Equivariant cohomology and the Cartan model

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1. Introduction

If a compact Lie group G acts on a manifold M, the space M/G of orbits of the action is usually a singular space. Nonetheless, it is often possible to develop a 'differential geometry' of the orbit space in terms of appropriately defined equivariant objects on M. In this article, we will be mostly concerned with 'differential forms on M/G'. A first idea would be to work with the complex of 'basic' forms on M, but for many purposes this complex turns out to be too small. A much more useful complex of *equivariant differential forms* on M was introduced by H. Cartan in 1950, in [2, Section 6]. In retrospect, Cartan's approach presented a differential form model for the equivariant cohomology of M, as defined by A. Borel [8] some ten years later. Borel's construction replaces the quotient M/G by a better behaved (but usually infinite-dimensional) *homotopy quotient* M_G, and Cartan's complex should be viewed as a model for forms on M_G.

One of the features of equivariant cohomology are the *localization formulas* for the integrals of equivariant cocycles. The first instance of such an integration formula was the 'exact stationary phase formula', discovered by Duistermaat-Heckman [12] in 1980. This formula was quickly recognized, by Berline-Vergne [5] and Atiyah-Bott [3], as a localization principle in equivariant cohomology. Today, equivariant localization is a basic tool in mathematical physics, with numerous applications.

In this article, we will begin with Borel's topological definition of equivariant cohomology. We then proceed to describe H. Cartan's more algebraic approach, and conclude with a discussion of localization principles.

As additional references for the material covered here, we particularly recommend the books by Berline-Getzler-Vergne [4] and Guillemin-Sternberg [17].

2. Borel's model of $H_G(M)$

Let G be a topological group. A *G-space* is a topological space M on which G acts by transformations $g \mapsto a_g$, in such a way that the action map

$$a\colon G \times M \to M \tag{1}$$

is continuous. An important special case of G-spaces are principal G-bundles $E \to B$, i.e. G-spaces locally isomorphic to products $U \times G$.

Definition 2.1. A *classifying bundle* for G is a principal G-bundle $EG \to BG$, with the following universal property: For any principal G-bundle $E \to B$, there is a map $f\colon B \to BG$, unique up to homotopy, such that E is isomorphic to the pull-back bundle f^*EG. The map f is known as a *classifying map* of the principal bundle.

To be precise, the base spaces of the principal bundles considered here must satisfy some technical condition. For a careful discussion, see Husemoller [18]. Classifying bundles exist for all G (by a construction due to Milnor [22]), and are unique up to G-homotopy equivalence.

It is a basic fact that principal G-bundles with contractible total space are classifying bundles.

Examples 2.2. (a) The bundle $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ is a classifying bundle for $G = \mathbb{Z}$.
(b) Let $\mathcal{H}$ be a separable complex Hilbert space, $\dim \mathcal{H} = \infty$. It is known that unit sphere $S(\mathcal{H})$ is contractible. It is thus a classifying $\mathrm{U}(1)$-bundle, with base the projective space $P(\mathcal{H})$. More generally, the Stiefel manifold $\mathrm{St}(k, \mathcal{H})$ of unitary k-frames is a classifying $U(k)$-bundle, with base the Grassmann manifold $\mathrm{Gr}(k, \mathcal{H})$ of k-planes.
(c) Any compact Lie group G arises as a closed subgroup of $U(k)$, for k sufficiently large. Hence, the Stiefel manifold $\mathrm{St}(k, \mathcal{H})$ also serves as a model for EG.
(d) The based loop group $G = L_0K$ of a connected Lie group K acts by gauge transformations on the space of connections $\mathcal{A}(S^1) = \Omega^1(S^1, \mathfrak{k})$. This is a classifying bundle for L_0K, with base K. The quotient map takes a connection to its holonomy.

For any commutative ring R (e.g. $\mathbb{Z}, \mathbb{R}, \mathbb{Z}_2$) let $H(\cdot; R)$ denote the (singular) cohomology with coefficients in R. Recall that $H(\cdot; R)$ is a graded commutative ring under cup product.

Definition 2.3. The *equivariant cohomology* $H_G(M) = H_G(M; R)$ of a G-space M is the cohomology ring of its *homotopy quotient* $M_G = EG \times_G M$,

$$H_G(M; R) = H(M_G; R). \tag{2}$$

Equivariant cohomology is a contravariant functor from the category of G-spaces to the category of R-modules. The G-map $M \to \mathrm{pt}$ induces an algebra homomorphism from $H_G(\mathrm{pt}) = H(BG)$ to $H_G(M)$. In this way, $H_G(M)$ is a module over the ring $H(BG)$.

Example 2.4. (Principal G-bundles) Suppose $E \to B$ is a principal G-bundle. The homotopy quotient E_G may be viewed as a bundle $E \times_G EG$ over B. Since the fiber is contractible, there is a homotopy equivalence

$$E_G \simeq B, \tag{3}$$

and therefore $H_G(E) = H(B)$.

Example 2.5. (Homogeneous spaces) If K is a closed subgroup of a Lie group G, the space EG may be viewed as a model for EK, with $BK = EG/K = EG \times_K (G/K)$. Hence,

$$H_G(G/K) = H(BK). \tag{4}$$

Let us briefly describe two of the main techniques for computing $H_G(M)$.

(1) *Leray spectral sequences.* If R is a field, the equivariant cohomology may be computed as the E_∞ term of the spectral sequence for the fibration $M_G \to BG$. If BG is simply connected (as is the case for all compact connected Lie groups), the E_2-term of the spectral sequence reads

$$E_2^{p,q} = H^p(BG) \otimes H^q(M). \tag{5}$$

(2) *Mayer-Vietoris sequences.* If $M = U_1 \cup U_2$ is a union of two G-invariant open subsets, there is a long exact sequence

$$\begin{aligned} \cdots \to H^k_G(M) \to H^k_G(U_1) \oplus H^k_G(U_2) \to \\ \to H^k_G(U_1 \cap U_2) \to H^{k+1}_G(M) \to \cdots \end{aligned}$$

More generally, associated to any G-invariant open cover there is a spectral sequence converging to $H_G(M)$.

