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Free boundaries in optimal transport and Monge-Ampère obstacle problems

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Abstract

Given compactly supported $0 \le f, g \in L^1(\mathbf{R}^n)$, the problem of transporting a fraction $m \le \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$ of the mass of f onto g as cheaply as possible is considered, where cost per unit mass transported is given by a cost function c, typically quadratic $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$. This question is shown to be equivalent to a double obstacle problem for the Monge-Ampère equation, for which sufficient conditions are given to guarantee uniqueness of the solution, such as f vanishing on $\operatorname{spt} g$ in the quadratic case. The part of f to be transported increases monotonically with m, and if $\operatorname{spt} f$ and $\operatorname{spt} g$ are separated by a hyperplane H, then this part will separated from the balance of f by a semiconcave Lipschitz graph over the hyperplane. If $f = f\chi_\Omega$ and $g = g\chi_\Lambda$ are bounded away from zero and infinity on separated strictly convex domains $\Omega, \Lambda \subset \mathbf{R}^n$, for the quadratic cost this graph is shown to be a $C^{1,\alpha}_{loc}$ hypersurface in Ω whose normal coincides with the direction transported; the optimal map between f and g is shown to be Hölder continuous up to this free boundary, and to those parts of the fixed boundary $\partial\Omega$ which map to locally convex parts of the path-connected target region.

1. Introduction

In the classical transportation problem of Monge [62] and Kantorovich [49], one is given a distribution $f(\mathbf{x})$ of iron mines throughout the countryside, and a distribution $g(\mathbf{y})$ of factories which require iron ore, and asked to decide which mines should supply ore to each factory in order to minimize the total

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transportation costs. Here the cost per ton of ore transported from $\mathbf{x}$ to $\mathbf{y} \in \mathbf{R}^n$ is given by a function which we usually take to be the distance squared $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$, while the problem is traditionally studied under the assumption that net production balances net consumption

$$\int_{\mathbf{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}^n} g(\mathbf{y})\, d\mathbf{y} < +\infty. \tag{1.1}$$

In the present work we examine the case in which the total production and consumption need not agree, but ask the question: If one wishes to utilize only a certain fraction $m \le \min\{\|f\|_1, \|g\|_1\}$ of production and consumption capacity, which mines should remain active and which factories should they supply if total transportation costs are to be a minimum? If the mines are separated from the factories, the unique solution turns out to be given by pair of domains $U, V \subset \mathbf{R}^n$, with U containing the active mines and V the active factories, together with a correspondence $\mathbf{s} : U \longrightarrow V$ mapping each active mine to the corresponding factory. The domains depend monotonically on m, and can be characterized as the non-contact regions in a double obstacle problem for the Monge-Ampère equation; they obey the mass balance relation

$$m = \int_U f(\mathbf{x})\, d\mathbf{x} = \int_V g(\mathbf{y})\, d\mathbf{y}, \tag{1.2}$$

together with the assertion that the optimal map between $f + (1 - \chi_V)g$ and $(1 - \chi_U)f + g$ coincides with the identity map $\mathbf{s}(\mathbf{x}) = \mathbf{x}$ outside of $U \cup V$. We go on to specify conditions on f and g (e.g. (1.4)–(1.5) with $\Omega, \Lambda \subset \mathbf{R}^n$ strictly convex) which are sufficient to ensure that U and V are path connected regions with $C^{1,\alpha}_{loc}$ smooth free boundaries, and that $\mathbf{s} : \overline{U} \longrightarrow \overline{V}$ is a homeomorphism (smoother on the interior if f and g are) which remains Hölder continuous up to the free, and part of the fixed, boundary.

Our approach relies on the duality ideas exploited by Brenier in his study [11][12] of the case of complete transfer $m = \|f\|_1 = \|g\|_1$, and on regularity results developed by Caffarelli for that case [15][14][16][17]. A main conclusion of Brenier was that for distance squared $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$, the optimal correspondence $\mathbf{y} = \mathbf{s}(\mathbf{x})$ between mines and factories could be uniquely characterized as the gradient $\mathbf{s} = \nabla\psi$ of a convex function $\psi : \mathbf{R}^n \longrightarrow \mathbf{R}$; c.f. parallel developments in Abdellaoui and Heinich [1], Cuesta-Albertos, Matrán, and Tuero-Díaz [25][26], Cullen and Purser [28], Knott and Smith [52][74], and Rüschendorf and Rachev [69], and alternative approaches in Caffarelli [16], Gangbo [37], and McCann [56]. Where smooth — and indeed almost everywhere [57, Remark 4.5] — this convex function ψ must satisfy the Monge-Ampère equation

$$\det\left[D^2_{ij}\psi(\mathbf{x})\right] = f(\mathbf{x})/g(\nabla\psi(\mathbf{x})). \tag{1.3}$$

If in addition, the production and consumption densities $f, g \geq 0$ are bounded above and below on their domains $\Omega, \Lambda \subset \mathbf{R}^n$ of support

$$\epsilon\chi_\Omega(\mathbf{x}) \leq f(\mathbf{x}) \leq \epsilon^{-1}\chi_\Omega(\mathbf{x}) \tag{1.4}$$

$$\epsilon\chi_\Lambda \quad \leq g \quad \leq \epsilon^{-1}\chi_\Lambda, \tag{1.5}$$

then Caffarelli has shown the map $\nabla\psi : \Omega \longrightarrow \Lambda$ to be injective [15] and Hölder continuous locally [14] provided $\Omega \subset \mathbf{R}^n$ is bounded and Λ is convex. Then $\psi \in C^{k+2,\beta}_{\mathrm{loc}}(\Omega)$ whenever f and g are $C^{k,\overline{\beta}}_{\mathrm{loc}}$ smooth for $\beta \in]0, \overline{\beta}[$ and $k = 0, 1, 2, \ldots$. Partial regularity could be extended to the boundary: $\psi \in C^{1,\alpha}(\overline{\Omega})$ [16] or $\psi \in C^{2,\alpha}(\overline{\Omega})$ [17], but only at the expense of assuming convexity and (in the latter case) smoothness of the domain Ω as well as the target Λ; c.f. Delanoë [29], Urbas [79], and Wolfson [82]. Since our partial transfer problem reduces to mapping $\chi_U f + (1 - \chi_V)g$ onto g, the interior regularity results can be invoked directly provided Λ is convex. Unfortunately, the boundary theory cannot be applied directly since the unknown domains $U \subset \Omega$ and $V \subset \Lambda$ generally fail to be convex. Our argument for extending Hölder estimates to the free boundary (and part of the fixed one) will couple the observation that free boundary never maps to fixed boundary, with a local version of Caffarelli's method, plus certain geometrical properties, such as an *interior ball condition* $\{\mathbf{x} \in \Omega \mid c(\mathbf{x}, \mathbf{s}(\mathbf{x}_0)) < c(\mathbf{x}_0, \mathbf{s}(\mathbf{x}_0))\} \subset U$ which holds for every $\mathbf{x}_0 \in U$, and provides a one-sided curvature bound at each point of the free boundary $\Omega \cap \partial U$. This ball condition implies the displacement $\mathbf{s}(\mathbf{x}) - \mathbf{x}$ is perpendicular to the free boundary, allowing us to conclude Hölder continuity of the free normal also. This discussion is developed in Section §7, which together with Appendix A contains a complete exposition of the $C^{1,\alpha}(\Omega)$ regularity theory (boundary and interior) when production and consumption are fixed and equal on two given convex sets. Unfortunately, the geometry we establish for the free boundary is not sufficient to decide whether higher regularity of the free normal and mapping nearby might follow from higher regularity of the data f and g, as it would for complete transfer between smooth uniformly convex domains [18]. This question remains open in the partial transfer case.

We mention that our partial transfer problem involves augmenting the Monge-Ampère equation (1.3) and inclusion $\nabla\psi(\mathbf{x}) \in \Lambda$ with Dirichlet free boundary data $\psi(\mathbf{x}) = |\mathbf{x}|^2/2$ on $\Omega \cap \partial U$. It goes without saying that the regularity discussion is specific to the quadratic cost $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$. In sharp contradistinction to the more familiar situation of complete transfer, the optimizer for the quadratic cost in the partial transfer problem will not generally optimize the bilinear cost $\tilde{c}(\mathbf{x}, \mathbf{y}) = -\langle \mathbf{x}, \mathbf{y} \rangle$. This is illustrated by the following simple example, which also indicates why new hypotheses are required to ensure uniqueness.

Example 1.1 (Transport between concentric balls). *Let* $f = \chi_{B_1}$ *and* $g = \chi_{B_n}$ *be the characteristic functions of balls centered at the origin, with radii* 1 *and* n *respectively. If we ask to transfer mass* $m < \|f\|_1$ *from* f *to* g *so as to minimize the quadratic cost* $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$*, the solution is far from unique: any function* $f_1 \le f \le g$ *with mass* m *can be transported to* $g_1 = f_1$ *at zero cost. On the other hand, the bilinear cost* $\tilde{c}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$ *is uniquely maximized when* $f_1 = (1 - \chi_{B_r})f$ *and* $g_1 = (1 - \chi_{B_R})g$ *are both chosen to be hollow spheres of mass* m*, and* $s_{\#} f_1 = g_1$ *maps monotonically outward along rays:* $s(\mathbf{x}) = k(|\mathbf{x}|)\mathbf{x}$ *with* $k(t) \ge 0$ *and* $k'(t) \ge 0$*.*

On the other hand, when $m = \min\{\|f\|_1, \|g\|_1\}$ then $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$ is uniquely minimized in this example and, we expect, more generally.

Regularity results for non-quadratic costs are a very recent development even in the context of the fixed boundary (complete transfer) problem, where Ma, Trudinger & Wang have identified a concavity condition on the Hessian of the cost, which — for smooth data and suitable domain geometry — yields regularity of the mapping [55] up to the boundary [77]. This condition is called (A3s) when it holds uniformly, and (A3w) otherwise. Loeper showed that whenever (A3w) fails, there are smooth data on perfectly suitable domains for which the optimal map is discontinuous. Conversely, when (A3s) holds, he gave a direct proof of Hölder continuity of the map $s : \Omega \longrightarrow \Lambda$, with Hölder exponent $\beta = 1/(4n-1)$, under very weak hypotheses on f and g [54]. Since our quadratic cost $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$ satisfies (A3w) but not (A3s), we cannot expect the Hölder continuity established below for partial transport to hold under general perturbations of the cost. It might be expected to hold for (A3w) perturbations, but since the affine invariance exploited below is specific to the quadratic cost, this question presumably requires a different approach to resolve. For (A3s) costs, Loeper's argument offers some hope of addressing the Hölder continuity of partial transport, a possibility currently being investigated with Y.-H. Kim [51].

