathfrak{o}}(\mathfrak{g})$ breaks up as a sum

$$\text{ad}_\xi = \text{ad}^\mathfrak{u}_\xi + \text{ad}^\mathfrak{p}_\xi, \quad \xi \in \mathfrak{u}$$

of operators $\text{ad}^\mathfrak{u}_\xi \in \hat{\mathfrak{o}}(\mathfrak{u})$ and $\text{ad}^\mathfrak{p}_\xi \in \hat{\mathfrak{o}}(\mathfrak{p})$. Accordingly,

$$\lambda(\xi) = \lambda^\mathfrak{u}(\xi) + \lambda^\mathfrak{p}(\xi), \quad \xi \in \mathfrak{u}$$

with $\lambda^\mathfrak{u}(\xi) \in \hat{\mathfrak{n}}^2(\mathfrak{u})$ and $\lambda^\mathfrak{p}(\xi) \in \hat{\mathfrak{n}}^2(\mathfrak{p})$. Denote by $\gamma'_u(\xi), \gamma'_p(\xi)$ their images under $q: \hat{\mathcal{W}}(\mathfrak{g}) \to \hat{\mathcal{W}}(\mathfrak{g})$. We have (cf. (7))

$$[\gamma'_p(\xi), \gamma'_p(\xi)] = \gamma'_p([\xi, \xi]) + \psi_{KP}^p(\xi, \xi),$$

where $\psi_{KP}^p(\xi, \zeta) = \psi_{KP}^p(\text{ad}_\xi^\mathfrak{p}, \text{ad}_\zeta^\mathfrak{p})$ defines a cocycle $\psi_{KP}^p \in \hat{\mathfrak{n}}^2(\mathfrak{u}^*)$. If $\rho^p = d\rho^p$ for some $\rho^p \in \mathfrak{u}_0^*$, then

$$\gamma_p(\xi) = \gamma'_p(\xi) + \langle \rho^p, \xi \rangle$$

gives a Lie algebra homomorphism $\mathfrak{u} \to \hat{\mathfrak{Cl}}(\mathfrak{p})$, generating the adjoint action of $\mathfrak{u}$. One obtains an algebra homomorphism $j: \mathcal{W}(\mathfrak{u}) \to \hat{\mathcal{W}}(\mathfrak{g})$, given on generators by

$$j(\xi) = \xi, \quad j(s(\xi)) = s(\xi) + \gamma_p^p(\xi), \quad \xi \in \mathfrak{u}.$$  

**Proposition 5.1.** The homomorphism $\mathcal{W}(\mathfrak{u}) \to \hat{\mathcal{W}}(\mathfrak{g})$ extends to an algebra homomorphism for the completion:

$$j: \hat{\mathcal{W}}(\mathfrak{u}) \to \hat{\mathcal{W}}(\mathfrak{g}).$$

It intertwines Lie derivatives and contraction by elements $\xi \in \mathfrak{u}$.

**Proof.** The first part follows by an argument parallel to that for Proposition 2.13. The second part follows from

$$j \circ L_\xi = j \circ [s(\xi) + \gamma'_u(\xi), \cdot] = [s(\xi) + \gamma'_p(\xi), \cdot] \circ j = L_\xi \circ j$$

and similarly $j \circ \iota_\xi = \frac{1}{2} j \circ [\xi, \cdot] = \frac{1}{2} [\xi, \cdot] \circ j = \iota_\xi \circ j$.  

\[\square\]
Proof. Since \( \iota_\xi = \frac{1}{\xi} [\xi, -] \), an element of \( \hat{W}(g) \) commutes with the generators \( j(\xi) \) for \( \xi \in u \) precisely if it lies in the \( u \)-horizontal subspace, given as the completion of \( U(g) \otimes \text{Cl}(p) \). The elements \( j(s(\xi)) = s(\xi) + \gamma_p^* (\xi) \) generate the \( u \)-action on that subspace. Hence, an element of \( \hat{W}(g) \) commutes with all \( j(\xi), j(s(\xi)) \) if and only if it is \( u \)-basic. \( \square \)

We will now make the stronger assumption that the Kac-Petersson classes of both \( g, u \) are zero. Let \( \rho \in g^*_0, \rho_u \in u^*_0 \) be elements such that

\[
\psi_{K P}^* = d\rho, \quad \psi_{K P}^u = d\rho_u,
\]

and take \( \rho_p := \rho|_{u_0} - \rho_u \in u^*_0 \) so that \( \psi_{K P}^p = d\rho_p \). Put

\[
\gamma(\xi) = \gamma'(\xi) + \langle \rho, \xi \rangle, \quad \gamma_u(\xi) = \gamma_u'(\xi) + \langle \rho_u, \xi \rangle
\]

for all \( \xi \in g, \xi \in u \), and let

\[
D = D' + \rho_p^* \in \hat{W}(g), \quad D_u = D_u' + \rho_u^* \in \hat{W}(u)
\]

be the cubic Dirac operators for \( g, u \). The commutator with these elements defines differentials on the two Weil algebras.