Example 2.6. Consider the standard $\mathrm{U}(1)$-action on S^2 by rotations. Cover S^2 by two open sets $U_\pm$, given as the complement of the south- and north pole, respectively. Since $U_+ \cap U_-$ retracts onto the equatorial circle, on which $\mathrm{U}(1)$ acts freely, its equivariant cohomology vanishes except in degree 0. On the other hand, $U_\pm$ retract onto the poles $p_\pm$. Hence, by the Mayer-Vietoris sequence the map $H^k_{\mathrm{U}(1)}(S^2) \cong H^k_{\mathrm{U}(1)}(p_+) \oplus H^k_{\mathrm{U}(1)}(p_-)$ given by pull-back to the fixed points is an isomorphism for $k > 0$. Since the pull-back map is a ring homomorphism, we conclude that $H_{\mathrm{U}(1)}(S^2; R)$ is the commutative ring generated by two elements $x_\pm$ of degree 2, subject to a single relation $x_+ x_- = 0$.

3. $\mathfrak{g}$-DIFFERENTIAL ALGEBRAS

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. A *G-manifold* is a manifold M together with a G-action such that the action map (1) is smooth. We would like to introduce a concept of *equivariant differential forms* on M. This complex should play the role of differential forms on the infinite-dimensional space M_G. In Cartan's approach, the starting point is an algebraic model for the differential forms on the classifying bundle EG.

The algebraic machinery will only depend on the *infinitesimal* action of G. It is therefore convenient to introduce the following concept.

Definition 3.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A *$\mathfrak{g}$-manifold* is a manifold M, together with a Lie algebra homomorphism $a\colon \mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto a_\xi$ into the Lie algebra of vector fields on M, such that the map $\mathfrak{g} \times M \to TM$, $(\xi, m) \mapsto a_\xi(m)$ is smooth.

Any G-manifold M becomes a $\mathfrak{g}$-manifold, by taking a_ξ to be the *generating vector field*

$$a_\xi(m) := \frac{d}{dt}\Big|_{t=0} a_{\exp(-t\xi)}(m). \tag{6}$$

Conversely, if G is simply connected, and M is a $\mathfrak{g}$-manifold for which all of the vector fields a_ξ are complete, the $\mathfrak{g}$-action integrates uniquely to an action of the group G.

The de Rham algebra $(\Omega(M), \mathrm{d})$ of differential forms on a $\mathfrak{g}$-manifold M carries graded derivations $L_\xi = L(a_\xi)$ (Lie derivatives, degree 0) and $\iota_\xi = \iota(a_\xi)$ (contractions, degree -1). One has the following graded commutation relations,

$$[\mathrm{d}, \mathrm{d}] = 0,\ [L_\xi, \mathrm{d}] = 0,\ [\iota_\xi, \mathrm{d}] = L_\xi, \tag{7}$$

$$[\iota_\xi, \iota_\eta] = 0,\ [L_\xi, L_\eta] = L_{[\xi,\eta]_\mathfrak{g}},\ [L_\xi, \iota_\eta] = \iota_{[\xi,\eta]_\mathfrak{g}}. \tag{8}$$

More generally, we define:

Definition 3.2. A *$\mathfrak{g}$-differential algebra* ($\mathfrak{g}$ − da) is a commutative graded algebra $A = \bigoplus_{n=0}^\infty A^n$, equipped with graded derivations $\mathrm{d}, L_\xi, \iota_\xi$ of degrees $1, 0, -1$ (where L_ξ, ι_ξ depend linearly on $\xi \in \mathfrak{g}$), satisfying the graded commutation relations (7), (8).

Definition 3.3. For any $\mathfrak{g}-\mathrm{da}$ A one defines the *horizontal subalgebra* $\mathcal{A}_{\mathrm{hor}} = \bigcap_\xi \ker(\iota_\xi)$, the invariant subalgebra $\mathcal{A}^{\mathfrak{g}} = \bigcap_\xi \ker(L_\xi)$, and the *basic subalgebra* $\mathcal{A}_{\mathrm{basic}} = \mathcal{A}_{\mathrm{hor}} \cap \mathcal{A}^{\mathfrak{g}}$.

Note that the basic subalgebra is a differential subcomplex of A.

Definition 3.4. A *connection* on a $\mathfrak{g}-\mathrm{da}$ is an invariant element $\theta \in A^1 \otimes \mathfrak{g}$, with the property $\iota_\xi \theta = \xi$. The *curvature* of a connection is the element $F^\theta \in A^2 \otimes \mathfrak{g}$ given as $F^\theta = \mathrm{d}\theta + \frac{1}{2}[\theta, \theta]_{\mathfrak{g}}$.

$\mathfrak{g}-\mathrm{da}$'s $\mathcal{A}$ admitting connections are the algebraic counterparts of (smooth) principal bundles, with $\mathcal{A}_{\mathrm{basic}}$ playing the role of the base of the principal bundle.