Kantorovich duality is of course quite general, see e.g. Kellerer [50], Rachev & Rüschendorf [66], or Villani [80], while unique characterizations of optimal maps for other costs have been investigated by Ahmad [3], Ambrosio & Rigot [9] Caffarelli [18], Gangbo & McCann [39][40][41], Gangbo & Święch [42], McCann [59] [60], Plakhov [65], Rüschendorf & Uckelmann [70], Uckelmann [78] in various geometries. Monge's cost $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ in particular has attracted recent attention from Ambrosio [5] & Pratelli [8], Ambrosio, Kircheim & Pratelli [7], Caffarelli, Feldman & McCann [19], DePascale, Evans & Pratelli [30], Feldman & McCann [36] [35], and Trudinger & Wang [76] following the work of Evans & Gangbo [33] and Sudakov [75]. Although the free boundary problem we pose has not been much studied, other transportation problems in which one measure is fixed and the second is selected by a variational principle have been examined extensively in the context of dynamical problems since the work of Otto [64] and Jordan, Kinderlehrer & Otto [48], see also e.g. Agueh [2], Ambrosio, Gigli & Savare [6], Cullen & Gangbo [27], Gianazza, Savare & Toscani [43], Savin [71]; in shape optimization since the work of Bouchitte, Buttazzo & Seppecher [10], see also e.g. Milakis [61] and Xia [83]; and in economics since the work of Rochet & Choné [67], see also e.g. Carlier [20], Carlier & Lachand-Robert [23], Carlier & Ekeland [22], Ekeland [32] and Buttazzo, Pratelli & Stepanov [13]. Obstacle problems for the Monge-Ampère equation have been considered by Chou & Wang [24], Dolbeault & Monneau [31], Lee [53], and Savin [72]; the formulation and boundary conditions of these single obstacle problems are quite different from the double obstacle problems analyzed below, even though some similar issues are addressed.

To formulate our problem more precisely, fix a pair of $L^1(\mathbf{R}^n)$ functions $f, g \geq 0$. Let $\Gamma_{\leq}(f, g)$ denote the set of non-negative Borel measures on $\mathbf{R}^n \times \mathbf{R}^n$ whose left and right marginals are dominated by $f(\mathbf{x})\, d\mathbf{x}$ and $g(\mathbf{y})\, d\mathbf{y}$ respectively:

$$\gamma[A \times \mathbf{R}^n] \leq \int_A f(\mathbf{x})\, d\mathbf{x} \quad \text{and} \quad \gamma[\mathbf{R}^n \times A] \leq \int_A g(\mathbf{y})\, d\mathbf{y} \tag{1.6}$$

for $\gamma \in \Gamma_{\leq}(f, g)$ and every Borel set $A \subset \mathbf{R}^n$. The cost functional to be minimized is

$$\mathcal{C}_\lambda(\gamma) := \int_{\mathbf{R}^n \times \mathbf{R}^n} [c(\mathbf{x}, \mathbf{y}) - \lambda]\, d\gamma(\mathbf{x}, \mathbf{y}), \tag{1.7}$$

with the minimum taken over all measures in $\Gamma_{\leq}(f, g)$ of fixed mass $\gamma[\mathbf{R}^n \times \mathbf{R}^n] = m$. For technical reasons it is easier to introduce a Lagrange multiplier $\lambda \geq 0$ conjugate to this constraint, and take the infimum over joint measures of all masses:

$$C_\lambda(f, g) := \inf_{\gamma \in \Gamma_{\leq}(f,g)} \mathcal{C}_\lambda(\gamma). \tag{1.8}$$

If the optimizer is unique, we denote it by γ_λ and its mass by $m(\lambda) := \gamma_\lambda[\mathbf{R}^n \times \mathbf{R}^n]$. It is then easily deduced that $m(\lambda) = -\partial C_\lambda(f, g)/\partial\lambda$ increases continuously from 0 to $\min\{\|f\|_1, \|g\|_1\}$ as λ is increased. Thus each mass m can be attained by selecting the appropriate value of $\lambda \geq 0$. Finally, we verify that only one measure in $\Gamma_{\leq}(f, g)$ with mass m is optimal, and characterize it as described.

The characterization of this optimal measure and its unicity are derived from a maximization problem dual to (1.8) (in the sense of linear programming or Kantorovich [49]). In fact, we check that

$$C_\lambda(f, g) = \sup_{\substack{u(\mathbf{x}) + v(\mathbf{y}) \leq c(\mathbf{x}, \mathbf{y}) - \lambda \\ u, v \leq 0}} \int_{\mathbf{R}^n} u(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} + \int_{\mathbf{R}^n} v(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}, \tag{1.9}$$

and use the optimal u and v to describe the active regions $U = \{\mathbf{x} \mid u(\mathbf{x}) < 0\}$ and $V = \{\mathbf{y} \mid v(\mathbf{y}) < 0\}$ and support of the optimal measure γ, where *support* refers to the smallest closed subset of $\mathbf{R}^n \times \mathbf{R}^n$ carrying the full mass of γ, denoted $\operatorname{spt}\gamma$. When $c(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2/2$, the minimization problem

$$\inf_{\substack{\psi(\mathbf{x})+\phi(\mathbf{y}) \ge \langle \mathbf{x},\, \mathbf{y}\rangle \\ \psi(\mathbf{x}) \ge (|\mathbf{x}|^2-\lambda)/2 \quad \phi(\mathbf{y}) \ge (|\mathbf{y}|^2-\lambda)/2}} \int_{\mathbf{R}^n} \psi(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} + \int_{\mathbf{R}^n} \phi(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} \tag{1.10}$$

is seen to be equivalent, in the sense that $\psi(\mathbf{x}) = (|\mathbf{x}|^2 - \lambda)/2 - u(x)$ and $\phi(\mathbf{y}) = (|\mathbf{y}|^2 - \lambda)/2 - v(\mathbf{y})$ optimize (1.10) precisely when (u, v) optimize (1.9). The difference in value between (1.10) and (1.9) is determined by the second moments and mass of f and g. As in Brenier [11], one may restrict the minimization to convex functions ψ and ϕ, since e.g. ψ can always be replaced by the convex function $\tilde{\psi}(\mathbf{x}) = \max\{\phi^*(\mathbf{x}), (|\mathbf{x}|^2 - \lambda)/2\}$ without increasing (1.10). Here ϕ^* denotes the Legendre-Fenchel transform

$$\phi^*(\mathbf{x}) := \sup_{\mathbf{y}\in\mathbf{R}^n} \langle \mathbf{x},\, \mathbf{y}\rangle - \phi(\mathbf{y}). \tag{1.11}$$

The optimal solutions to (1.10) and (1.8) are related by

$$\begin{aligned} \gamma[\{(\mathbf{x}, \nabla\psi(\mathbf{x})) \mid \mathbf{x} \in U\}] &= \gamma[\mathbf{R}^n \times \mathbf{R}^n] \\ &= \int_U f(\mathbf{x}) d\mathbf{x} \end{aligned}$$

where $U = \{\mathbf{x} \in \mathbf{R}^n \mid \psi(\mathbf{x}) > (|\mathbf{x}|^2 - \lambda)/2\}$, so the convex function ψ determines both the support of γ (essentially the graph of $s = \nabla\psi$) and its left marginal $f\chi_U$ — i.e. the active mines and correspondence between these mines and factories. Note the appearance of the Dirichlet condition $\psi(\mathbf{x}) = (|\mathbf{x}|^2 - \lambda)/2$ implicitly satisfied along the free boundary $\Omega \cap \partial U$, and the condition $\nabla\psi(\mathbf{x}) = \mathbf{x}$ implied throughout $\mathbf{R}^n \setminus U$.

The remainder of this manuscript is organized as follows. The next section derives a duality theory for the partial transfer problem with quite general cost functions $c(\mathbf{x},\mathbf{y})$, giving sufficient conditions for uniqueness of the optimizer. A third section demonstrates monotone dependence of the active domains U and V on the amount m of mass transferred. For costs of the form $c(\mathbf{x},\mathbf{y}) = h(\mathbf{x}) - \langle \mathbf{x},\, \mathbf{y}\rangle + k(\mathbf{y})$, a fourth section formulates a double obstacle problem for the Monge-Ampère equation which it shows to be equivalent; the Lagrange multiplier λ controlling the optimal mass parameterizes the distance between the upper and lower obstacles. For the quadratic cost $c(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2/2$, a fifth section addresses semi-concavity of the free boundary, when f and g are compactly supported on opposite sides of a hyperplane. The last two sections address interior and boundary regularity for the optimal mapping, under the assumption that $f = f\chi_\Omega$ and $g = g\chi_\Lambda$ are bounded away from zero and infinity (1.4)–(1.5) on separated strictly convex domains $\Omega, \Lambda \subset \mathbf{R}^n$. An appendix is included which makes the boundary regularity analysis essentially self-contained.

2. Duality and uniqueness for partial transfer

For quite general costs, the solution to the partial mass transfer problem (1.8) is typically (c.f. [32]) derived from the solution of a complete mass transfer problem constructed as follows: Attach an *isolated* point $\hat{\infty}$ to $\mathbf{R}^n$, and extend the cost function

$$\hat{c}(\mathbf{x},\mathbf{y}) := \begin{cases} c(\mathbf{x},\mathbf{y}) - \lambda & \text{if } \mathbf{x} \neq \hat{\infty} \text{ and } \mathbf{y} \neq \hat{\infty} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

and measures $d\mu(\mathbf{x}) := f(\mathbf{x})\, d\mathbf{x}$ and $d\nu(\mathbf{y}) = g(\mathbf{y})\, d\mathbf{y}$ to $\hat{\mathbf{R}}^n := \mathbf{R}^n \cup \{\hat{\infty}\}$ by adding a Dirac mass isolated at infinity: $\hat{\mu} = \mu + \|g\|_{L^1}\delta_{\hat{\infty}}$ and $\hat{\nu} = \nu + \|f\|_{L^1}\delta_{\hat{\infty}}$. The measures $\hat{\mu}$ and $\hat{\nu}$ now have the same total mass, and we can ask to minimize the integral of the cost function $\hat{c}$ against joint measures with these marginals:

$$\Gamma(\hat{\mu},\hat{\nu}) := \left\{ 0 \leq \hat{\gamma} \text{ on } \hat{\mathbf{R}}^n \times \hat{\mathbf{R}}^n \;\middle|\; \begin{array}{rl} \hat{\mu}[U] = & \hat{\gamma}[U \times \hat{\mathbf{R}}^n] \\ \hat{\nu}[U] = & \hat{\gamma}[\hat{\mathbf{R}}^n \times U] \end{array} \text{ for Borel } U \subset \hat{\mathbf{R}}^n \right\}.$$

