

# Higher order asymptotics of fast diffusion in Euclidean space: a dynamical systems approach

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- 2 History and goals
- 3 Methods and results
  - The dynamical systems approach
  - Main results
  - A few key ideas
- 4 Acknowledgements

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- spreads at rate  $\tau^{1/2}$ ; asymptotic to spreading Gaussian

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## POROUS MEDIUM REGIME ( $m > 1$ )

- fluid in rock; population spreading; temperature dependent conductivity
- rate of diffusion  $\rho^{m-1}$  varies **directly** with density  $\rho$  of diffusing material
- compactly supported  $\rho_0$  remains compactly supported at  $\tau > 0$

## Motivation: dissipative fluids

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} = \nabla P(\rho) - b\mathbf{v} \quad (1)$$

- if drag negligible ( $b \ll 1$ ), (1) couples with continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2)$$

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- polytropic equation of state  $P(\rho) = \frac{b}{m-1} \rho^m$  gives nonlinear diffusion (3)

## Fast diffusion regime ( $m < 1$ )

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- compactly supported  $\rho_0(y)$  spread at rate  $\tau^\beta$  with  $\beta = \frac{1}{2}(1 + \frac{n}{p})$
- develop tails of algebraic thickness  $\rho(\tau, y) \sim (\frac{\sqrt{k\tau}}{|y|})^{n+p}$  as  $y \rightarrow \infty$
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# A sample result

McCann-Slepcev ('06,  $p \geq n$ ) if  $\|\rho_0\|_1 = \|\tilde{\rho}_0\|_1$  then for all  $\delta > 0$

$$\|\rho(\tau, \cdot) - \tilde{\rho}(\tau, \cdot)\|_1 = \frac{a}{\tau^\beta} |z_0 - \tilde{z}_0| + O\left(\frac{1}{\tau^{1-\delta}}\right) \quad \text{as } \tau \rightarrow \infty$$

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- for the higher dimensions see Koch ('99,  $-p > n \geq 2$ )

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Denzler-Koch-McCann ( $p > 0$ )  $= O(\tau^{-1})$  as  $\tau \rightarrow \infty$

for some translate of  $\tilde{\rho}_0$

## A (finite dimensional) dynamical systems approach

$$x'(t) = -V(x(t)) \in \mathbf{R}^n \quad \text{with} \quad x(0) = x_0$$

LINEARIZE around fixed point  $V(x_\infty) = 0$  to get:

$$(x(t) - x_\infty)' = -DV(x_\infty)(x(t) - x_\infty)' + O(x(t) - x_\infty)^2$$

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If  $\sigma(DV(x_\infty)) = \{\lambda_1 \leq \dots \leq \lambda_n\}$  with eigenvectors  $\hat{\phi}_i$ , as  $t \rightarrow \infty$  expect

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NB: True if  $V(x) = DE(x)$ , since  $DV(x) = \text{Hess}_x E$  is then *self-adjoint*.

- to **simplify**, only strive for asymptotics to order  $O(e^{-2\lambda_1 t})$
- *differentiability* of  $V(x_0)$  or  $x(t) = X(t, x_0)$  wrt  $x_0 \in \mathbf{R}^n$  is crucial

# New coordinates: a time-dependent rescaling of space

Viewed from receding distance  $\tau^\beta$ , our spreading profile

$$\rho(\tau, y) = \frac{1}{(1 + 2p\tau)^{n\beta}} u\left(\frac{1}{2p} \log(1 + 2p\tau), \frac{y}{(1 + 2p\tau)^\beta}\right)$$

iff

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{m} \Delta(u^m) + \frac{2}{1 - m} \nabla \cdot (ux)$$

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The latter dynamics fix the minimizer of  $E(u)$ : a Barenblatt profile

$$u_B(x) = (B + |x|^2)^{-(n+p)/2}$$

for a suitable choice of  $B > 0$  (and fixed total mass).

# Gradient flow structure and linearization

Otto '01 Rescaled dynamics is the *gradient flow* (steepest descent) of  $E(u)$  with respect to an infinite-dimensional heuristic Riemannian structure whose intrinsic distance is the  $L^2$  Wasserstein transportation metric  $d_2$

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Denzler-McCann '03,'05 **Diagonalize** the Hessian  $D^2E(u_B)$  of this entropy on the Hilbert tangent space  $W^{1,2}(\mathbf{R}^n, u_B(x)dx)$  at  $u_B$ , by separating eigenfunctions  $\phi(x) = Y(\frac{x}{|x|})R(|x|)$  into *angular* and *radial* parts (*spherical harmonics* and *hypergeometric functions*, respectively):

$$\sigma(D^2E(u_B)) = \{\lambda_{\ell k}\}_{\{k, \ell \in \mathbf{N} \mid \ell + 2k < (p/2) + 1\}} \cup [\lambda_0^{cts}, \infty[$$