4. Weil algebra

The Weil algebra $W\mathfrak{g}$ is the algebraic analogue to the classifying bundle EG. Similar to EG, it may be characterized by a universal property:

Theorem 4.1. *There exists a* $\mathfrak{g}-\mathrm{da}$ $W\mathfrak{g}$ *with connection* θ_W*, having the following universal property: If* $\mathcal{A}$ *is a* $\mathfrak{g}-\mathrm{da}$ *with connection* θ*, there is a unique algebra homomorphism* $c\colon W\mathfrak{g} \to \mathcal{A}$ *taking* θ_W *to* θ*.*

Clearly, the universal property characterizes $W\mathfrak{g}$ up to a unique isomorphism. To get an explicit construction, choose a basis $\{e_a\}$ of $\mathfrak{g}$, with dual basis $\{e^a\}$ of $\mathfrak{g}^*$. Let $y^a \in \wedge^1\mathfrak{g}^*$ be the corresponding generators of the exterior algebra, and $v^a \in S^1\mathfrak{g}^*$ the generators of the symmetric algebra. Let

$$W^n\mathfrak{g} = \bigoplus_{2i+j=n} S^i\mathfrak{g}^* \otimes \wedge^j\mathfrak{g}^* \tag{9}$$

carry the differential,

$$\mathrm{d}y^a = v^a + \frac{1}{2} f^a_{bc} y^b y^c \tag{10}$$

$$\mathrm{d}v^a = -f^a_{bc} v^b y^c \tag{11}$$

where $f^a_{bc} = \langle e^a, [e_b, e_c]_{\mathfrak{g}}\rangle$ are the structure constants of $\mathfrak{g}$. Define the contractions $\iota_a = \iota_{e_a}$ by

$$\iota_a y^b = \delta^b_a, \quad \iota_a v^b = 0, \tag{12}$$

and let $L_a = [\mathrm{d}, \iota_a]$. Then L_a are the generators for the adjoint action on $W\mathfrak{g}$. The element $\theta_W = y^a \otimes e_a \in W^1\mathfrak{g} \otimes \mathfrak{g}$ is a connection on $W\mathfrak{g}$. Notice that we could also use y^a, $\mathrm{d}y^a$ as generators of $W\mathfrak{g}$. This identifies $W\mathfrak{g}$ with the *Koszul algebra*, and implies:

Theorem 4.2. $W\mathfrak{g}$ *is acyclic. That is, the inclusion* $\mathbb{R} \to W\mathfrak{g}$ *is a homotopy equivalence.*

Acyclicity of $W\mathfrak{g}$ corresponds to the contractibility of the total space of EG.

The basic subalgebra of $W\mathfrak{g}$ is equal to $(S\mathfrak{g}^*)^{\mathfrak{g}}$, and the differential restrict to zero on this subalgebra, since d changes parity. Hence, if $\mathcal{A}$ is a $\mathfrak{g}-\mathrm{da}$ with connection, the characteristic homomorphism $c\colon W\mathfrak{g} \to \mathcal{A}$ induces an algebra homomorphism, $(S\mathfrak{g}^*)^{\mathfrak{g}} \to H(\mathcal{A}_{\mathrm{basic}})$. This homomorphism is independent of θ:

Theorem 4.3. *Suppose* θ_0, θ_1 *are two connections on a* $\mathfrak{g}-\mathrm{da}$ $\mathcal{A}$*. Then their characteristic homomorphisms* $c_0, c_1\colon W\mathfrak{g} \to \mathcal{A}$ *are* $\mathfrak{g}$*-homotopic. That is, there is a chain homotopy intertwining contractions and Lie derivatives.*

Remark 4.4. One obtains other interesting examples of $\mathfrak{g}-\mathrm{da}$'s if one drops the commutativity assumption from the definition. For instance, suppose $\mathfrak{g}$ carries an invariant scalar product. Let $\mathrm{Cl}(\mathfrak{g})$ be the corresponding Clifford algebra, and $U(\mathfrak{g})$ the enveloping algebra. The *non-commutative Weil algebra* [2]

$$\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}) \tag{13}$$

is a (non-commutative) $\mathfrak{g}-\mathrm{da}$, with the derivations d, L_a, ι_a defined on generators by the same formulas as for $W\mathfrak{g}$.

5. Equivariant cohomology of $\mathfrak{g}$-da's

In analogy to $H_G(M) := H(M_G)$, we now declare:

Definition 5.1. The *equivariant cohomology algebra* of a $\mathfrak{g}-\mathrm{da}$ $\mathcal{A}$ is the cohomology of the differential algebra $\mathcal{A}_{\mathfrak{g}} := (W\mathfrak{g} \otimes \mathcal{A})_{\mathrm{basic}}$,

$$H_{\mathfrak{g}}(\mathcal{A}) := H(\mathcal{A}_{\mathfrak{g}}) \tag{14}$$

The equivariant cohomology $H_{\mathfrak{g}}(\mathcal{A})$ has functorial properties parallel to those of $H_G(M)$. In particular, $H_{\mathfrak{g}}(A)$ is a module over

$$H_{\mathfrak{g}}(\{0\}) = H((W\mathfrak{g})_{\mathrm{basic}}) = (S\mathfrak{g}^*)^{\mathfrak{g}}. \tag{15}$$

Theorem 5.2. *Suppose* $\mathcal{A}$ *is a* $\mathfrak{g}-\mathrm{da}$ *with connection* θ*, and let* $c\colon W\mathfrak{g} \to \mathcal{A}$ *be the characteristic homomorphism. Then*

$$W\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A}, \; w \otimes x \mapsto c(w)\, x \tag{16}$$

is a $\mathfrak{g}$*-homotopy equivalence, with* $\mathfrak{g}$*-homotopy inverse the inclusion*

$$\mathcal{A} \to W\mathfrak{g} \otimes \mathcal{A}, \; x \mapsto 1 \otimes x. \tag{17}$$

In particular, there is a canonical isomorphism,

$$H(\mathcal{A}_{\mathrm{basic}}) \cong H_{\mathfrak{g}}(\mathcal{A}). \tag{18}$$

Proof. By Theorem 4.3, the automorphism $w \otimes x \mapsto 1 \otimes c(w)\, x$ of $W\mathfrak{g} \otimes \mathcal{A}$ is $\mathfrak{g}$-homotopic to the identity map. □