Lemma 5.3. The map \( j: \hat{W}(u) \to \hat{W}(g) \) is a homomorphism of \( u \)-differential algebras.

Proof. It remains to show that the map \( j \) intertwines differentials. It suffices to check on generators. For \( \xi \in u \),

\[
j(d\xi) = j(s_u(\xi) + \gamma_u(\xi)) = s(\xi) + \gamma_p(\xi) + \gamma_u(\xi) = s(\xi) + \gamma(\xi) = d j(\xi),
\]

and similarly \( j(ds_u(\xi)) = d j(s_u(\xi)) \). \( \square \)

We define the relative cubic Dirac operator \( D_{g,u} \) as a difference,

\[
D_{g,u} = D - j(D_u).
\]

Proposition 5.4. The element \( D_{g,u} \) lies in \( \hat{W}(g, u) \), and squares to an element of the center of \( \hat{W}(g, u) \). Explicitly,

\[
D_{g,u}^2 = \text{Cas}_g - j(\text{Cas}_u) + \frac{1}{24} \text{tr}_{g_0}(\text{Cas}_{g_0}) - \frac{1}{24} \text{tr}_{u_0}(\text{Cas}_{u_0}) + B(\rho_p^*, \rho^*) - B(\rho_u^*, \rho_u^*).
\]

Proof. Using that \( j \) intertwines contractions \( \iota_\xi, \xi \in u \), we find

\[
\iota_\xi D_{g,u} = \iota_\xi D - j(\iota_\xi D_u)
\]

\[
= s(\xi) + \gamma(\xi) - j(s_u(\xi) + \gamma_u(\xi))
\]

\[
= \gamma(\xi) - \gamma_p(\xi) - \gamma_u(\xi) = 0.
\]
Thus $\mathcal{D}_{g,u}$ is $u$-horizontal, and it is clearly $u$-invariant as well. Thus $\mathcal{D}_{g,u} \in \hat{\mathcal{W}}(g,u)$. In particular, $\mathcal{D}_{g,u}$ commutes with $j(\mathcal{D}_u)$. Consequently, $[\mathcal{D}, \mathcal{D}] = j([\mathcal{D}_u, j(\mathcal{D}_u)]) + [\mathcal{D}_{g,u}, \mathcal{D}_{g,u}]$, that is

$$\mathcal{D}_{g,u}^2 = \mathcal{D}^2 - j(\mathcal{D}_u^2).$$

Now use Corollary 4.2.

\[\square\]

6. Application to Kac-Moody algebras

In his paper [10], Kostant used the cubic Dirac operator $\mathcal{D}_{g,u}$ to prove generalized Weyl character formulas for any pair of a semi-simple Lie algebra $g$ and equal rank subalgebra $u$. In this Section, we show that much of this theory carries over to symmetrizable Kac-Moody algebras, with only minor adjustments.

6.1. Notation and basic facts. Let us recall some notation and basic facts; our main references are the books by Kac [5] and Kumar [13].

Let $A = (a_{ij})_{1 \leq i,j \leq l}$ be a generalized Cartan matrix, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $A$. Thus $\mathfrak{h}$ is a vector space of dimension $2l - \mathrm{rk}(A)$, and $\Pi = \{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{h}^*$ (the set of simple roots) and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \subset \mathfrak{h}$ (the corresponding co-roots) satisfy $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$. The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra generated by elements $h \in \mathfrak{h}$ and elements $e_j, f_j$ for $j = 1, \ldots, l$, subject to relations

$$[h, e_i] = \langle \alpha_i, h \rangle e_i, \quad [h, f_i] = -\langle \alpha_i, h \rangle f_i, \quad [h, h'] = 0, \quad [e_i, f_j] = \delta_{ij} \alpha_i^\vee,$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0, \quad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0, \quad i \neq j.$$