A bijection between $\gamma \in \Gamma_{\leq}(f,g)$ and $\hat{\gamma} \in \Gamma(\hat{\mu},\hat{\nu})$ is given by

$$\hat{\gamma} = \gamma + (f - f_1) \otimes \delta_{\hat{\infty}} + \delta_{\hat{\infty}} \otimes (g - g_1) + \gamma[\mathbf{R}^d \times \mathbf{R}^d]\delta_{(\hat{\infty},\hat{\infty})}, \tag{2.2}$$

where $f_1 \leq f$ and $g_1 \leq g$ represent the marginals of $\gamma \in \Gamma_{\leq}(f,g)$. Since the point at infinity acts as a tariff-free reservoir (2.1), it is easy to see the infimum (1.8) agrees with

$$\inf_{\hat{\gamma} \in \Gamma(\hat{\mu},\hat{\nu})} \int_{\hat{\mathbf{R}}^n} \hat{c}(\mathbf{x},\mathbf{y}) d\hat{\gamma}(\mathbf{x},\mathbf{y}), \tag{2.3}$$

and γ optimizes (1.8) if and only if it coincides with the restriction of a minimizing $\hat{\gamma}$ to $\mathbf{R}^n \times \mathbf{R}^n$. Under very mild assumptions, this allows us to invoke the standard duality theory (2.10), in the form of the following lemmas and corollaries; see e.g. [40] [50] [66] [80]. The interior ball condition (2.9) deduced for the active domain plays a critical role in the developments which follow. Proposition 2.9 then identifies conditions (2.16) on the cost function to make the optimal transfer unique. Injectivity of $\mathbf{y} \longrightarrow \nabla_{\mathbf{x}} c(\mathbf{x}_0, \mathbf{y})$ is a familiar criterion from Gangbo [38], Carlier [21], and Ma, Trudinger & Wang for uniqueness of total transfer; it follows from strict convexity in Caffarelli [18] and Gangbo & McCann [39]. What is new to the setting of partial transfer is the requirement that this map be non-vanishing. For the quadratic cost $c(\mathbf{x},\mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2$, f and g must therefore be disjointly supported for our uniqueness criterion to apply. Here the support $\operatorname{spt} f$ of a measure shall always refer to the smallest closed set containing full mass.

DEFINITION 2.1 (c-CONCAVITY AND c-TRANSFORM). *A function* $\hat{u} : \hat{\mathbf{R}}^n \longrightarrow [-\infty, \infty[$ *is* $\hat{c}$-concave *if it is not identically* $-\infty$ *on* $\operatorname{spt}\hat{\mu}$ *but satisfies*
(2.4)
$$\hat{u}(\mathbf{x}) = \inf_{\mathbf{y}\in \operatorname{spt}\hat{\nu}} \hat{c}(\mathbf{x},\mathbf{y}) - \hat{u}_{\hat{c}}(\mathbf{y}) =: \hat{u}_{\hat{c}\hat{c}} \quad \text{where} \quad \hat{u}_{\hat{c}}(\mathbf{y}) := \inf_{\mathbf{x}\in \operatorname{spt}\hat{\mu}} \hat{c}(\mathbf{x},\mathbf{y}) - \hat{u}(\mathbf{x}).$$

LEMMA 2.2 (TOPOLOGICAL PRELIMINARIES). *Fix* $0 \le f, g \in L^1(\mathbf{R}^n)$. *Then* $\Gamma_{\le}(f,g)$ *is weak-$*$ compact in the Banach space dual* $C_\infty(\mathbf{R}^n \times \mathbf{R}^n)^*$. *If the cost*
$$c(\mathbf{x},\mathbf{y}) \ge u(\mathbf{x}) + \mathbf{v}(\mathbf{y}) \tag{2.5}$$
is continuous and dominates a sum $u(\mathbf{x}) + v(\mathbf{y})$ *with* $uf \in L^1(\mathbf{R}^n)$ *and* $vg \in L^1(\mathbf{R}^n)$, *then* $\mathcal{C}_\lambda : \Gamma_{\le}(f,g) \longrightarrow]-\infty, \infty]$ *is weak-$*$ lower semicontinuous and well-defined.*

Proof: Here $C_\infty(\mathbf{R}^n \times \mathbf{R}^n)$ denotes the continuous functions which vanish at ∞ normed by the supremum norm, i.e. the closed subspace of $L^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ generated by compactly supported continuous functions. The norm of a positive measure $\gamma \in \Gamma_{\le}(f,g)$ in the Banach space dual $C_\infty(\mathbf{R}^n \times \mathbf{R}^n)^*$ coincides with its mass $\|\gamma\|_{C_\infty^*} = \gamma[\mathbf{R}^d \times \mathbf{R}^d]$. Thus $\Gamma_{\le}(f,g)$ is bounded by $\min\{\|f\|_{L^1}, \|g\|_{L^1}\}$, and weak-$*$ pre-compact by the Banach-Alaoglu theorem. Any sequence $\gamma_n \in \Gamma_{\le}(f,g)$ has a weak-$*$ convergent subsequence $\gamma_n \in \Gamma(f_n, g_n)$ whose marginals $f_n \to f_\infty$ and $g_n \to g_\infty$ also converge weak-$*$ in $C_\infty(\mathbf{R}^n)$. Now $\gamma_n \to \gamma_\infty \in \Gamma(f_\infty, g_\infty)$ according to [56, Proposition 9(ii)]. Since $f_\infty \le f$ and $g_\infty \le g$ we have $\Gamma_{\le}(f,g)$ weak-$*$ compact.

Fixing $\lambda = 0$ for the moment, we may assume $u, v \le 0$. We also assume both f have g have positive mass, since otherwise the lemma is trivially true. Now extend u and v to $\hat{\mathbf{R}}^n$ by taking $\hat{u}(\hat{\infty}) = 0 = \hat{v}(\hat{\infty})$ so that (2.11) holds. Since $\hat{\mu}[\hat{\infty}] = \|g\|_1$ in (2.2), we deduce $\hat{u}_{\hat{c}} \le 0$ from (2.4), and $\hat{u}_{\hat{c}}(\hat{\infty}) = 0$ from $u \le 0$. Replacing $\hat{v}$ by $\hat{u}_{\hat{c}} \ge \hat{v}$ and then $\hat{u}$ by $\hat{u}_{\hat{c}\hat{c}}$, it therefore costs no generality to assume $u \le 0$ is a $\hat{c}$-concave function and $v = \hat{u}_{\hat{c}} \le 0$ in the hypotheses (2.5). Moreover, $\hat{u}$ and $\hat{v}$ are infima of continuous functions (2.4), hence upper semicontinuous. The cost function $\tilde{c}(\mathbf{x},\mathbf{y}) := \hat{c}(\mathbf{x},\mathbf{y}) - \hat{u}(\mathbf{x}) - \hat{v}(\mathbf{y}) \ge 0$ is now bounded below, and lower semi-continuous, so the associated integral
$$\tilde{\mathcal{C}}(\hat{\gamma}) := \int_{\hat{\mathbf{R}}^n \times \hat{\mathbf{R}}^n} \tilde{c}(\mathbf{x},\mathbf{y}) d\hat{\gamma}(\mathbf{x},\mathbf{y})$$
is well-defined and weak-$*$ lower semicontinuous on $\Gamma(\hat{\mu}, \hat{\nu})$ by a monotone convergence theorem argument in which the cost $\tilde{c}(\mathbf{x},\mathbf{y})$ is approximated from below by a continuous cost vanishing at $\infty (\ne \hat{\infty})$. Now $\tilde{\mathcal{C}}(\hat{\gamma})$ differs from $\mathcal{C}_\lambda(\gamma)$ by the finite constant
$$0 \ge \int_{\hat{\mathbf{R}}^n} \hat{u}(\mathbf{x}) d\hat{\mu}(\mathbf{x}) + \int_{\hat{\mathbf{R}}^n} \hat{v}(\mathbf{y}) d\hat{\nu}(\mathbf{y}) > -\infty$$

so $\mathcal{C}_\lambda(\gamma) > -\infty$ and lower semicontinuous on $\Gamma_\leq(f,g)$ for $\lambda = 0$.

Applying the same lemma to both $c(\mathbf{x},\mathbf{y}) = \pm 1$ shows that the mass $m(\gamma) := \gamma[\mathbf{R}^d \times \mathbf{R}^d]$ is a weak-$*$ continuous function on $\Gamma_\leq(f,g)$. Thus weak-$*$ lower semi-continuity of $\mathcal{C}_0(\gamma)$ extends equally well to $\mathcal{C}_\lambda(\gamma) = \mathcal{C}_0(\gamma) - \lambda m(\gamma)$ for $\lambda \neq 0$. QED.

Lemma 2.3 (Optimality criterion). *Let f, g and c satisfy the hypotheses of Lemma 2.2. Then the infimum (2.3) is finite, attained, and there exists a $\hat{c}$-concave function $\hat{u} : \hat{\mathbf{R}}^n \longrightarrow [-\infty, \infty[$ such that every optimal measure $\hat{\gamma} \in \Gamma(\hat{\mu},\hat{\nu})$ satisfies*

$$\operatorname{spt}\hat{\gamma} \subset \{(\mathbf{x},\mathbf{y}) \in \hat{\mathbf{R}}^n \times \hat{\mathbf{R}}^n \mid \hat{u}(\mathbf{x}) + \hat{u}_{\hat{c}}(\mathbf{y}) = \hat{c}(\mathbf{x},\mathbf{y})\} =: \partial_{\hat{c}}\hat{u}. \tag{2.6}$$

Proof: According to Lemma 2.2, the cost function $\mathcal{C}_\lambda(\gamma)$ is weak-$*$ lower semicontinuous on the compact set $\Gamma_\leq(f,g)$. Thus the infimum (1.8), or equivalently (2.3), is attained in $]-\infty, 0]$; it is non-positive since $\hat{\gamma} = \mu \otimes \delta_{\hat{\infty}} + \delta_{\hat{\infty}} \otimes \nu$ is a competitor with zero cost (2.1). For a cost $\hat{c}(\mathbf{x},\mathbf{y}) \geq 0$ on $X \times Y := \operatorname{spt}\hat{\mu} \times \operatorname{spt}\hat{\nu}$, Gangbo and McCann [40, §2] construct a single $\hat{c}$-concave function $\hat{u} : X \longrightarrow [-\infty, \infty[$ such that

$$\operatorname{spt}\hat{\gamma} \subset \partial_{\hat{c}}\hat{u} := \{(\mathbf{x},\mathbf{y}) \in X \times Y \mid \mathbf{x} \in \arg\min_{\mathbf{z}\in X} \hat{c}(\mathbf{z},\mathbf{y}) - \hat{u}(\mathbf{z})\}$$

holds simultaneously for every optimizer $\hat{\gamma} \in \Gamma(\hat{\mu},\hat{\nu})$. Non-negativity of $\hat{c}$ is used only to ensure $\mathcal{C}_\lambda(\hat{\gamma})$ is well-defined in their proof; since we have this instead from Lemma 2.2, their proof extends to our signed costs also. This gives the desired identity (2.6). Since $\hat{u}$ is $\hat{c}$-concave, Rachev and Rüschendorf [66, §3.3.5] assert $\hat{u} = \hat{u}_{\hat{c}\hat{c}}$. The functions $\hat{u}$ and $\hat{u}_{\hat{c}}$ can be extended to all of $\hat{\mathbf{R}}^n$ via (2.4) without modifying their values on $X \times Y$. QED.