The above definition of the complex $\mathcal{A}_{\mathfrak{g}}$ is often referred to as the *Weil model* of equivariant cohomology, while the term *Cartan model* is reserved for a slightly different description of $\mathcal{A}_{\mathfrak{g}}$. Identify the space $(S\mathfrak{g}^* \otimes \mathcal{A})^{\mathfrak{g}}$ with the algebra of equivariant $\mathcal{A}$-valued polynomial functions $\alpha\colon \mathfrak{g} \to \mathcal{A}$. Define a differential $\mathrm{d}_{\mathfrak{g}}$ on this space by setting

$$(\mathrm{d}_{\mathfrak{g}}\alpha)(\xi) = \mathrm{d}(\alpha(\xi)) - \iota_\xi \alpha(\xi). \tag{19}$$

Theorem 5.3 (H. Cartan)**.** *The natural projection $W\mathfrak{g} \otimes \mathcal{A} \to S\mathfrak{g}^* \otimes \mathcal{A}$ restricts to an isomorphism of differential algebras, $\mathcal{A}_{\mathfrak{g}} \cong (S\mathfrak{g}^* \otimes \mathcal{A})^{\mathfrak{g}}$.*

Suppose $\mathcal{A}$ carries a connection θ. The $\mathfrak{g}$-homotopy equivalence (16) induces a homotopy equivalence $\mathcal{A}_{\mathfrak{g}} \to \mathcal{A}_{\mathrm{basic}}$ of the basic subcomplexes. By explicit calculation, the corresponding map for the Cartan model is given by,

$$(S\mathfrak{g}^* \otimes \mathcal{A})^{\mathfrak{g}} \to \mathcal{A}_{\mathrm{basic}},\ \alpha \mapsto P^\theta_{\mathrm{hor}}(\alpha(F^\theta)). \tag{20}$$

Here $\alpha(F^\theta) \in \mathcal{A}^{\mathfrak{g}}$ is the result of substituting the curvature of θ, and $P_{\mathrm{hor}} : \mathcal{A} \to \mathcal{A}_{\mathrm{hor}}$ is horizontal projection. On elements of $(S\mathfrak{g}^*)^{\mathfrak{g}} \subset (S\mathfrak{g}^* \otimes \mathcal{A})^{\mathfrak{g}}$, the map (20) specializes to the Chern-Weil homomorphism.

There is an algebraic counterpart of the Leray spectral sequence, as follows. Introduce a filtration,

$$F^p \mathcal{A}^{p+q}_{\mathfrak{g}} := \bigoplus_{2i \ge p} (S^i\mathfrak{g}^* \otimes \mathcal{A}^q)^{\mathfrak{g}} \tag{21}$$

Since second term in the equivariant differential (19) raises the filtration degree by 2, it follows that

$$E_2^{p,q} = (S^{p/2}\mathfrak{g}^*)^{\mathfrak{g}} \otimes H^q(\mathcal{A}), \tag{22}$$

for p even, $E_2^{p,q} = 0$ for p odd. In fortunate cases, the spectral sequence collapses at the E_2-stage (see below).

6. Equivariant de Rham theory

We will now restrict ourselves to the case that $\mathcal{A} = \Omega(M)$ is the algebra of differential forms on a G-manifold, where G is compact and connected.

Theorem 6.1 (Equivariant de Rham theorem)**.** *Suppose G is a* compact, connected *Lie group, and that M is a G-manifold. Then there is a canonical isomorphism,*

$$H_G(M;\mathbb{R}) \cong H_{\mathfrak{g}}(\Omega(M)), \tag{23}$$

where the left hand side is the equivariant cohomology as defined by the Borel construction.

Motivated by this result, we will change our notation slightly and write

$$\Omega_G(M) = (S\mathfrak{g}^* \otimes \Omega(M))^G \tag{24}$$

for the Cartan complex of *equivariant differential forms*, and d_G for the equivariant differential (19).

Remark 6.2. Theorem 6.1 fails, in general, for non-compact Lie groups G. A differential form model for the non-compact case was developed by Getzler [13].

Example 6.3. Let (M,ω) be a symplectic manifold, and $a\colon G \to \mathrm{Diff}(M)$ a Hamiltonian group action. That is, a preserves the symplectic form, $a_g^*\omega = \omega$, and there exists an equivariant moment map $\Phi\colon M \to \mathfrak{g}^*$ such that $\iota_\xi\omega + \mathrm{d}\langle\Phi,\xi\rangle = 0$. Then the *equivariant symplectic form* $\omega_G(\xi) := \omega + \langle\Phi,\xi\rangle$ is equivariantly closed.

Example 6.4. Let G be a Lie group, and denote by

$$\theta^L = g^{-1}\mathrm{d}g,\ \theta^R = \mathrm{d}g g^{-1} \tag{25}$$

the left-, right-invariant Maurer-Cartan forms. Suppose $\mathfrak{g} = \mathrm{Lie}(G)$ carries an invariant scalar product $\cdot$, and consider the closed 3-form

$$\phi = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L]. \tag{26}$$

Then

$$\phi_G(\xi) = \phi + \frac{1}{2}(\theta^L + \theta^R)\cdot\xi \tag{27}$$

is a closed equivariant extension for the conjugation action of G. More generally, transgression gives explicit differential forms ϕ_j generating the cohomology ring $H(G) = (\wedge\mathfrak{g}^*)^G$. Closed equivariant extensions of these forms were obtained by Jeffrey [19], using a construction of Bott-Shulman.