The non-zero weights $\alpha \in \mathfrak{h}^*$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ are called the roots, the corresponding root spaces are denoted $\mathfrak{g}_\alpha$. The set $\Delta$ of roots is contained in the lattice $Q = \bigoplus_{j=1}^l \mathbb{Z} \alpha_j \subset \mathfrak{h}^*$. Let $Q^+ = \bigoplus_{j=1}^l \mathbb{Z}_{\geq 0} \alpha_j$, and put $\Delta^+ = \Delta \cap Q^+$ and $\Delta^- = -\Delta^+$. One has $\Delta = \Delta^+ \cup \Delta^-.$

Let $W$ be the Weyl group of $\mathfrak{g}$, i.e. the group of transformations of $\mathfrak{h}$ generated by the simple reflections $\xi \mapsto \xi - \langle \alpha_j, \xi \rangle \alpha_j^\vee$. The dual action of $W$ as a reflection group on $\mathfrak{h}^*$ preserves $\Delta$. Let $\Delta^{re}$ be the set of real roots, i.e. roots that are $W$-conjugate to roots in $\Pi$, and let $\Delta^{im}$ be its complement, the imaginary roots. For $\alpha \in \Delta^{re}$ one has $\dim \mathfrak{g}_\alpha = 1$.

The length $\ell(w)$ of a Weyl group element may be characterized as the cardinality of the set

$$\Delta^+_w = \Delta^+ \cap w\Delta^-$$

of positive roots that become negative under $w^{-1}$ [13, Lemma 1.3.14]. We remark that $\Delta^+_w \subset \Delta^{re}$ [5, §5.2].

Fix a real subspace $\mathfrak{h}^c \subset \mathfrak{h}$ containing $\Pi^\vee$. Let $C \subset \mathfrak{h}^c$ be the dominant chamber and $X$ the Tits cone [5, §3.12]. Thus $C$ is the set of all $\xi \in \mathfrak{h}^c$ such that $\langle \alpha, \xi \rangle \geq 0$ for all $\alpha \in \Pi$, while $X$ is characterized by the property that $\langle \alpha, \xi \rangle < 0$ for at most finitely many $\alpha \in \Delta$. The $W$-action preserves $X$, and $C$ is a fundamental domain in the sense that every $W$-orbit in $X$ intersects $C$ in a unique point.

For any $\mu = \sum_{j=1}^l k_j \alpha_j \in Q$ one defines $\mathrm{ht}(\mu) = \sum_{j=1}^l k_j$. The principal grading on $\mathfrak{g}$ is defined by letting $\mathfrak{g}_i$ for $i \neq 0$ be the direct sum of root spaces $\mathfrak{g}_\alpha$ with $\mathrm{ht}(\alpha) = i$, and $\mathfrak{g}_0 = \mathfrak{h}$. Letting $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, it follows that $\mathfrak{g}_+ = \mathfrak{n}_+$ and $\mathfrak{g}_- = \mathfrak{n}_- \oplus \mathfrak{h}$. 

6.2. The Kac-Peterson cocycle. Suppose from now on that $A$ is symmetrizable, that is, there exists a diagonal matrix $D = \text{diag}(\epsilon_1, \ldots, \epsilon_l)$ such that $D^{-1}A$ is symmetric. In this case, $\mathfrak{g}$ carries a non-degenerate symmetric invariant bilinear form $B$ with the property $B(\alpha_j', \xi) = \epsilon_j(\alpha_j, \xi)$, $\xi \in \mathfrak{h}$ [5, §2.2]. One refers to $B$ as a standard bilinear form. Choose $\rho \in \mathfrak{h}^*$ with $\langle \rho, \alpha_j' \rangle = 1$ for $j = 1, \ldots, l$.

**Proposition 6.1.** The Kac-Peterson cocycle of the symmetrizable Kac-Moody algebra $\mathfrak{g}$ is exact. In fact, \[ \psi_{KP} = d\rho. \]

**Proof.** Use $B$ to define $\text{Cas}_{\mathfrak{g}}'$. As shown by Kac [5, Theorem 2.6] the operator $\text{Cas}_{\mathfrak{g}} := \text{Cas}_{\mathfrak{g}}' + 2\rho^*$ is $\mathfrak{g}$-invariant. By Corollary 2.8 above this is equivalent to $\psi_{KP} = d\rho$. \[ \square \]

6.3. Regular subalgebras. We now introduce a suitable class of ‘equal rank’ subalgebras. Following Morita and Naito [17, 18], consider a linearly independent subset $\Pi_\mathfrak{u} \subset \Delta^{re,+}$ with the property that the difference of any two elements in $\Pi_\mathfrak{u}$ is not a root. We denote by $\mathfrak{u} \subset \mathfrak{g}$ the Lie subalgebra generated by $\mathfrak{h}$ together with the root spaces $\mathfrak{g}_{\pm,\beta}$ for $\beta \in \Pi_\mathfrak{u}$. Let $\mathfrak{p} = \mathfrak{u}^\perp$, so that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$.