Corollary 2.4 (Active versus inactive locations). *Take f, g, c as in Lemma 2.2 and $\lambda \in \mathbf{R}$. Suppose $\gamma_\lambda \in \Gamma(f_1, g_1)$ minimizes (1.8). Then $(\mathbf{x}_1,\mathbf{y}_1) \in \operatorname{spt}\gamma_\lambda$ implies $c(\mathbf{x}_1,\mathbf{y}_1) \leq \lambda$. If $\mathbf{x}_0 \in \operatorname{spt}[f - f_1]$ and/or $\mathbf{y}_0 \in \operatorname{spt}[g - g_1]$ also exist, then $c(\mathbf{x}_1,\mathbf{y}_1) \leq \min\{c(\mathbf{x}_0,\mathbf{y}_1), c(\mathbf{x}_1,\mathbf{y}_0)\}$ and if both exist $\lambda \leq c(\mathbf{x}_0,\mathbf{y}_0)$. Thus U is disjoint from $\operatorname{spt}[f - f_1]$, and V is disjoint from $\operatorname{spt}[g - g_1]$, where*

$$U := \bigcup_{(\mathbf{x}_1,\mathbf{y}_1)\in\operatorname{spt}\gamma_\lambda} \{\mathbf{x} \in \mathbf{R}^n \mid c(\mathbf{x},\mathbf{y}_1) < c(\mathbf{x}_1,\mathbf{y}_1)\} \quad \text{and}$$

$$V := \bigcup_{(\mathbf{x}_1,\mathbf{y}_1)\in\operatorname{spt}\gamma_\lambda} \{\mathbf{y} \in \mathbf{R}^n \mid c(\mathbf{x}_1,\mathbf{y}) < c(\mathbf{x}_1,\mathbf{y}_1)\}. \tag{2.7}$$

Proof: Let $\hat{\gamma}$ from (2.2) extend $\gamma := \gamma_\lambda$. For all $(\mathbf{x}_0,\mathbf{y}_0), (\mathbf{x}_1,\mathbf{y}_1) \in \operatorname{spt}\hat{\gamma}$, the standard monotonicity inequality

$$\hat{c}(\mathbf{x}_0,\mathbf{y}_0) + \hat{c}(\mathbf{x}_1,\mathbf{y}_1) \leq \hat{c}(\mathbf{x}_0,\mathbf{y}_1) + \hat{c}(\mathbf{x}_1,\mathbf{y}_0) \tag{2.8}$$

is easily deduced from (2.4) and (2.6); see e.g. [40, §2.7]. Given $(\mathbf{x}_1, \mathbf{y}_1) \in \operatorname{spt}\gamma$, we have $(\mathbf{x}_0, \mathbf{y}_0) = (\hat{\infty}, \hat{\infty}) \in \operatorname{spt}\hat{\gamma}$ from (2.2), whence $c(\mathbf{x}_1, \mathbf{y}_1) \le \lambda$ from (2.8) and (2.1). If there exists $\mathbf{x}_0 \in \operatorname{spt}[f - f_1]$, then $(\mathbf{x}_0, \hat{\infty}) \in \operatorname{spt}\hat{\gamma}$ so taking $\mathbf{y}_0 = \hat{\infty}$ yields $\hat{c}(\mathbf{x}_1, \mathbf{y}_1) \le \hat{c}(\mathbf{x}_0, \mathbf{y}_1)$ in (2.8). Similarly $\mathbf{y}_0 \in \operatorname{spt}[g - g_1]$ implies $c(\mathbf{x}_1, \mathbf{y}_1) \le c(\mathbf{x}_1, \mathbf{y}_0)$. Finally, applying (2.8) to the pair of points $(\mathbf{x}_0, \hat{\infty}), (\hat{\infty}, \mathbf{y}_0) \in \operatorname{spt}\hat{\gamma}$, yields $\lambda \le c(\mathbf{x}_0, \mathbf{y}_0)$.

Turning to (2.7), we see that $\mathbf{x} \in U \cap \operatorname{spt}[f - f_1]$ implies the contradiction

$$c(\mathbf{x}, \mathbf{y}_1) < c(\mathbf{x}_1, \mathbf{y}_1) \le \lambda \le c(\mathbf{x}, \mathbf{y}_1),$$

so U is disjoint from $\operatorname{spt}[f - f_1]$. Similarly, V is disjoint from $\operatorname{spt}[g - g_1]$, by symmetry under interchange of $\mathbf{x} \leftrightarrow \mathbf{y}$ and $f \leftrightarrow g$. QED.

Example 2.5 (Interior ball condition). *Taking $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^p$ with $p > 0$, and $B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{x} - \mathbf{y}| < r\}$ in the preceding corollary yields*

$$U = \bigcup_{(\mathbf{x},\mathbf{y}) \in \operatorname{spt}\gamma_\lambda} B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \text{ and } V = \bigcup_{(\mathbf{x},\mathbf{y}) \in \operatorname{spt}\gamma_\lambda} B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{x}). \tag{2.9}$$

Corollary 2.6 (Kantorovich duality). *Fix f, g and c satisfying the hypotheses of Lemma 2.2 and $\lambda \in \mathbf{R}$. Then the maximum and minimum below are attained — by any $\hat{\gamma} \in \Gamma(\hat{\mu}, \hat{\nu})$ and $(\hat{u}, \hat{v}) = (\hat{u}_{\hat{c}\hat{c}}, \hat{u}_{\hat{c}})$ satisfying (2.6) with $\hat{c} \in L^1(d\hat{\gamma})$:*

(2.10)
$$\max_{\substack{(\hat{u},\hat{v}) \in L^1(d\hat{\mu} \times d\hat{\nu}) \\ \hat{u}(\mathbf{x}) + \hat{v}(\mathbf{y}) \le \hat{c}(\mathbf{x},\mathbf{y})}} \int_{\hat{\mathbf{R}}^n} \hat{u}(\mathbf{x}) d\hat{\mu}(\mathbf{x}) + \int_{\hat{\mathbf{R}}^n} \hat{v}(\mathbf{y}) d\hat{\nu}(\mathbf{y}) = \min_{\hat{\gamma} \in \Gamma(\hat{\mu},\hat{\nu})} \int_{\hat{\mathbf{R}}^n \times \hat{\mathbf{R}}^n} \hat{c}(\mathbf{x}, \mathbf{y}) d\hat{\gamma}(\mathbf{x}, \mathbf{y}).$$

Proof: Let $\hat{u} \in L^1(d\hat{\mu})$ and $\hat{v} \in L^1(d\hat{\nu})$ be functions satisfying

$$\hat{u}(\mathbf{x}) + \hat{v}(\mathbf{y}) \le \hat{c}(\mathbf{x}, \mathbf{y}), \tag{2.11}$$

Integrating (2.11) against any $\hat{\gamma} \in \Gamma(\hat{\mu}, \hat{\nu})$ shows the infimum dominates the supremum in (2.10); we have only to exhibit a case of equality to conclude the proof.

Choose any $\hat{\gamma} \in \Gamma(\hat{\mu}, \hat{\nu})$ with $\hat{c} \in L^1(d\hat{\gamma})$ and $\hat{u}$ satisfying (2.4) and (2.6). These exist by the preceding lemma, and setting $\hat{v} = \hat{u}_{\hat{c}}$ implies (2.11). Moreover $\hat{u}(\mathbf{x}) + \hat{v}(\mathbf{y}) = \hat{c}(\mathbf{x}, \mathbf{y}) \in \mathbf{R}$ holds throughout $\operatorname{spt}\hat{\gamma}$. In particular $\hat{u}$ is real valued $\hat{\mu}$-a.e., and $\hat{v}$ is real valued $\hat{\nu}$-a.e. Extend uf and $vg \in L^1(\mathbf{R}^n)$ from (2.5) to vanish at $\hat{\infty}$. Defining $\tilde{u} := \hat{u} - u$, $\tilde{v} := \hat{v} - v = \tilde{u}_{\tilde{c}}$ and $\tilde{c}(\mathbf{x}, \mathbf{y}) = \hat{c}(\mathbf{x}, \mathbf{y}) - u(\mathbf{x}) - v(\mathbf{y}) \ge 0$ yields $\tilde{u}(\mathbf{x}) + \tilde{v}(\mathbf{y}) \le \tilde{c}(\mathbf{x}, \mathbf{y})$ with equality on $\operatorname{spt}\hat{\gamma}$. In particular, $\tilde{u}(\mathbf{x}) + \tilde{v}(\mathbf{y}) \ge 0$ on $\operatorname{spt}\hat{\gamma}$ shows $\tilde{u}$ and $\tilde{v}$ bounded below $\hat{\mu}$ and $\hat{\nu}$-a.e., respectively. This means $\int \tilde{u} d\hat{\mu}$ and $\int \tilde{v} d\hat{\nu}$ do not diverge to $-\infty$. Integrating (2.11) now yields

$$\int_{\hat{\mathbf{R}}^n} \tilde{u}(\mathbf{x}) d\hat{\mu}(\mathbf{x}) + \int_{\hat{\mathbf{R}}^n} \tilde{u}_{\tilde{c}}(\mathbf{y}) d\hat{\nu}(\mathbf{y}) = \int_{\hat{\mathbf{R}}^n \times \hat{\mathbf{R}}^n} \tilde{c}(\mathbf{x}, \mathbf{y}) d\hat{\gamma}(\mathbf{x}, \mathbf{y}) < \infty, \tag{2.12}$$

showing $\tilde{u} \in L^1(d\hat{\mu})$ and $\tilde{v} \in L^1(d\hat{\nu})$. Subtracting the finite integral of $uf + vg$ from both sides demonstrates that a finite equality in (2.10) is achieved by $\hat{\gamma}$ and $(\hat{u}, \hat{v}) = (\hat{u}_{\hat{c}\hat{c}}, \hat{u}_{\hat{c}}) \in L^1(d\hat{\mu} \times d\hat{\nu})$. QED.