A G-manifold is called *equivariantly formal* if

$$H_G(M) = (S\mathfrak{g}^*)^G \otimes H(M) \tag{28}$$

as an $(S\mathfrak{g}^*)^G$-module. Equivalently, this is the condition that the spectral sequence (22) for $H_G(M)$ collapses at the E_2-term. M is equivariantly formal under any of the following conditions, (1) $H^q(M) = 0$ for

q odd, (2) the map $H_G(M) \to H(M)$ is onto, (3) M admits a G-invariant Morse function with only even indices, (4) M is a symplectic manifold and the G-action is Hamiltonian. (The last fact is a theorem due to Ginzburg and Kirwan, see e.g. [14]).

Example 6.5. The conjugation action of a compact Lie group is equivariantly formal, by criterion (2). In this case (28) is an isomorphism of algebras.

It is important to note that (28) is not an algebra isomorphism, in general. Already the rotation action of $G = \mathrm{U}(1)$ on $M = S^2$, discussed in Example 2.6, provides a counter-example.

Theorem 6.6 (Injectivity). *Suppose T is a compact torus, and M is T-equivariantly formal. Then the pull-back map $H_T(M) \to H_T(M^T)$ to the fixed point set is injective.*

Since the pull-back map to the fixed point set is an algebra homomorphism, one can sometimes use this result to determine the algebra structure on $H_T(M)$: Let $\alpha_r \in H(M)$ be generators of the ordinary cohomology algebra, and let $(\alpha_r)_T$ be equivariant extensions. Denote by $x_r \in H_T(M^T)$ the pull-backs of $(\alpha_r)_T$ to the fixed point set, and let y_i be a basis of $\mathfrak{t}^*$, viewed as elements of $S\mathfrak{t}^* \subset H_T(M^T)$. Then $H_T(M)$ is isomorphic to the subalgebra of $H_T(M^T)$ generated by the x_r and y_j.

The case of non-Abelian compact groups G may be reduced to maximal torus T using the following result. Observe that for any G-manifold M, there is a natural action of the Weyl group $W = N(T)/T$ on $H_T(M)$.

Theorem 6.7. *The natural restriction map*

$$H_G(M;\mathbb{R}) \to H_T(M;\mathbb{R})^W \tag{29}$$

onto the Weyl group invariants is an algebra isomorphism.

Remark 6.8. The Cartan complex (24) may be viewed as a small model for the differential forms on the infinite-dimensional space M_G. In the non-commutative case, there exists an even 'smaller' Cartan model, with underlying complex $(S\mathfrak{g}^*)^G \otimes \Omega(M)^G$, involving only invariant differential forms on M. See e.g. [1, 15].

7. Equivariant characteristic forms

Let G be a compact Lie group, and $E \to B$ a principal G-bundle with connection $\theta \in \Omega^1(E) \otimes \mathfrak{g}$. Suppose the principal G-action commutes with the action of a compact Lie group K on E, and that θ is K-invariant. The *K-equivariant curvature* of θ is defined as follows,

$$F_K^\theta = \mathrm{d}_K\theta + \tfrac{1}{2}[\theta,\theta] \in \Omega_K^2(E) \otimes \mathfrak{g}.$$

By the equivariant version of (20), there is a canonical chain map,

$$\Omega_{K\times G}(E) \to \Omega_K(B), \tag{30}$$

defined by substituting the K-equivariant curvature for the $\mathfrak{g}$-variable, followed by horizontal projection with respect to θ. The *Cartan map* (30) is homotopy inverse to the pull-back map from $\Omega_K(B)$ to $\Omega_{K\times G}(B)$.

Example 7.1. The complex $\Omega_{K\times G}(E)$ contains a subcomplex $(S\mathfrak{g}^*)^G$. The restriction of (30) By construction, is the *equivariant Chern-Weil map*

$$(S\mathfrak{g}^*)^G \to \Omega_K(B) \tag{31}$$

Forms in the image of (31) are equivariantly closed; they are called the *K-equivariant characteristic forms* of E.

Example 7.2. Similarly, if $\mathcal{V} \to B$ is a K-equivariant vector bundle with structure group $G \subset \mathrm{GL}(k)$, one defines the K-equivariant characteristic forms of $\mathcal{V}$ to be those of the corresponding bundle of G-frames in $\mathcal{V}$. For instance, suppose $\mathcal{V}$ is an oriented K-equivariant vector bundle of even rank k, with an invariant metric and compatible connection. The Pfaffian defines an invariant polynomial on $\mathfrak{so}(k)$,

$$\zeta \mapsto \det{}^{1/2}(\zeta/2\pi) \tag{32}$$

(equal to 0 if k is odd). The K-equivariant characteristic form of degree k on B determined by (32) is known as the *equivariant Euler form*

$$\mathrm{Eul}_K(\mathcal{V}) \in \Omega_K^k(B). \tag{33}$$

Similarly, one defines equivariant Pontrjagin forms of $\mathcal{V}$, and (for Hermitian vector bundles) equivariant Chern forms.

Example 7.3. Suppose G is a maximal rank subgroup of the compact Lie group K. The bundle $K \to K/G$ admits a unique K-invariant connection. Hence, one obtains a canonical chain map $(S\mathfrak{g}^*)^G \to \Omega_K(K/G)$, realizing the isomorphism $H_K(K/G) \cong (S\mathfrak{g}^*)^G$. In particular, any G-invariant element of $\mathfrak{g}^*$ defines a closed K-equivariant 2-form on K/G. For instance, symplectic forms on co-adjoint orbits are obtained in this way.

Suppose M is a G-manifold, and let $Q = E \times_G M$ be the associated bundle. For any K-invariant connection on E one obtains a chain map

$$\Omega_G(M) \to \Omega_{K\times G}(E \times M) \to \Omega_K(Q), \tag{34}$$

by composing the pull-back to $E \times M$ with the Cartan map for the principal bundle $E \times M \to Q$.