**Examples 6.2.** (a) If $\Pi_\mathfrak{u} = \emptyset$ one obtains $\mathfrak{u} = \mathfrak{h}$. (b) Suppose $\mathfrak{g}$ is an affine Kac-Moody algebra, i.e. the double extension of a loop algebra $\mathfrak{k}[z, z^{-1}]$ of a semi-simple Lie algebra $\mathfrak{k}$. Let $\mathfrak{l} \subset \mathfrak{k}$ be an equal rank subalgebra of $\mathfrak{k}$. Let $\Pi_\mathfrak{l} \subset \Delta^{+}_{\mathfrak{k}}$ be the simple roots of $\mathfrak{l}$, and $\Pi_\mathfrak{u} \subset \Delta^{+}$ the corresponding affine roots. Then $\mathfrak{u} = \mathfrak{k}[z, z^{-1}]$. This is the setting considered in Landweber’s paper [14].

It was shown in [17, 18] that $\mathfrak{u}$ is a direct sum (as Lie algebras) of a symmetrizable Kac-Moody algebra $\hat{\mathfrak{u}}$ with a subalgebra of $\mathfrak{h}$.\footnote{In fact, Naito [18] constructs an explicit subspace $\mathfrak{h} \subset \mathfrak{h}$ such that the Lie algebra $\hat{\mathfrak{g}}$ generated by $\mathfrak{h}$ and the $\mathfrak{g}_{\pm,\beta}$, $\beta \in \Pi_\mathfrak{u}$ is a Kac-Moody algebra. He also considers subsets $\Pi_\mathfrak{u}$ that do not necessarily consist of real roots, and finds that the resulting $\hat{\mathfrak{u}}$ is a symmetrizable generalized Kac-Moody algebra.}

Furthermore, the standard bilinear form $B$ on $\mathfrak{g}$ restricts to a standard bilinear form on $\hat{\mathfrak{u}}$.

For any root $\alpha \in \Delta$ put $n_\alpha = \dim \mathfrak{u}_\alpha$ and $n_\mathfrak{p}(\alpha) = \dim(\mathfrak{p}_\alpha)$. Thus $n(\alpha) = n_\mathfrak{u}(\alpha) + n_\mathfrak{p}(\alpha)$ is the multiplicity of $\alpha$ in $\mathfrak{g}$. Let $\Delta_\mathfrak{u}$ (resp. $\Delta_\mathfrak{p}$) be the set of roots such that $n(\alpha) > 0$ (resp. $n_\mathfrak{p}(\alpha) > 0$). Thus $\Delta_\mathfrak{u}$ is the set of roots of $\mathfrak{u}$. Let $W_\mathfrak{u} \subset W$ be the Weyl group of $\mathfrak{u}$ (generated by reflections for elements of $\Pi_\mathfrak{u}$), and define a subset

\[ W_\mathfrak{p} = \{ w \in W \mid w^{-1}\Delta_\mathfrak{u}^+ \subset \Delta^+ \}. \]

**Lemma 6.3.** We have $w \in W_\mathfrak{p} \iff \Delta_\mathfrak{u}^+ \subset \Delta_\mathfrak{p}$. Every $w \in W$ can be uniquely written as a product $w = w_1w_2$ with $w_1 \in W_\mathfrak{u}$ and $w_2 \in W_\mathfrak{p}$.

**Proof.** By definition, $w \in W_\mathfrak{p}$ if and only if the intersection $\Delta_\mathfrak{u}^+ \cap w\Delta^- = \Delta_\mathfrak{u} \cap \Delta_\mathfrak{u}^+$ is empty. Since $\Delta_\mathfrak{u}^+$ consists of real roots, this means $\Delta_\mathfrak{u}^+ \subset \Delta_\mathfrak{p}$. For the second claim, let $C_\mathfrak{u} \subset X_\mathfrak{u}$ be the chamber and Tits cone for $\mathfrak{u}$. One has $w \in W_\mathfrak{p}$ if and only if $w^{-1}\Delta_\mathfrak{u}^+ \subset \Delta_\mathfrak{p}$, if and only if $wC \subset C_\mathfrak{u}$. Let $w \in W$ be given. Then $wC \subset X_\mathfrak{u}$ is contained in a unique chamber of $\mathfrak{u}$. Hence there is a unique $w_1 \in W_\mathfrak{u}$ such that $wC \subset w_1C_\mathfrak{u}$. Equivalently, $w_2 := w^{-1}w \in W_\mathfrak{p}$. \[ \square \]
We have a decomposition $p = p_+ \oplus p_-$, where $p_\pm = p \cap n_\pm$. The splitting defines a spinor module $S_p = \bigwedge p_-$ over $\mathrm{Cl}(p)$, where the elements of $p_+$ act by contraction and those of $p_-$ by exterior multiplication. The Clifford action on this module extends to the completion $\tilde{\mathrm{Cl}}(S_p)$.