Corollary 2.7 (Duality for partial-transfer). *The hypotheses of Lemma 2.2 also imply*

(2.13)
$$\max_{\substack{u(\mathbf{x})+v(\mathbf{y})\leq c(\mathbf{x},\mathbf{y})-\lambda \\ u(\mathbf{x}),v(\mathbf{y})\leq 0}} \int_{\mathbf{R}^n} u(\mathbf{x})f(\mathbf{x})\,d\mathbf{x} + \int_{\mathbf{R}^n} v(\mathbf{y})g(\mathbf{y})\,d\mathbf{y} = \min_{\gamma\in\Gamma_{\leq}(f,g)} \int_{\mathbf{R}^n\times\mathbf{R}^n} (c-\lambda)\,d\gamma,$$

and the maximum is attained by the restriction to $\mathbf{R}^d \times \mathbf{R}^d$ *of* $(\hat{u} - \hat{u}(\hat{\infty}), \hat{u}_{\hat{c}} + \hat{u}(\hat{\infty}))$ *from Lemma 2.3.*

Proof: The restriction of $\hat{\gamma} \in \Gamma(\hat{\mu}, \hat{\nu})$ to $\mathbf{R}^n$ gives a measure in $\Gamma_{\leq}(f, g)$ and the associated costs are the same since transportation to and from the isolated reservoir is free (2.1). Moreover, each $\gamma \in \Gamma_{\leq}(f, g)$ extends uniquely to a measure $\hat{\gamma} \in \Gamma(\hat{\mu}, \hat{\nu})$: if γ has marginals $f' \leq f$ and $g' \leq g$, then $\hat{\gamma}$ will have density $f - f'$ and $g - g'$ on $\mathbf{R}^n \times \{\hat{\infty}\}$ and $\{\hat{\infty}\} \times \mathbf{R}^n$, plus an isolated atom of weight $\|f'\|_{L^1(\mathbf{R}^n)} = \|g'\|_{L^1(\mathbf{R}^n)}$ at $(\hat{\infty}, \hat{\infty})$. Thus the minima in (2.10) and (2.13) coincide; it remains to show the same for the maxima.

Any competitors (u, v) in (2.13) can be extended to $\hat{\mathbf{R}}^n$ by taking $u(\hat{\infty}) = 0 = v(\hat{\infty})$; this non-positive extension satisfies (2.11) because of (2.1). The maximum (2.10) over the larger class of competitors can only dominate (2.13). Conversely, the lemma and corollary preceding are unchanged if $(\hat{u}, \hat{v})$ are replaced by $(\hat{u} + k, \hat{v} - k)$ for $k \in \mathbf{R}$. Since $\hat{u}(\hat{\infty}) > -\infty$ for a finite objective, we are free to assume $\hat{u}(\hat{\infty}) = 0$, in which case (2.1) and (2.11) imply $\hat{v}(\mathbf{y}) \leq 0$ throughout $\hat{\mathbf{R}}^n$. At $\mathbf{y} = \hat{\infty}$, the only constraint is that $\hat{v}(\hat{\infty}) \leq \inf_{\mathbf{x}\in\hat{\mathbf{R}}^n} -\hat{u}(\mathbf{x}) =: -u_{max}$, and equality can be assumed to hold for the maximizing $(\hat{u}, \hat{v})$. Thus

$$\int_{\hat{\mathbf{R}}^n} \hat{u}\,d\hat{\mu} + \int_{\hat{\mathbf{R}}^n} \hat{v}\,d\hat{\nu} = \int_{\mathbf{R}^n} \hat{v}g + \int_{\mathbf{R}^n} \hat{u}f\,d\mathbf{x} - u_{max}\|f\|_{L^1(\mathbf{R}^n)}, \tag{2.14}$$

and the sum of the last two terms is not positive since $u_{max} \geq u(\hat{\infty}) \geq 0$. Replacing u by $\min\{u, 0\}$ pointwise always increases the objective (2.14), and makes it easier to satisfy the constraint (2.11). Therefore we conclude $u_{max} = 0$, so the objective functionals in our two maximizations agree. Since the restriction (u, v) of $(\hat{u}, \hat{v})$ to $\mathbf{R}^n$ now satisfies the constraints of (2.13), it is clear that the latter maximization dominates (2.10). Hence the two maximum values coincide, and the latter is attained by the restriction to $\mathbf{R}^d \times \mathbf{R}^d$ of $(\hat{u}, \hat{u}_{\hat{c}})$ after normalizing $\hat{u}(\hat{\infty}) = 0 = \hat{u}_{\hat{c}}(\hat{\infty})$ as described. QED.

To address uniqueness, mappings, and the regularity which follows, we shall need the notion of a pushed-forward measure. Given a measure space

$(X,\hat\mu)$ and a measurable space Y, each measurable map $\mathbf{s} : X \longrightarrow Y$ induces a measure $\mathbf{s}_\#\hat\mu$ on Y, called the *push-forward* of $\hat\mu$ through $\mathbf{s}$, and defined by $\mathbf{s}_\#[V] = \hat\mu[\mathbf{s}^{-1}(V)]$ for each measurable set $V \subset Y$. If $\xi : Y \longrightarrow [-\infty,\infty]$ is measurable, it is not hard to check

$$\int_Y \xi \, d(\mathbf{s}_\#\hat\mu) = \int_X \xi(\mathbf{s}(\mathbf{x}))\, d\hat\mu(\mathbf{x}), \tag{2.15}$$

when either integral is well-defined. As an example, the projection $\pi : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ given by $\pi(\mathbf{x},\mathbf{y}) = \mathbf{x}$ pushes forward $\gamma \in \Gamma(\hat\mu,\hat\nu)$ to its left marginal $\hat\mu = \pi_\#\gamma$. We shall use the notation $\mathrm{dom}\,\xi := \{\mathbf{x} \in X \mid |\xi(\mathbf{x})| < \infty\}$, and if $X = \mathbf{R}^n$ then $\mathrm{dom}\,\nabla\xi \subset \mathbf{R}^n$ will denote the points of differentiability of ξ.

Lemma 2.8 (Uniqueness of transportation). *Taking f, g, c as in Lemma 2.2, assume every $\hat c$-concave function $\hat u : \hat{\mathbf{R}}^n \longrightarrow [-\infty,\infty[$ has the property that for f-a.e. $\mathbf{x}_0 \in \mathrm{dom}\,\hat u \setminus \{\hat\infty\}$, the equation $\hat u(\mathbf{x}_0) + \hat u_c(\mathbf{y}) = \hat c(\mathbf{x}_0,\mathbf{y})$ has at most one solution $\mathbf{y} = \mathbf{s}_{\hat u}(\mathbf{x}_0)$ in $\mathrm{spt}\,\hat\nu$. Then a unique measure $\hat\gamma \in \Gamma(\hat\mu,\hat\nu)$ of finite cost $\hat c \in L^1(d\hat\gamma)$ has restriction to $\mathbf{R}^n \times \hat{\mathbf{R}}^n$ given by $\hat\gamma_1 = (\mathbf{id} \times \mathbf{s}_{\hat u})_\# f$ with $\hat u$ a $\hat c$-concave function, and $\hat u_{\hat c} = \hat u(\hat\infty) = 0$ on $\mathrm{spt}\,[\hat\nu - \mathbf{s}_{\hat u\#} f]$. This $\hat\gamma$ uniquely minimizes (2.3).*

Proof: Lemma 2.3 asserts the existence of at least one optimal measure $\hat\gamma \in \Gamma(\hat\mu,\hat\nu)$ with (2.3) finite, and provides a $\hat c$-concave function $\hat u$ such that all optimal measures are supported inside $\partial_{\hat c}\hat u$. It costs no generality to assume $\hat u(\hat\infty) = 0$. The projection $\pi(\mathrm{spt}\,\hat\gamma)$ under $\pi(\mathbf{x},\mathbf{y}) = \mathbf{x}$ is σ-compact, and since $(\mathbf{x},\mathbf{y}) \in \mathrm{spt}\,\hat\gamma \subset \partial_{\hat c}\hat u$ implies $u(\mathbf{x})$ finite, $\pi(\mathrm{spt}\,\hat\gamma) \subset \mathrm{dom}\,\hat u$. By hypothesis, some Borel set $S \subset \pi(\mathrm{spt}\,\hat\gamma) \setminus \{\hat\infty\}$ containing the full mass of f, admits a map $\mathbf{s}_{\hat u} : S \longrightarrow \mathrm{spt}\,\nu$ such that $S \times \hat{\mathbf{R}}^n \cap \mathrm{spt}\,\hat\gamma = G := \{(\mathbf{x}, \mathbf{s}_{\hat u}(\mathbf{x})) \mid \mathbf{x} \in S\}$. Note that $\mathbf{s}_{\hat u}$ depends on $\hat u$ but not on $\hat\gamma$, except possibly through the precise choice of domain S. The graph G is clearly Borel. Since $\pi : G \longrightarrow \hat{\mathbf{R}}^n$ is Lipschitz and univalent, Federer [34, §2.2.10, p. 67] shows $\mathbf{s}_{\hat u}^{-1}(B) = \pi(G \cap (\mathbf{R}^n \times B))$ is Borel whenever $B \subset \hat{\mathbf{R}}^n$ is. The map $\mathbf{s}_{\hat u}$ is therefore Borel. By [41, Lemma 2.4] we conclude the restriction $\hat\gamma_1 := \hat\gamma|_{\mathbf{R}^n \times \hat{\mathbf{R}}^n}$ is given by $\hat\gamma_1 = (\mathbf{id} \times \mathbf{s}_{\hat u})_\# f$. Since $\hat\gamma_2 := \hat\gamma - \hat\gamma_1$ is supported on $\{\hat\infty\} \times \hat{\mathbf{R}}^n$, it is completely determined by its right marginal $\hat\nu_2 := \hat\nu - \mathbf{s}_{\hat u\#} f$. If a second measure $\hat\gamma'$ minimizes (2.3), the same argument shows $\hat\gamma' = (\mathbf{id} \times \mathbf{s}'_{\hat u})_\# f + \delta_{\hat\infty} \otimes (\hat\nu - \mathbf{s}'_{\hat u\#} f)$, where $\mathbf{s}'_{\hat u} = \mathbf{s}_{\hat u}$ on the intersection of their domains $S \cap S'$. Since this intersection carries the full mass of f, we conclude that finiteness of (2.3) implies the optimizer $\hat\gamma' = \hat\gamma$ is unique. Finally, since $(\mathbf{x},\mathbf{y}) \in \mathrm{spt}\,\hat\gamma_2 = \{\hat\infty\} \times \mathrm{spt}\,\hat\nu_2$ implies $\hat u(\mathbf{x}) + \hat u_{\hat c}(\mathbf{y}) = 0$ with $\mathbf{x} = \hat\infty$, we conclude $\hat u_{\hat c}(\mathbf{y}) = -\hat u(\hat\infty) = 0$ throughout $\mathrm{spt}\,\hat\nu_2$.

If any other $\hat c$-concave function $\hat u'$ with $\hat u'_{\hat c} = 0 = \hat u'(\hat\infty)$ holding on $\mathrm{spt}\,[\hat\nu - \mathbf{s}_{\hat u\#} f]$ induces a measure $\hat\gamma' = (\mathbf{id} \times \mathbf{s}_{\hat u'})_\# f + \delta_{\hat\infty} \otimes (\hat\nu - \mathbf{s}_{\hat u'\#} f)$ in $\Gamma(\hat\mu,\hat\nu)$, we conclude $\hat\gamma'$-a.e. $(\mathbf{x},\mathbf{y})$ belongs to $\partial_{\hat c}\hat u'$. Integrating the equality $\hat u(\mathbf{x}) + \hat u_{\hat c}(\mathbf{y}) = \hat c(\mathbf{x},\mathbf{y}) \in L^1(d\hat\gamma')$ against $\hat\gamma'$, we conclude (2.12) holds and is finite, as in the last part of the proof of Corollary 2.6. Now duality shows $\hat\gamma'$ to be optimal, hence to coincide with the unique minimizer $\hat\gamma$ in (2.10). QED.