Example 7.4. Suppose (M, ω) is a Hamiltonian G-manifold, with moment map $\Phi\colon M \to \mathfrak{g}^*$. The image of $\omega_G = \omega + \Phi$ under the map (34) defines a closed K-equivariant 2-form on Q. This construction is of importance in symplectic geometry, where it arises in the context of Sternberg's *minimal coupling*.

8. Equivariant Thom forms

Let $\pi\colon \mathcal{V} \to B$ be a G-equivariant oriented real vector bundle of rank k over a compact base B. There is a canonical chain map, called *fiber integration*

$$\pi_*\colon \Omega^\bullet(\mathcal{V})_{cp} \to \Omega^{\bullet-k}(B) \tag{35}$$

where the subscript indicates 'compact support'. It is characterized by the following properties: (i) for a form of degree k, the value of its fiber integral at $x \in B$ is equal to the integral over the fiber $\mathcal{V}_x$, and (ii)

$$\pi_*(\alpha \wedge \pi^*\beta) = \pi_*\alpha \wedge \beta \tag{36}$$

for all $\alpha \in \Omega(\mathcal{V})_{cp}$ and $\beta \in \Omega(B)$. Fiber integration extends to G-equivariant differential forms, and commutes with the equivariant differential.

Theorem 8.1 (Equivariant Thom isomorphism)**.** *Fiber integration defines an isomorphism,*

$$H_G^{\bullet+k}(\mathcal{V})_{cp} \to H_G^\bullet(B). \tag{37}$$

An equivariant *Thom form* for a G-vector bundle is a cocycle $\mathrm{Th}_G(\mathcal{V}) \in \Omega_G^k(\mathcal{V})_{cp}$, with the property,

$$\pi_* \mathrm{Th}_G(\mathcal{V}) = 1. \tag{38}$$

Given $\mathrm{Th}_G(\mathcal{V})$, the inverse to (37) is realized on the level of differential forms as

$$\Omega_G^\bullet(B) \to \Omega_G^{\bullet+k}(E), \quad \alpha \mapsto \mathrm{Th}_G(\mathcal{V}) \wedge \pi^*\alpha \tag{39}$$

A beautiful 'universal' construction of Thom forms was obtained by Mathai-Quillen [21]. Using (34), it suffices to describe an $\mathrm{SO}(k)$-equivariant Thom form for the trivial bundle $\mathbb{R}^k \to \{0\}$. Using multi-index notation for ordered subsets $I \subset \{1, \ldots, k\}$, write

$$\mathrm{Th}_{\mathrm{SO}(k)}(\mathbb{R}^k)(\zeta) = \frac{e^{-\|x\|^2}}{\pi^{k/2}} \sum_I \epsilon_I \mathrm{det}^{1/2}\big(\frac{\zeta_I}{2}\big)\, (\mathrm{d}x)^{I^c}. \tag{40}$$

Here the sum is over all subsets I with $|I|$ even, and I^c is the complement of I. The matrix ζ_I is obtained from ζ by deleting all rows and columns that are not in I, and $\det^{1/2}$ is defined as a Pfaffian. Finally, ϵ_I is the sign of the shuffle permutation defined by I, that is, $(\mathrm{d}x)^I (\mathrm{d}x)^{I^c} = \epsilon_I \mathrm{d}x_1 \cdots \mathrm{d}x_k$. As shown by Mathai-Quillen the form (40) is equivariantly closed, and clearly (38) holds since the top degree part is just a Gaussian. If k is even, the Mathai-Quillen formula can also be written, on the open dense where $\zeta \in \mathfrak{so}(k)$ is invertible,

$$\mathrm{Th}_{\mathrm{SO}(k)}(\mathbb{R}^k)(\zeta) = \mathrm{det}^{1/2}\big(\tfrac{\zeta}{2\pi}\big)\, e^{-\|x\|^2 - \langle dx, \zeta^{-1}(dx)\rangle} \tag{41}$$

The form $\mathrm{Th}_{\mathrm{SO}(k)}(\mathbb{R}^k)$ given by these formulas does not have compact support, but is rapidly decreasing at infinity. One obtains a compactly supported Thom form, by applying an $\mathrm{SO}(k)$-equivariant diffeomorphism from $\mathbb{R}^k$ onto some open ball of finite radius.

Note that the pull-back of (40) to the origin is equal to $\det^{1/2}(\frac{\zeta}{2\pi})$ (equal to 0 if k is odd). This implies:

Theorem 8.2. *Let $\iota\colon B \to \mathcal{V}$ denote the inclusion of the zero section. Then*

$$\iota^* \mathrm{Th}_G(\mathcal{V}) = \mathrm{Eul}_G(\mathcal{V}), \tag{42}$$

where $\mathrm{Eul}_G(\mathcal{V}) \in \Omega_G^k(B)$ is the equivariant Euler form.

Suppose next that M is a G-manifold, and S a closed G-invariant submanifold with oriented normal bundle ν_S. Choose a G-equivariant tubular neighborhood embedding,

$$\nu_S \to U \subset M, \tag{43}$$

and let $\mathrm{PD}_G(S) \in \Omega_G(M)_{cp}$ be the image of $\mathrm{Th}_G(\mathcal{V})$ under this embedding. The form $\mathrm{PD}_G(S)$ has the property,

$$\int_M \mathrm{PD}_G(S) \wedge \alpha = \int_S \iota_S^* \alpha, \tag{44}$$

for all *closed* equivariant forms $\alpha \in \Omega_G(M)$. It is called an *equivariant Poincaré dual* of S. By construction, the pull-back to S is the equivariant Euler form,

$$\iota_S^* \mathrm{PD}_G(S) = \mathrm{Eul}_G(\nu_S). \tag{45}$$

Equivariant Poincaré duality takes transversal intersections of G-manifolds to wedge products, similar to the non-equivariant case.