Fix $\rho_u \in \mathfrak{h}^*$ with $\langle \rho_u, \beta' \rangle = 1$ for all $\beta \in \Pi_u$. Let $\rho_p = \rho|_u - \rho_u$, defining a Lie algebra homomorphism $\gamma_p = \gamma'_p + \rho_p \colon u \to \tilde{\mathrm{Cl}}(p)$. By composition with the spinor action one obtains an integrable $u$-representation

$\pi_5 \colon u \to \text{End}(S_p)$.

**Proposition 6.4.** The restriction of $\pi_5$ to $\mathfrak{h} \subset u$ differs from the adjoint representation of $\mathfrak{h}$ by a $\rho_p$-shift:

$$\pi_5(\xi) = \langle \rho_p, \xi \rangle + \text{ad}(\xi), \quad \xi \in \mathfrak{h}.$$  

Hence, the weights for the action of $\mathfrak{h}$ on $S_p$ are of the form

$$\rho_p - \sum_{\alpha \in \Delta_p^+} k_\alpha \alpha,$$

where $0 \leq k_\alpha \leq n_p(\alpha)$. The parity of the corresponding weight space is $\sum k_\alpha \mod 2$. For all $w \in W_p$, the element

$$w \rho_p - \rho_u$$

is a weight of $S_p$, of multiplicity 1. The parity of the weight space $S_p$ equals $l(w) \mod 2$.

**Proof.** For each $\alpha \in \Delta_p^+$, fix a basis $e^{(s)}_{\alpha}$, $s = 1, \ldots, n_p(\alpha)$ of $p_\alpha$, and let $e^{(s)}_{-\alpha}$ be the $B$-dual basis of $p_-\alpha$. By definition, we have $\gamma_p(\xi) = \langle \rho_p, \xi \rangle + \gamma'_p(\xi)$ with

$$\gamma'_p(\xi) = -\frac{1}{2} \sum_{\alpha \in \Delta_p^+} \sum_{s=1}^{n_p(\alpha)} \langle \alpha, \xi \rangle e^{(s)}_{-\alpha} e^{(s)}_{\alpha}.$$  

The action of $\gamma'_p(\xi)$ on the spinor module is just the adjoint action of $\xi$. This proves the first assertion. It is now straightforward to read off the weights of the action on $S_p$. For all $w \in W$ one has $\rho - w \rho = \sum_{\alpha \in \Delta_w^+} \alpha$ (cf. [13, Corollary 1.3.22]). If $w \in W_p$, so that $\Delta_w^+ \subset \Delta_p^+$, it follows that $w \rho - \rho_u = w \rho - \rho + \rho_p = \rho_p - \sum_{\alpha \in \Delta_w^+} \alpha$ is a weight of $S_p$. We now use

$$S_{\mathfrak{h}^\perp} = S_p \otimes S_{u \cap \mathfrak{h}^\perp}$$

as modules over $\mathrm{Cl}(\mathfrak{h}^\perp) = \mathrm{Cl}(\mathfrak{p}) \otimes \mathrm{Cl}(u \cap \mathfrak{h}^\perp)$. Hence, the tensor product with a generator of the line $(S_{u \cap \mathfrak{h}^\perp})_{\rho_u}$ defines an isomorphism of the weight space $(S_p)_{w \rho - \rho_u}$ with $(S_{\mathfrak{h}^\perp})_{w \rho}$; but the latter is 1-dimensional, and its parity is given by $l(w) \mod 2$ (cf. [13, Lemma 3.2.6]).

---

### 6.4. Action of the cubic Dirac operator.

The subalgebra $u$ inherits a $\mathbb{Z}$-grading from $g$, with $u_i$ the direct sum of root spaces $u_\alpha$ for $\alpha = \sum r k_r \beta_r$ and $i = \sum r m_r$. It is thus the grading of type $m = (m_1, \ldots, m_r)$ [5, §1.5] with $m_r = \mathrm{ht} (\beta_r)$. Let $\mathcal{W}(u)$ be the completion of the quantum Weil algebra for this grading. (It is just the same as the completion defined by the principal grading of $u$).