# Gradient flow structure and linearization

Otto '01 Rescaled dynamics is the *gradient flow* (steepest descent) of  $E(u)$  with respect to an infinite-dimensional heuristic Riemannian structure whose intrinsic distance is the  $L^2$  Wasserstein transportation metric  $d_2$

Denzler-McCann '03,'05 **Diagonalize** the Hessian  $D^2E(u_B)$  of this entropy on the Hilbert tangent space  $W^{1,2}(\mathbf{R}^n, u_B(x)dx)$  at  $u_B$ , by separating eigenfunctions  $\phi(x) = Y(\frac{x}{|x|})R(|x|)$  into *angular* and *radial* parts (*spherical harmonics* and *hypergeometric functions*, respectively):

$$\sigma(D^2E(u_B)) = \{\lambda_{\ell k}\}_{\{k, \ell \in \mathbf{N} \mid \ell + 2k < (p/2) + 1\}} \cup [\lambda_0^{cts}, \infty[$$

$$\lambda_{\ell k} = (\ell + 2k)p + n\ell + 4k(1 - \ell - k)$$

$$\lambda_0^{cts} = \left(\frac{p}{2} + 1\right)^2$$

$D^2E(u_B)\phi = \lambda_{\ell k}\phi$  implies  $\phi(x)$  a degree  $\ell + 2k$  polynomial  $\phi_{\ell k}(x)$

# Slowest dynamically invariant submanifolds

Multiplicity degree  
(=dimension), of  $\phi$ , eigenvalue  $\leftrightarrow$  invariant manifold

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- $\ell$  determines eigenvalue multiplicity (dimension of invariant manifold)
- for each  $\ell \in \mathbf{N}$ , the amplitude  $u_{\ell 0}(x)$  can be deduced from  $u_0(x)$  explicitly, since harmonic moments are invariant under the dynamics.
- amplitudes of the higher radial modes  $k \geq 1$  corresponding to each angular momentum subspace  $\ell$  are not generally explicit.

## Theorem (Higher-order asymptotics in weighted spaces)

Fix  $p = 2(1 - m)^{-1} - n > 2$  and  $2\lambda_{01} > \Lambda \in [\lambda_{01}, \lambda_0^{cts}]$ . If  $u(t, x)$  is a solution with compactly supported initial data  $u_0 = \rho_0$  and center of mass at the origin, then there exist a sequence of polynomials  $\{\phi_{\ell k}(x)\}$ , each element of which either vanishes or has degree  $\ell + 2k \in ]1, \frac{p}{2} + 1[$ , such that

$$\left\| \left[ \frac{u(t, x)}{u_B(x)} - 1 \right] \right\|$$

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as  $t \rightarrow \infty$ , where the sum is over non-negative integers  $k, \ell \in \mathbf{N}$  for which  $\lambda_{\ell k} = (\ell + 2k)p + n\ell + 4k(1 - \ell - k)$  lies in the interval  $]0, \Lambda[$  (and for which  $\ell \leq 1$  if  $n = 1$ ).

## A few key ideas

- To stay a uniform distance from the singularity at zero of  $u \mapsto u^m$ , reexpress **nonlinear** dynamics in terms of the *relative density*  $v(t, x) := u(t, x)/u_B(x)$ .

$$v_t = \frac{1}{m} \nabla \cdot [(B + |x|^2) \nabla v^m] + \text{lot}(Dv, v, |x|, B)$$

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- To cure **degenerate** parabolicity of dynamics **linearized** at  $v(t, x) = 1$ , view  $\mathbf{R}^n$  as a (conformally flat) Riemannian manifold  $(M, g)$  with the so-called *cigar* metric

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- combine DeGiorgi-Nash-Moser with implicit function theorem to get **differentiability** of  $v_0 \in C^{k, \alpha}(M) \cap B_\epsilon(1) \mapsto v \in C^{k, \alpha}([0, \infty[ \times M)$

A few more details:  $(v - 1)_t = -L(v - 1) + o(v - 1)$

where  $L : C^{k,\alpha}(M) \rightarrow C^{k,\alpha}(M)$  given by

$$\begin{aligned} L_\eta &= (\cosh s)^{-\eta} \circ L \circ (\cosh s)^\eta \\ &= -\Delta_{(M, ds^2)} + 2\left(\frac{p}{2} - 1 - \eta\right) \tanh s \frac{\partial}{\partial s} + \left(\frac{p}{2} + 1\right)^2 \\ &\quad - \left(\frac{p}{2} - 1 - \eta\right)^2 - \left(\left(\frac{n}{2} + \frac{p}{2} + 1\right)^2 - \left(\frac{n}{2} + \frac{p}{2} - 1 - \eta\right)^2\right) \frac{1}{\cosh^2 s} \end{aligned}$$

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here

$$H\phi = (B + |x|^2) \Delta_{\mathbf{R}^n} - (p + n)x \cdot \nabla \phi$$

so also  $L \circ \frac{1}{B+|x|^2} = \frac{1}{B+|x|^2} \circ H$

Thank you!