Remark 8.3. In general, the $(S\mathfrak{g}^*)^G$-submodule generated by Poincaré duals of G-invariant submanifold is strictly smaller than $H_G(M)$. In this sense the terminology 'duality' is misleading.

9. Localization Theorem

In this Section, T will denote a torus. Suppose M is a compact oriented T-manifold. For any component F of the fixed point set of T, the action of T on ν_F fixes only the zero section F. This implies that the normal bundle ν_F has even rank and is orientable. Fix an orientation, and give F the induced orientation.

Since T is compact, the list of stabilizer groups of points in M is finite. Call $\xi \in \mathfrak{t}$ *generic* if it is not in the Lie algebra of any of these stabilizers, other than T itself. In this case, value $\mathrm{Eul}_T(\nu_F, \xi)$ of the equivariant Euler form is invertible as an element of $\Omega(F)$.

Theorem 9.1 (Integration formula)**.** *Suppose M is a compact oriented T-manifold, where T is a torus. Let $\alpha \in \Omega_T(M)$ be a closed equivariant form, and let $\xi \in \mathfrak{t}$ be generic. Then*

$$\int_M \alpha(\xi) = \sum_F \int_F \frac{\iota_F^* \alpha(\xi)}{\mathrm{Eul}_T(\nu_F, \xi)} \tag{46}$$

where the sum is over the connected components of the fixed point set.

Rather than fixing ξ, one can also view (46) as an equality of rational functions of $\xi \in \mathfrak{t}$.

Remark 9.2. The integration formula was obtained in 1983 by Berline-Vergne [5], based on ideas of Bott [9]. The topological counterpart, as a "localization principle" was proved independently by Atiyah-Bott [3]. More abstract versions of the localization theorem in equivariant cohomology had been proved earlier by Borel, Chiang-Skjelbred and others.

Remark 9.3. If $\alpha = \mathrm{PD}_T(F) \wedge \beta$, where β is equivariantly closed, the integration formula is immediate from the property (44) of Poincaré duals. The essence of the proof is to reduce to this case.

Remark 9.4. The localization contributions are particularly nice if $F = \{p\}$ is isolated (which can onely happen if $\dim M$ is even). In this case, $\iota_F^* \alpha(\xi)$ is simply the value of the function $\alpha_{[0]}(\xi)$ at p. For the Euler form one has

$$\mathrm{Eul}(\nu_F, \xi) = (-1)^{\dim M/2} \prod \langle \mu_j(p), \xi \rangle \tag{47}$$

where $\mu_j(p) \in \mathfrak{t}^*$ are the (real) weights of the action on the tangent space T_pM. (Here we have chosen an isomorphism $T_pM \cong \mathbb{C}^l$ compatible with the orientation.) Hence, if all fixed points are isolated,

$$\int_M \alpha(\xi) = (-1)^{\dim M/2} \sum_p \frac{\alpha_{[0]}(\xi)(p)}{\prod_j \langle \mu_j(p), \xi \rangle} \tag{48}$$

Example 9.5. Let M be a compact oriented manifold, and $e(M) = \int_M \mathrm{Eul}(TM)$ its Euler characteristic. Suppose a torus T acts on M. Then

$$e(M) = \sum_F e(F), \tag{49}$$

where the sum is over the fixed point set of T. This follows from the integral of the *equivariant* Euler form $\alpha(\xi) = \mathrm{Eul}_T(M, \xi)$, by letting $\xi \to 0$ in the localization formula. In particular, if M admits a circle action with isolated fixed points, the number of fixed points is equal to the Euler characteristic.

In a similar fashion, the localization formula gives interesting expressions for other characteristic numbers of manifolds and vector bundles, in the presence of a circle action. Some of these formulas were discovered prior to the localization formula, see in particular Bott [9].

Example 9.6. In this example, we show that for a simply connected, simple Lie group G the 3-form $\phi \in \Omega^3(G)$ defined in (26) is integral, provided $\cdot$ is taken to be the *basic inner product* (for which the length squared of the short co-roots equals 2). Since any such G is known to contains an $\mathrm{SU}(2)$ subgroup, it suffices to prove this for $G = \mathrm{SU}(2)$. Consider the conjugation action of the maximal torus $T \cong \mathrm{U}(1)$, consisting of diagonal matrices. The fixed point set for this action is T itself. The normal bundle ν_F is trivial, with T acting on the fiber $\mathfrak{g}/\mathfrak{t}$ by the negative root $-\alpha$. Hence, $\mathrm{Eul}(\nu_F, \xi) = \langle \alpha, \xi \rangle$. Let $\check{\alpha} \in \mathfrak{t}$ be the co-root, defined by $\langle \alpha, \check{\alpha} \rangle = 2$. By definition, $\langle \alpha, \xi \rangle = \check{\alpha} \cdot \xi$ for all $\xi \in \mathfrak{t}$. Let us integrate the T-equivariant extension $\phi_T(\xi)$ (cf. (27)). Its pull-back to T is $\theta^T \cdot \xi$, where $\theta^T \in \Omega(T, \mathfrak{t})$ is the Maurer-Cartan form. The integral $\int_T \theta^T$ is a generator of the integral lattice, i.e. it equals $\check{\alpha}$. Thus

$$\int_{\mathrm{SU}(2)} \phi_T(\xi) = \frac{\int_T \theta^T \cdot \xi}{\langle \alpha, \xi \rangle} = \frac{\check{\alpha} \cdot \xi}{\langle \alpha, \xi \rangle} = 1. \tag{50}$$

It follows that $\int_{\mathrm{SU}(2)} \phi = 1$.