Let $P \subset \mathfrak{h}^*$ be the weight lattice of $g$, and $P^+ \subset P$ the dominant weights. Thus $\mu \in P$ if and only if $\langle \mu, \alpha_j^\vee \rangle \in \mathbb{Z}$ for $j = 1, \ldots, l$, and $\mu \in P^+$ if these pairings are all non-negative.
For any $\mu \in P^+$ let $L(\mu)$ be the irreducible integrable representation of $\mathfrak{g}$ of highest weight $\mu$. By [5, §11.4], $L(\mu)$ carries a unique (up to scalar) Hermitian form for which the elements of the real form of $\mathfrak{g}$ are represented as skew-adjoint operators. The weights $\nu$ of $L(\mu)$ satisfy $\mu - \nu \in Q^+$, hence there is a $\mathbb{Z}$-grading on $L(\mu)$ such that elements of $L(\mu)_{\nu}$ have degree $j = -\text{ht}(\mu - \nu)$. The $\mathfrak{g}$-action is compatible with the gradings, i.e. the action map $\mathfrak{g} \otimes L(\mu) \to L(\mu)$ preserves gradings. The spinor module $S_p = \wedge p_-$ carries the $\mathbb{Z}$-grading defined by the $\mathbb{Z}$-grading on $p_-$, and the module action $\text{Cl}(p) \otimes S_p \to S_p$ preserves gradings.

The action of $\mathcal{W}(\mathfrak{g}, u)$ on the graded vector space $L(\mu) \otimes S_p$ extends to an action of the completion $\hat{\mathcal{W}}(\mathfrak{g}, u)$. We denote by

$$\mathcal{D}_{L(\mu)} \in \hat{\text{End}}(L(\mu) \otimes S_p)$$

the image of $\mathcal{D}_{\mathfrak{g}, u}$ under this representation. Then $\mathcal{D}_{L(\mu)}$ is an odd, skew-adjoint operator.

Since $\mathcal{D}_{L(\mu)}$ commutes with the diagonal action of $u$ on $L(\mu) \otimes S_p$, its kernel $\ker(\mathcal{D}_{L(\mu)})$ is a $\mathbb{Z}_2$-graded $u$-representation.

Let $P^+_u \subset P_u \subset \mathfrak{b}^*$ be the set of dominant weights for $u$. For any $\nu \in P^+_u$, let $M(\nu)$ be the corresponding irreducible highest weight representation of $u$. Parallel to [10, Theorem 4.24] we have:

**Theorem 6.5.** The kernel of the operator $\mathcal{D}_{L(\mu)}$ is a direct sum,

$$\ker(\mathcal{D}_{L(\mu)}) = \bigoplus_{w \in W_p} M(w(\mu + \rho) - \rho_u).$$

Here the even (resp. odd) part of the kernel is the sum over the $w \in W_p$ such that $l(w)$ is even (resp. odd).

**Proof.** Given an integrable $u$-representation, and any $u$-dominant weight $\nu \in P^+_u$, let the subscript $[\nu]$ denote the corresponding isotypical subspace. We are interested in $\ker(\mathcal{D}_{L(\mu)})[\nu]$. Since $\mathcal{D}_{L(\mu)}$ is skew-adjoint, its kernel coincides with that of its square:

$$\ker(\mathcal{D}_{L(\mu)}) = \ker(\mathcal{D}^2_{L(\mu)}).$$

The action of $\text{Cas}_\mathfrak{g}$ on $L(\mu)$ is as a scalar $B(\mu + \rho, \mu + \rho) - B(\rho, \rho)$, and similarly for the action of $\text{Cas}_u$ on $M(\nu)$. Hence

$$\mathcal{D}^2_{L(\mu)} = B(\mu + \rho, \mu + \rho) - j(\text{Cas}_u) - B(\rho_u, \rho_u)$$

acts on $(L(\mu) \otimes S_p)[\nu]$ as a scalar, $B(\mu + \rho, \mu + \rho) - B(\nu + \rho_u, \nu + \rho_u)$. This shows that

$$\ker(\mathcal{D}_{L(\mu)})[\nu] = \bigoplus_{\nu} (L(\mu) \otimes S_p)[\nu],$$

where the sum $\bigoplus_{\nu}$ is over all $\nu \in \Delta_u$ satisfying $B(\mu + \rho, \mu + \rho) = B(\nu + \rho_u, \nu + \rho_u)$. We want to identify this sum as a sum over $W_p$.