The following theorem gives conditions on the cost which guarantee uniqueness of partial transfer. These conditions suffice for the present purpose, though we have no doubt that various refinements are possible and desirable, e.g. for eliminating the compact support assumption from Theorem 4.3. We say a function $\xi : \mathbf{R}^n \longrightarrow [-\infty, \infty]$ is *superdifferentiable* at $\mathbf{x}_0 \in \mathbf{R}^n$ if there exists $\mathbf{p} \in \mathbf{R}^n$ such that

$$\limsup_{\mathbf{x}\to\mathbf{x}_0} \frac{\xi(\mathbf{x}) - \xi(\mathbf{x}_0) - \langle \mathbf{p}, \mathbf{x} - \mathbf{x}_0\rangle}{|\mathbf{x} - \mathbf{x}_0|} \le 0.$$

Any concave function $\xi : \mathbf{R}^n \longrightarrow [-\infty, \infty[$ is superdifferentiable on $\operatorname{int}[\operatorname{dom}\xi]$.

Proposition 2.9 (Ensuring uniqueness of partial transfer). *Fix $0 \le f, g \in L^1(\mathbf{R}^n)$. Assume $c(\mathbf{x}, \mathbf{y})$ is Lipschitz and superdifferentiable on the interior of* $\operatorname{conv}[\operatorname{spt} f \times \operatorname{spt} g]$. *Suppose for f-a.e. $\mathbf{x}_0 \in \mathbf{R}^n$ the map*

$$\mathbf{y} \in D \longrightarrow \nabla_{\mathbf{x}} c(\mathbf{x}_0, \mathbf{y}) \text{ is non-vanishing and injective} \tag{2.16}$$

on the set $D \subset \operatorname{spt} g$ where it is well-defined. Then — with the possible exception of the lower bounds (2.5) all hypotheses of Lemma 2.8 are satisfied.

Proof: McShane's theorem gives a global extension of $\hat c(\mathbf{x}, \mathbf{y})$ to $\mathbf{R}^n \times \mathbf{R}^n$ with Lipschitz constant L. Let $\hat u$ be a $\hat c$-concave function. Then (2.4) expresses $\hat u$ as the infimum of a family of Lipschitz functions of $\mathbf{x}$, with $|\nabla_{\mathbf{x}} \hat c(\mathbf{x}, \mathbf{y})| \le L$. According to e.g. [81, §10.26], $\hat u$ is real-valued and has the same Lipschitz constant L, since the alternative $\hat u := -\infty$ is not $\hat c$-concave. Now $\hat u$ is differentiable f-a.e. by Rademacher's theorem. Choose $\mathbf{x}_0 \in \operatorname{dom}\nabla\hat u \cap \operatorname{int}[\operatorname{conv}[\operatorname{spt} f]]$. Recall $\hat c(\mathbf{x}_0, \mathbf{y}) - \hat u(\mathbf{x}_0) - \hat u_{\hat c}(\mathbf{y}) \ge 0$ from (2.4). Suppose $\mathbf{y}_0 \in \operatorname{spt}\hat\nu$ produces equality. Our non-negative function then attains a local minimum with respect to both variables, so $\mathbf{0} \in \mathbf{R}^n$ is a subgradient of $\hat c(\mathbf{x}, \mathbf{y}_0) - \hat u(\mathbf{x})$ at $\mathbf{x}_0$. Since $\mathbf{x}_0 \in \operatorname{dom}\nabla\hat u$, it follows that $-\nabla\hat u(\mathbf{x}_0)$ is a subgradient for $h(\mathbf{x}) := \hat c(\mathbf{x}, \mathbf{y}_0)$ at $\mathbf{x}_0 \ne \hat\infty$. Now $h(\mathbf{x})$ is also superdifferentiable, so its derivative exists and $\nabla_{\mathbf{x}} \hat c(\mathbf{x}_0, \mathbf{y}_0) = \nabla\hat u(\mathbf{x}_0)$. If $\mathbf{y}_0 = \hat\infty$ we have $\nabla_{\mathbf{x}} \hat c(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$ from (2.1). Thus $\nabla\hat u(\mathbf{x}_0) \ne \mathbf{0}$ implies $\mathbf{y}_0 \in \operatorname{spt} g$, in which case the injectivity hypothesis (2.16) determines $\mathbf{y}_0 \in \operatorname{spt} g$ uniquely in terms of $\hat u$ and $\mathbf{x}_0$. On the other hand, if $\nabla\hat u(\mathbf{x}_0) = \mathbf{0}$ we can only have $\mathbf{y}_0 \in \{\hat\infty\} = \operatorname{spt}\hat\nu \setminus \operatorname{spt} g$ by the non-vanishing restriction on $\nabla_{\mathbf{x}} c$. Either way, $\mathbf{y}_0$ is uniquely determined, so the hypotheses of Lemma 2.8 are verified. QED.

Example 2.10 (Square distance). *Taking $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$, the condition (2.16) for uniqueness becomes that f vanish a.e. on* $\operatorname{spt} g$. *In particular, the mass distributions f and g must be mutually singular. Some such* *condition is obviously necessary, as the example* $f = g = \chi_\Omega$ *with* $m < \mathrm{vol}\,[\Omega]$ *shows. When* $\mathrm{spt}\, f$ *is separated from* $\mathrm{spt}\, g$ *by a positive distance, (2.9) becomes an interior ball condition for the active sets* U *and* V*. The bounds (2.5) hold whenever* f *and* g *have finite second moments.*

Corollary 2.11 (Lagrange multiplier for transport amount). *Take* f, g *and* c *as in Lemma 2.3. Suppose for each* λ *the optimizer* $\gamma_\lambda \in \Gamma_\leq(f,g)$ *in (1.8) is unique. Then* $\lambda \longrightarrow \gamma_\lambda$ *is a weak-$*$ continuous curve in* $C_\infty(\mathbf{R}^n \times \mathbf{R}^n)^*$ *and* $-\mathcal{C}_\lambda(\gamma_\lambda)$ *is a convex differentiable function of* $\lambda \in \mathbf{R}$ *whose slope* $-d\mathcal{C}_\lambda(\gamma_\lambda)/d\lambda = m(\lambda) := \gamma_\lambda[\mathbf{R}^d \times \mathbf{R}^d]$ *ranges continuously from* $m(-\infty) = 0$ *to* $m(\infty) = \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$*. Each distinct slope* $m(\lambda) = m(\lambda')$ *corresponds to a unique measure* $\gamma_\lambda = \gamma_{\lambda'}$*. The extremal slopes are attained for finite* λ *if* $c(\mathbf{x}, \mathbf{y})$ *is bounded on* $\mathrm{spt}\, f \times \mathrm{spt}\, g$*.*

Proof: Define $\mathcal{C}_\lambda(f,g) := \mathcal{C}_\lambda(\gamma_\lambda)$. Let us first argue the weak-$*$ continuity of the curve $\lambda \longrightarrow \gamma_\lambda$ of optimal measures. The chain of ineqaulities

$$\begin{aligned} \mathcal{C}_\lambda(\gamma_{\lambda+\delta}) - \delta m(\lambda+\delta) &= \mathcal{C}_{\lambda+\delta}(f,g) \\ &\leq \mathcal{C}_{\lambda+\delta}(\gamma_\lambda) = \mathcal{C}_\lambda(f,g) - \delta m(\lambda) \end{aligned}$$

shows that as $\delta \to 0$, the energy $\mathcal{C}_\lambda(\gamma_{\lambda+\delta})$ converges to its minimum value $\mathcal{C}_\lambda(f,g)$. Since our curve lies in the compact set $\Gamma_\leq(f,g)$ of Lemma 2.2, every sequence $\delta(n) \to 0$ admits a convergent subsequence $\gamma_{\lambda+\delta(n(k))} \to \gamma_\infty$. The lower semi-continuity of the same lemma guarantees γ_∞ is a minimizer, hence $\gamma_\infty = \gamma_\lambda$ by our uniqueness hypothesis. This shows continuity of the curve of measures at $\lambda \in \mathbf{R}$. As remarked at the conclusion of the proof of Lemma 2.2, the mass functional $\Gamma_\leq(f,g) \longrightarrow \gamma[\mathbf{R}^n \times \mathbf{R}^n]$ is weak-$*$ continuous, so we have continuity of $m(\lambda)$ as well.

Formulas (1.7)–(1.8) express the minimal cost as an infimum of non-increasing affine function $\mathcal{C}_\lambda(\gamma)$ of λ, hence $\mathcal{C}_\lambda(f,g)$ is concave non-increasing on $\lambda \in \mathbf{R}$. Writing the difference quotient in two ways,

$$\begin{aligned} -m(\lambda+\delta) &\leq \frac{\mathcal{C}_\lambda(\gamma_{\lambda+\delta}) - \mathcal{C}_\lambda(f,g) - \delta m(\lambda+\delta)}{\delta} \\ &= \frac{\mathcal{C}_{\lambda+\delta}(f,g) - \mathcal{C}_\lambda(f,g)}{\delta} \\ &= \frac{\mathcal{C}_{\lambda+\delta}(f,g) - \mathcal{C}_{\lambda+\delta}(\gamma_\lambda) - \delta m(\lambda)}{\delta} \leq -m(\lambda) \end{aligned}$$

the limit $\delta \to 0$ shows the continuous function $-m(\lambda)$ to be the slope of $\mathcal{C}_\lambda(f,g)$.

If the same slope $m(\lambda) = m(\lambda+\delta)$ is attained for two different Lagrange multipliers $\lambda \neq \lambda + \delta$, this means that the corresponding optimizers γ_λ and $\gamma_{\lambda+\delta}$ have the same mass. From $\mathcal{C}_\lambda(\gamma_\lambda) \leq \mathcal{C}_\lambda(\gamma_{\lambda+\delta})$ and $\mathcal{C}_{\lambda+\delta}(\gamma_{\lambda+\delta}) \leq \mathcal{C}_{\lambda+\delta}(\gamma_\lambda)$ they must also have the same cost: $\int c\, d\gamma_\lambda = \int c\, d\gamma_{\lambda+\delta}$. But then the last two inequalities become equalities, and the hypothesized uniqueness of minimizer for $\mathcal{C}_\lambda(\cdot)$ implies $\gamma_{\lambda+\delta} = \gamma_\lambda$.