10. Duistermaat-Heckman formulas

In this Section we discuss the Duistermaat-Heckman formula, for the case of isolated fixed points. Let T be a torus, and (M,ω) a compact Hamiltonian T-space, with moment map $\Phi\colon M \to \mathfrak{t}^*$. Denote by $\omega_T = \omega + \Phi$ the equivariant extension of ω. Assuming isolated fixed points, the localization formula gives, for all integers $k \geq 0$,

$$(51) \qquad \int_M (\omega + \langle \Phi, \xi\rangle)^k = (-1)^n \sum_p \frac{\langle \Phi(p), \xi\rangle^k}{\prod_j \langle \mu_j(p), \xi\rangle}.$$

where $n = \frac{1}{2}\dim M$. Note that both sides are homogeneous of degree $k-n$ in ξ, but the terms on the right hand side are only rational functions while the left hand side is a polynomial. For $k = n$ both sides are independent of ξ, and compute the integral $\int_M \omega^n$. For $k < n$, the integral (51) is zero, and the cancellation of the terms on the right hand side gives identities among the weights $\mu_j(p)$. (51) also implies

$$(52) \qquad \int_M e^{\omega + \langle \Phi, \xi\rangle} = (-1)^n \sum_p \frac{e^{\langle \Phi(p), \xi\rangle}}{\prod_j \langle \mu_j(p), \xi\rangle}.$$

Assume in particular that $T = \mathrm{U}(1)$, and let $\xi = t\xi_0$ where ξ_0 is the generator of the integral lattice in $\mathfrak{t}$. Identify $\mathfrak{t} \cong \mathbb{R}$ in such a way that ξ_0 corresponds to $1 \in \mathbb{R}$. Then $H = \langle \Phi, \xi_0\rangle$ is a Hamiltonian function with periodic flow. Write $a_j(p) = \langle \mu_j(p), \xi_0\rangle \in \mathbb{Z}$. Then (52) reads,

$$(53) \qquad \int_M e^{tH}\,\frac{\omega^n}{n!} = \frac{(-1)^n}{t^n} \sum_p \frac{e^{tH(p)}}{\prod_j a_j(p)}.$$

The right hand side of (53) is the leading term for the *stationary phase approximation* of the integral on the left. For this reason, Formula (52) is known as the Duistermaat-Heckman *exact stationary phase theorem.*

Formula (52) has the following consequence for the push-forward of the Liouville measure under the moment map, the so-called *Duistermaat-Heckman measure* $H_*(\frac{\omega^n}{n!})$. Let Θ be the Heaviside measure (i.e. the characteristic measure of the positive real axis).

Theorem 10.1 (Duistermaat-Heckman)**.** *The push-forward $H_*(\frac{\omega^n}{n!})$ is piecewise polynomial measure of degree $n-1$, with singularities at the set of all $H(p)$ for fixed points p of the action. One has the formula,*

$$(54) \qquad H_*(\frac{\omega^n}{n!}) = \sum_p \frac{(\lambda - H(p))^{n-1}}{\prod_j a_j(p)}\,\Theta(\lambda - H(p))$$

Proof. It is enough to show that the Laplace transforms of the two sides are equal. Multiplying by $e^{t\lambda}$ and integrating over λ (take $t < 0$ to ensure convergence of the integral), the resulting identity is just (53). □

Remark 10.2. The Theorem generalizes to Hamiltonian actions of higher rank tori, and also to non-isolated fixed points. See the paper by Guillemin-Lerman-Sternberg [16] for a detailed discussion of this formula and of its 'quantum analogue'.

11. Equivariant index theory

By definition, the Cartan model consists of equivariant forms $\alpha(\xi)$ with *polynomial* dependence on the equivariant parameter ξ. However, the integration formula holds in much greater generality. For instance, one may consider generalized Cartan complexes [20], where the parameter ξ varies in some invariant open subset of $\mathfrak{g}$, and the polynomial dependence is replaced by smooth dependence. The use of these more general complexes in equivariant index theory was pioneered by Berline and Vergne, see also [4].

Assume that M is an even-dimensional, compact oriented Riemannian manifold, equipped with a Spin-c structure. According to the Atiyah-Singer theorem, the index of the corresponding Dirac operator D is given by the formula,

$$(55) \qquad \mathrm{ind}(D) = \int_M \hat{A}(M) e^{c/2}.$$

Here c is the curvature 2-form of the complex line bundle associated to the Spin-c structure, and $\hat{A}(M)$ is the $\hat{A}$-form. Recall that $\hat{A}(M)$ is obtained by substituting the curvature form in the formal power series expansion of the function $\hat{A}(x) = \det^{1/2}(\frac{x/2}{\sinh(x/2)})$ on $\mathfrak{so}(n)$.

Suppose now that a compact, connected Lie group G acts on M by isometries, and that the action lifts to the Spin-c bundle. Replacing curvatures with equivariant curvatures, one defines the equivariant form $\hat{A}(M)(\xi)$ and the form $c(\xi)$. Note that $\hat{A}(\xi)$ is only defined for ξ in a sufficiently small neighborhood of 0, since the function $\hat{A}(x)$ is not analytic for all x.

The G-index of the equivariant Spin-c Dirac operator is a virtual character $g \mapsto \mathrm{ind}(D)(g)$ of the group G. For $g = \exp\xi$ sufficiently small it is given by the formula,

$$(56) \qquad \mathrm{ind}(D)(\exp\xi) = \int_M \hat{A}(M)(\xi) e^{c(\xi)/2}.$$

For ξ sufficiently small, the fixed point set of g coincides with the set of zeroes of the vector field a_ξ. The localization formula reproduces the Atiyah-Segal formula for $\mathrm{ind}(D)(g)$, as an integral over M^g.

Berline-Vergne [6] gave similar formulas for the equivariant index of any G-equivariant elliptic operator, and more generally [7] for operators that are *transversally elliptic* in the sense of Atiyah.

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