Suppose $\nu$ is any weight with $(L(\mu) \otimes S_p)_\nu \neq 0$. We will show $B(\nu + \rho_u, \nu + \rho_u) \leq B(\mu + \rho, \mu + \rho)$. By [5, Prop. 11.4(b)], an element $\nu \in P_u$ for which equality holds is automatically in $P^+_u$, and the multiplicity of $M(\nu)$ in $L(\mu) \otimes S_p$ is then equal to the dimension of the highest weight space $(L(\mu) \otimes S_p)_\nu$. Write $\nu = \nu_1 + \nu_2$ where $L(\mu)_{\nu_1}$ and $(S_p)_{\nu_2}$ are non-zero. By our description of the set of weights of $S_p$, the element $\nu_2 + \rho_u$ is among the weights of the $\mathfrak{g}$-representation $L(\rho)$, and in particular lies in the dual Tits cone $X^\vee$ of $\mathfrak{g}$. Since the Tits cone is convex, and $\nu_1 \in X^\vee$, it follows that $\nu_1 + (\nu_2 + \rho_u) = \nu + \rho_u \in X^\vee$. 


Consequently, there exists \( w \in W \) such that \( w^{-1}(\nu + \rho_u) \in C^\nu \subset \mathfrak{h}^* \). Since \( \nu_2 + \rho_u \) is a weight of \( L(\rho) \), so is its image under \( w^{-1} \). Hence
\[
\kappa_2 = \rho - w^{-1}(\nu_2 + \rho_u) \in Q^+.
\]
On the other hand, since \( w^{-1}\nu_1 \) is a weight of \( L(\mu) \), we also have \( \kappa_1 = \mu - w^{-1}\nu_1 \in Q^+ \). Adding, we obtain
\[
\mu + \rho = \kappa + w^{-1}(\nu + \rho_u),
\]
with \( \kappa = \kappa_1 + \kappa_2 \in Q^+ \). Since the pairing of \( \kappa \) with \( w^{-1}(\nu + \rho_u) \in C^\nu \) is non-negative, the inequality \( B(\mu + \rho, \mu + \rho) \geq B(\nu + \rho_u, \nu + \rho_u) \) follows. Equality holds if and only if \( \kappa = 0 \), i.e. \( \kappa_1 = 0 \) and \( \kappa_2 = 0 \), i.e. \( \nu_2 = w\rho - \rho_u \) and \( \nu_1 = w\mu \). The \( \mathfrak{h} \)-weight spaces \( (S_p)_{w\rho - \rho_u} \) and \( L(\mu)_{w\nu} \) are 1-dimensional, hence so is their tensor product, \( (L(\mu) \otimes S_p)_\nu \). It follows that \( \nu \) appears with multiplicity 1.

This shows that \( M(\nu) \) appears in \( \ker(D_{L(\mu)}) \) if and only if it can be written in the form \( \nu = w(\mu + \rho) - \rho_u \), for some \( w \in W_p \), and in this case it appears with multiplicity 1. Note finally that \( w \) with this property is unique, since \( \mu + \rho \) is regular. The parity of the \( \nu \)-isotypical component follows since \( (S_p)_{w\rho - \rho_u} \) has parity equal to that of \( l(w) \). \( \square \)

The weights
\[
\nu = w(\mu + \rho) - \rho_u, \ w \in W_p
\]
are referred to as the multiplet corresponding to \( \mu \). Note that for given \( \mu \), the value of the quadratic Casimir \( \text{Cas}_u \) on the representations \( M(w(\mu + \rho) - \rho_u) \) is given by the constant value \( B(\mu + \rho, \mu + \rho) - B(\rho_u, \rho_u) \), independent of \( w \).

6.5. Characters. For any weight \( \nu \in \mathfrak{h}^* \), we write \( e(\nu) \) for the corresponding formal exponential. We will regard the spinor module as a super representation, using the usual \( \mathbb{Z}_2 \)-grading of the exterior algebra. The even and odd part are denoted \( S^0_p \) and \( S^1_p \), and its formal character \( c_h(S_p) = \sum_\nu (\dim(S^0_p)_\nu - \dim(S^1_p)_\nu) e(\nu) \). Here \( (S^0_p)_\nu \) and \( (S^1_p)_\nu \) are the \( \mathfrak{h} \)-weight spaces, and \( e(\nu) \) is the formal character defined by \( \nu \) (cf. [5, §10.2]).