If the function $c(\mathbf{x}, \mathbf{y})$ is bounded, then taking λ negative enough ensures $c(\mathbf{x}, \mathbf{y}) - \lambda > 0$, so the infimum (1.8) is attained by $\gamma_\lambda = 0$. Similarly, taking λ positive enough so $c(\mathbf{x}, \mathbf{y}) - \lambda < 0$ ensures $m(\lambda) = \min\{\|f\|_1, \|g\|_1\}$: unless equality holds in $f_1 \leq f$ or $g_1 \leq g$, Corollary 2.4 contradicts $c(\mathbf{x}_0, \mathbf{y}_0) < \lambda$. If $c(\mathbf{x}, \mathbf{y})$ is unbounded, then given any $\epsilon > 0$, taking $R > 0$ sufficiently large ensures both f and g have mass less than ϵ outside the ball $B_R(\mathbf{0}) \subset \mathbf{R}^n$. Taking λ extreme enough we can force $c(\mathbf{x}, \mathbf{y}) - \lambda$ to have the sign of our choice on $B_R(\mathbf{0}) \times B_R(\mathbf{0})$. A positive sign ensures $m(\lambda) < \epsilon$ while a negative sign ensures $m(\lambda) + \epsilon > \min\{\|f\|_1, \|g\|_1\}$. QED.

3. Monotone expansion of active regions

Given distributions $0 \leq f, g \in L^1(\mathbf{R}^n)$ of compact support and a continuous cost function $c(\mathbf{x}, \mathbf{y})$, let γ_λ denote the minimizer of the constrained optimization problem (1.8); clearly γ_λ minimizes transportation costs among all transfer schemes which transport mass $m(\lambda) = \gamma_\lambda[\mathbf{R}^n \times \mathbf{R}^n]$ from f to g. We turn now to showing that the marginals $f_\lambda \leq f$ and $g_\lambda \leq g$ of $\gamma_\lambda \in \Gamma(f_\lambda, g_\lambda)$ depend monotonically on $\lambda \in \mathbf{R}$, or equivalently (by results of the preceding section) on the amount $m = m(\lambda)$ of mass transferred.

It is convenient to address this question for discrete measures $\mu, \nu \geq 0$ on $\mathbf{R}^n$, which approximate the desired distributions in the continuum limit $\mu \to f$ and $\nu \to g$. Given finite sets $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^n$ with cardinality $P = \#(X)$ and $Q = \#(Y)$, let us therefore consider the problem of choosing $M \leq \min\{P, Q\}$ distinct points $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\} \subset X$ and M distinct points $\{\mathbf{y}_1, \ldots, \mathbf{y}_M\} \subset Y$, which minimize the sum

$$\sum_{i=1}^{M} c(\mathbf{x}_i, \mathbf{y}_i)$$

among such choices. Letting $\Gamma^M_\leq(\mu, \nu) \subset \Gamma_\leq(\mu, \nu)$ denote the set of mass $M = \gamma[\mathbf{R}^n \times \mathbf{R}^n]$ joint measures whose left and right marginals are dominated by μ and ν respectively. Given

$$\mu = \sum_{\mathbf{x} \in X} \delta_{\mathbf{x}} \quad \text{and} \quad \nu = \sum_{\mathbf{y} \in Y} \delta_{\mathbf{y}}, \tag{3.1}$$

the problem described above is equivalent to finding an extremal measure $\gamma^M = \sum_{i=1}^M \delta_{(\mathbf{x}_i, \mathbf{y}_i)}$ in $\Gamma^M_\leq(\mu, \nu)$ which minimizes $\mathcal{C}_\lambda(\gamma)$ among such choices. Our first proposition asserts that the marginals of γ^M depend monotonically on M.

PROPOSITION 3.1 (DISCRETE MONOTONICITY OF ACTIVE REGION). *Fix disjoint sets $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^n$ of finite cardinality $P = \#(X)$ and*

$Q = \#(Y)$ and the corresponding discrete measures (3.1). For each integer $M \leq \min\{P, Q\}$, let Γ^M_{ext} denote the set of extremal measures in $\Gamma^M_{\leq}(\mu, \nu)$ which minimize $\mathcal{C}_\lambda(\gamma)$. Fix $\gamma^M \in \Gamma^M_{ext}$ and denote its marginals by μ^M and ν^M. If $M < \min\{P, Q\}$ there exists $\gamma^{M+1} \in \Gamma^{M+1}_{ext}$ with $\mu^{M+1} \geq \mu^M$ and $\nu^{M+1} \geq \nu^M$; similarly, if $M > 0$ there exists $\gamma^{M-1} \in \Gamma^{M-1}_{ext}$ whose marginals $\mu^{M-1} \leq \mu^M$ and $\nu^{M-1} \leq \nu^M$ are dominated by those of γ^M.

Proof: Fix $\gamma^M \in \Gamma^M_{ext}$ and $\gamma^{M+1} \in \Gamma^{M+1}_{ext}$ for $M < \min\{P, Q\}$. Let μ^M and ν^M denote the left and right marginals of γ^M, and $X^M := \operatorname{spt} \mu^M$, $Y^M := \operatorname{spt} \nu^M$ and $J^M := \operatorname{spt} \gamma^M$ their respective supports. Extremality of γ^M in $\Gamma^M_{\leq}(\mu, \nu)$ implies $\gamma^M[(\mathbf{x}, \mathbf{y})] = 1$ for all $(\mathbf{x}, \mathbf{y}) \in J^M$, or equivalently $\gamma^M = \sum_{(\mathbf{x},\mathbf{y}) \in J^M} \delta_{(\mathbf{x},\mathbf{y})}$. Since $\mu[\mathbf{x}] = 1$ for each $\mathbf{x} \in \operatorname{spt} \mu$, we conclude both X^M and Y^M have M points.

By induction on the number of points in $(X^M \setminus X^{M+1}) \cup (Y^M \setminus Y^{M+1})$, we shall show it is possible to construct $\gamma' \in \Gamma^{M+1}_{ext}$ whose marginals dominate those of γ^M, and $\gamma \in \Gamma^M_{ext}$ whose marginals are dominated by those of γ^{M+1}.

If the set named above is empty, we take $\gamma = \gamma^M$ and $\gamma' = \gamma^{M+1}$ and are done. The inductive hypothesis asserts that γ and γ' exist provided $(X^M \setminus X^{M+1}) \cup (Y^M \setminus Y^{M+1})$ has less than j points. Let us therefore assume that $(X^M \setminus X^{M+1}) \cup (Y^M \setminus Y^{M+1})$ has precisely $j > 0$ points.

We define a successor function $\sigma : X^M \cup Y^{M+1} \longrightarrow X^{M+1} \cup Y^M$ by $\sigma(\mathbf{x}) = \mathbf{y}$ if $(\mathbf{x}, \mathbf{y}) \in J^M$ and $\sigma(\mathbf{y}) = \mathbf{x}$ if $(\mathbf{x}, \mathbf{y}) \in J^{M+1}$. This function is well-defined since X and Y are disjoint. It is injective and surjective, since $\sigma^{-1}(\mathbf{y}) := \mathbf{x}$ if $(\mathbf{x}, \mathbf{y}) \in J^M$ and $\sigma^{-1}(\mathbf{x}) := \mathbf{y}$ if $(\mathbf{x}, \mathbf{y}) \in J^{M+1}$ gives a well-defined inverse to σ. Notice that orbits of σ partition $X^M \cup X^{M+1} \cup Y^M \cup Y^{M+1}$ into equivalence classes. Those orbits which are not periodic have length less than $4M + 2$, and can only start in $(X^M \setminus X^{M+1}) \cup (Y^{M+1} \setminus Y^M)$ and end in $(X^{M+1} \setminus X^M) \cup (Y^M \setminus Y^{M+1})$.

Since $j > 0$, the set $(X^M \setminus X^{M+1}) \cup (Y^M \setminus Y^{M+1})$ is non-empty. So σ has at least one orbit which is not periodic and is distinguished by the fact that it either starts in $X^M \setminus X^{M+1}$ or ends in $Y^M \setminus Y^{M+1}$ (or both). Notice the elements of this (and all) orbits of σ alternate between X and Y. We will separate our discussion into two cases depending on whether the distinguished orbit consists of an even or odd number of elements. If the distinguished orbit has $2k+1$ elements, then it starts in $X^M \setminus X^{M+1}$ and ends in $X^{M+1} \setminus X^M$, or else it starts in $Y^{M+1} \setminus Y^M$ and ends in $Y^M \setminus Y^{M+1}$ — a case which can be handled similarly (by symmetry). Assuming the former without loss of generality, the orbit consists of a sequence of points $\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \ldots \mathbf{x}_k, \mathbf{y}_k, \mathbf{x}_{k+1}$, with $(\mathbf{x}_i, \mathbf{y}_i) \in J^M$ for $1 \leq i \leq k$ and $(\mathbf{x}_{i+1}, \mathbf{y}_i) \in J^{M+1}$ for each $1 \leq i \leq k$.

Then $\mathbf{x}_{k+1} \in X^{M+1} \setminus X^M$ is not represented in γ^M, whose optimality implies

$$\sum_{i=1}^{k} c(\mathbf{x}_i, \mathbf{y}_i) \leq \sum_{i=1}^{k} c(\mathbf{x}_{i+1}, \mathbf{y}_i).$$

The reverse inequality follows from optimality of γ^{M+1} since the orbit starts with a point $\mathbf{x}_0 \in X^M \setminus X^{M+1}$ not present in γ^{M+1}. This shows the measures $\gamma^+ \leq \gamma^M$ and $\gamma^- \leq \gamma^{M+1}$ defined by

$$\gamma^+ := \sum_{i=1}^{k} \delta_{(\mathbf{x}_i, \mathbf{y}_i)} \quad \text{and} \quad \gamma^- := \sum_{i=1}^{k} \delta_{(\mathbf{x}_{i+1}, \mathbf{y}_i)}$$

have the same right marginals and the same cost; their left marginals differ by $\delta_{\mathbf{x}_0} - \delta_{\mathbf{x}_{k+1}}$ — the points which appear in X^M and X^{M+1} respectively, but not both. Thus $\gamma := \gamma^M + \gamma^- - \gamma^+ \in \Gamma^M_{ext}$ and $\gamma' := \gamma^{M+1} + \gamma^+ - \gamma^- \in \Gamma^{M+1}_{ext}$. Moreover, $\operatorname{spt}\gamma \setminus J^{M+1}$ has $j-k$ points, so the inductive hypothesis yields an element of Γ^M_{ext} whose marginals are dominated by those of γ^{M+1}. Similarly, $J^M \setminus \operatorname{spt}\gamma'$ has $j-k$ points, so induction again yields an element of Γ^{M+1}_{ext} whose marginals dominate those of γ^M.