**Proposition 6.6.** The super character of the spin representation of \( u \) on \( p \) is given by the formula
\[
\text{ch}(S_p) = e(\rho_p) \prod_{\alpha \in \Delta_+^p} (1 - e(-\alpha))^{n_p(\alpha)}.
\]

**Proof.** For each root space \( p_{-\alpha} \), the character of the adjoint action of \( \mathfrak{h} \) on \( \wedge p_{-\alpha} \) equals \( (1 - e(-\alpha))^{n_p(\alpha)} \). The character of the adjoint action on \( \wedge p_{-\alpha} = \bigotimes_{\alpha \in \Delta_+^p} \wedge p_{-\alpha} \) is the product of the characters on \( \wedge p_{-\alpha} \). By Proposition 6.4 the action of \( \mathfrak{h} \) as a subalgebra of \( u \) differs from the adjoint action by a \( \rho_p \)-shift accounting for an extra factor \( e(\rho_p) \). \( \square \)

Consider \( L(\mu) \otimes S_p \) as a super representation of \( u \). Its formal super character is
\[
\text{ch}(L(\mu) \otimes S_p) = \text{ch}(L(\mu)) \text{ch}(S_p).
\]

On the other hand, since \( D_{L(\mu)} \) is an odd skew-adjoint operator on this space, this coincides with
\[
\text{ch}(\ker(D_{L(\mu)})) = \sum_{w \in p} (-1)^{l(w)} \text{ch}(M(w(\mu + \rho) - \rho_u)).
\]
This gives the generalized Weyl-Kac character formula,
\[
\chi(L(\mu)) = \sum_{w \in W} (-1)^{l(w)} \chi(M(w(\mu + \rho) - \rho)) \frac{e(\rho_p)}{\prod_{\alpha \in \Delta_p^+} (1 - e(\alpha))^{\eta_p(\alpha)}},
\]
valid for quadratic subalgebras \( u \subset g \) of the form considered above. For \( u = h \) one recovers the usual Weyl-Kac character formula [5, §10.4] for symmetrizable Kac-Moody algebras. Note that the Weyl-Kac character formula also holds for the non-symmetrizable case, see Kumar [13, Chapter 3.2]. We do not know how to treat this general case using cubic Dirac operators.

**Example 6.7.** As a concrete example, consider the Kac-Moody algebra of hyperbolic type, associated to the generalized Cartan matrix
\[
\begin{pmatrix}
2 & -3 \\
-3 & 2
\end{pmatrix}
\]
(cf. [5, Exercise 5.28]). The Weyl group \( W \) is generated by the reflections \( r_1, r_2 \) corresponding to \( \alpha_1, \alpha_2 \). The set \( P^+ \) of dominant weights is generated by \( \varpi_1 = -\frac{1}{5}(2\alpha_1 + 3\alpha_2) \) and \( \varpi_2 = -\frac{1}{5}(2\alpha_2 + 3\alpha_1) \). One has \( \rho = \varpi_1 + \varpi_2 = -(\alpha_1 + \alpha_2) \).

Put \( \Pi_u = \{ \beta_1, \beta_2 \} \) with
\[
\beta_1 = \alpha_1, \quad \beta_2 = r_2(\alpha_1) = \alpha_1 + 3\alpha_2.
\]
Since \( \beta_2 - \beta_1 = 3\alpha_2 \) is not a root, \( \Pi_u \) is the set of simple roots for a Kac-Moody Lie subalgebra \( u \subset g \). One finds that \( \rho_u = \varpi_1 \), and the fundamental \( u \)-weights spanning \( P_u^+ \) are \( \tau_1 = \varpi_1 - \frac{1}{5}\varpi_2 \) and \( \tau_2 = \frac{1}{3}\varpi_2 \).

The Weyl group \( W_u \) is generated by the reflections defined by \( \beta_1, \beta_2 \), i.e. by \( r_1 \) and \( r_2r_1r_2 \). A general element of \( W_u \) is thus a word in \( r_1, r_2 \), with an even number of \( r_2 \)'s. One has
\[
W_p = \{ 1, r_2 \},
\]
giving duplets of \( u \)-representations. Write weights \( \mu \in P^+ \) in the form \( \mu = k_1 \varpi_1 + k_2 \varpi_2 \). Then the corresponding duplet is given by the weights
\[
\mu + \rho - \rho_u = k_1 \varpi_1 + (k_2 + 1)\varpi_2 = k_1 \tau_1 + (k_1 + 3k_2 + 3)\tau_2,
\]
\[
r_2(\mu + \rho) - \rho_u = (k_1 + 3(k_2 + 1))\varpi_2 - (k_2 + 1)\varpi_2 = (k_1 + 3k_2 + 3)\tau_1 + k_2\tau_2.
\]

**References**


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