We turn now to the case that the distinguished orbit has an even number $2k$ of elements, with $k \geq 1$ as before. In this case the orbit consists of a sequence of points $\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \ldots \mathbf{x}_k, \mathbf{y}_k$ starting with $\mathbf{x}_1 \in X^M \setminus X^{M+1}$ and ending with $\mathbf{y}_k \in Y^M \setminus Y^{M+1}$. Here $(\mathbf{x}_i, \mathbf{y}_i) \in J^M$ for all $1 \leq i \leq k$ and $(\mathbf{x}_{i+1}, \mathbf{y}_i) \in J^{M+1}$ for $1 \leq i < k$, so the orbit includes one fewer couplet from J^M than from J^{M+1}. Since $\#(J^{M+1}) > \#(J^M)$, at least one orbit of σ has more couplets from J^{M+1} than from J^M; it must begin in $Y^{M+1} \setminus Y^M$ and end in $X^{M+1} \setminus X^M$, thus consisting of a sequence of points $\mathbf{y}'_0, \mathbf{x}'_1, \mathbf{y}_{1'}, \ldots, \mathbf{x}'_\ell, \mathbf{y}'_\ell, \mathbf{x}'_{\ell+1}$ of length $2\ell+2$, with $(\mathbf{x}'_i, \mathbf{y}'_i) \in J^M$ if $1 \leq i \leq \ell$, and $(\mathbf{x}'_{i+1}, \mathbf{y}'_i) \in J^{M+1}$ for $0 \leq i \leq \ell$. Here $\ell \geq 0$. Optimality of γ^M implies

$$\sum_{i=1}^{k} c(\mathbf{x}_i, \mathbf{y}_i) + \sum_{i=1}^{\ell} c(\mathbf{x}'_i, \mathbf{y}'_i) \leq \sum_{i=1}^{k-1} c(\mathbf{x}_{i+1}, \mathbf{y}_i) + \sum_{i=0}^{\ell} c(\mathbf{x}'_{i+1}, \mathbf{y}'_i),$$

while the reverse inequality follows from optimality of γ^{M+1}. This shows the measures $\gamma^+ \leq \gamma^M$ and $\gamma^- \leq \gamma^{M+1}$ defined by

$$\gamma^+ := \sum_{i=1}^{k} \delta_{(\mathbf{x}_i, \mathbf{y}_i)} + \sum_{i=1}^{\ell} \delta_{(\mathbf{x}'_i, \mathbf{y}'_i)} \quad \text{and} \quad \gamma^- := \sum_{i=1}^{k-1} \delta_{(\mathbf{x}_{i+1}, \mathbf{y}_i)} + \sum_{i=0}^{\ell} \delta_{(\mathbf{x}'_{i+1}, \mathbf{y}'_i)}$$

again have the same cost; their left marginals differ by $\delta_{\mathbf{x}_1} - \delta_{\mathbf{x}'_{\ell+1}}$ and their right marginals by $\delta_{\mathbf{y}_k} - \delta_{\mathbf{y}'_0}$. Once again we find $\gamma := \gamma^M + \gamma^- - \gamma^+ \in \Gamma^M_{ext}$ and $\gamma' := \gamma^{M+1} + \gamma^+ - \gamma^- \in \Gamma^{M+1}_{ext}$. Moreover, $\operatorname{spt}\gamma \setminus J^{M+1}$ has $j-k-\ell$ points, so the inductive hypothesis yields an element of Γ^M_{ext} whose marginals are dominated by those of γ^{M+1}. Similarly, $J^M \setminus \operatorname{spt}\gamma'$ has $j-k-\ell$ points, so induction again yields an element of Γ_{ext}^{M+1} whose marginals dominate those of γ^M. This establishes the proposition. QED.

To address densities $f, g \in L^1(\mathbf{R}^n)$ with respect to Lebesgue, let $\Gamma_{\leq}^m(f,g) \subset \Gamma_{\leq}(f,g)$ denote the joint measures $\gamma \geq 0$ of total mass $m = \gamma[\mathbf{R}^n \times \mathbf{R}^n]$ whose marginals are dominated by f and g. We shall need to recall two elementary lemmas from functional analysis. Let $C_\infty(\mathbf{R}^n)$ denotes the Banach space of continuous functions which tend to zero at ∞ equipped with the supremum norm. Its dual $C_\infty(\mathbf{R}^n)$ consists of measures with finite total mass, normed by total variation.

LEMMA 3.2 (COMPACTNESS). *Fix a sequence of joint measures $\gamma_k \in \Gamma_{\leq}^{m_k}(\mu_k, \nu_k)$. Suppose the marginal bounds $\mu_k \geq 0$ and $\nu_k \geq 0$ converge weak-$*$ in $C_\infty(\mathbf{R}^n)^*$ to respective limits μ and ν as $k \to \infty$. If $\mu_k[\mathbf{R}^n] \to \mu[\mathbf{R}^n]$ and $\nu_k[\mathbf{R}^n] \to \nu[\mathbf{R}^n]$, then a subsequence of γ_k converges weak-$*$ in $C_\infty(\mathbf{R}^n \times \mathbf{R}^n)^*$ to some limit $\gamma \in \Gamma_{\leq}(\mu, \nu)$. Moreover, the marginals and mass of γ_k converge weak-$*$ to those of γ.*

Proof: Fix sequences γ_k, $\mu_k \to \mu$ and $\nu_k \to \nu$ satisfying the hypotheses of the lemma. Let $\overline{\mathbf{R}}^n$ denote the one point compactification of $\mathbf{R}^n$. Let $\hat{\mu}$ denote the extension of μ to $\overline{\mathbf{R}}^n$ which vanishes on infinity, and $\hat{\gamma}_k$ the extension of γ_k to $\overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n$ which vanishes on both $\{\infty\} \times \overline{\mathbf{R}}^n$ and $\overline{\mathbf{R}}^n \times \{\infty\}$. It follows that $\hat{\mu}_k$ and $\hat{\nu}_k$ are the marginals of $\hat{\gamma}_k$. Since any continuous function $\xi \in C(\overline{\mathbf{R}}^n)$ can be decomposed as a constant plus $\xi - \xi(\infty) \in C_\infty(\mathbf{R}^n)$, we deduce $\hat{\mu}_k \to \hat{\mu}$ and $\hat{\nu}_k \to \hat{\nu}$ from the hypothesized conservation of mass $\mu_k[\mathbf{R}^n] \to \mu[\mathbf{R}^n]$ and $\nu_k[\mathbf{R}^n] \to \nu[\mathbf{R}^n]$.

Choose a uniform bound R for the total variation of the measures μ_k and ν_k (and hence γ_k). The ball of radius R in the dual space $C(\overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n)^*$ is compact by the Banach-Alaoglu theorem. The weak-$*$ topology is metrizable on this ball, so the sequence $\hat{\gamma}_k$ admits a weak-$*$ convergent subsequence. We abandon the original sequence and denote the convergent subsequence by $\hat{\gamma}_k \to \hat{\gamma}$. Since $1 \in C(\overline{\mathbf{R}}^n) \subset C(\overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n)$, the marginals and mass of $\hat{\gamma}_k$ converge to those of $\hat{\gamma}$.

We need to check that $\hat{\gamma}$ assigns no mass to infinity. Let $\pi^i_\#(\hat{\gamma})$ denote the marginals of $\hat{\gamma}$, where $\pi^1(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ and $\pi^2(\mathbf{x}, \mathbf{y}) = \mathbf{y}$. Taking $k \to \infty$ in the hypotheses $\pi^1_\#(\hat{\gamma}_k) \leq \hat{\mu}_k$ yields $\pi^1_\#(\hat{\gamma}) \leq \hat{\mu}$, and $\pi^2_\#(\hat{\gamma}) \leq \hat{\nu}$ similarly. Since $\hat{\mu}[\{\infty\}] = 0 = \hat{\nu}[\{\infty\}]$, we conclude $\hat{\gamma}$ vanishes on $\{\infty\} \times \overline{\mathbf{R}}^n$ and $\overline{\mathbf{R}}^n \times \{\infty\}$.

Weak-$*$ convergence of γ_k to the restriction γ of $\hat{\gamma}$ to $\mathbf{R}^n \times \mathbf{R}^n$ follows from $C_\infty(\mathbf{R}^n \times \mathbf{R}^n) \subset C(\overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n)$, as does weak-$*$ convergence of the marginals $\pi^i_\#(\gamma_k) \to \pi^i(\gamma) = \pi^i(\hat{\gamma})$ from $C_\infty(\mathbf{R}^n) \subset C(\overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n)$. Thus $\gamma \in \Gamma_{\leq}(\mu, \nu)$ as desired, and $\gamma[\mathbf{R}^n \times \mathbf{R}^n] = \hat{\gamma}[\mathbf{R}^n \times \mathbf{R}^n]$ is the limit of the masses of the γ_k. QED.

Lemma 3.3 (Optimality survives limits). *Let $\mu_k, \nu_k \geq 0$ be measures which converge weak-$*$ to $f, g \in L^1(\mathbf{R}^n)$, with $\mu_k[\mathbf{R}^n] \to \|f\|_{L^1}$ and $\nu_k[\mathbf{R}^n] \to \|g\|_{L^1}$ as $k \to \infty$. Take $c_k(\mathbf{x}, \mathbf{y})$ continuous converging uniformly to $c_0 \in C_\infty(\mathbf{R}^n \times \mathbf{R}^n)$, and $\gamma_k \to \gamma_0$ weak-$*$ with $m_k := \gamma_k[\mathbf{R}^n \times \mathbf{R}^n] \leq m_0$. If γ_k minimizes the associated cost function $\mathcal{C}_0^k(\gamma)$ on $\Gamma_{\leq}^{m_k}(\mu_k, \nu_k)$ for each $k \geq 1$, then γ_0 minimizes $\mathcal{C}_0^0(\gamma)$ on $\Gamma_{\leq}^{m_0}(f, g)$.*

Proof: Take $\gamma_k \in \Gamma_{\leq}^{m_k}(\mu_k, \nu_k)$ converging weak-$*$ to γ_0 and $c_k \to c_0 \in C_\infty(\mathbf{R}^n \times \mathbf{R}^n)$ uniformly as hypothesized. Then $m_k \to m_0 := \gamma_0[\mathbf{R}^n \times \mathbf{R}^n]$ by the preceding lemma. For each $\tilde{\gamma} \in \Gamma_{\leq}^{m_0}(f, g)$ we claim $\mathcal{C}_0^0(\tilde{\gamma}) \geq \mathcal{C}_0^0(\gamma_0)$. We assume f and g are non-zero since otherwise there is nothing to prove.

Define $d\mu_0(\mathbf{x}) := f(\mathbf{x})d\mathbf{x}$ and $d\nu_0(\mathbf{y}) := g(\mathbf{y})d\mathbf{y}$ and probability measures $\hat{\mu}_k = \mu_k/\mu_k[\mathbf{R}^n]$ and $\hat{\nu}_k = \nu_k/\nu_k[\mathbf{R}^n]$ for each $k \geq 0$. By Brenier's theorem [12], there exist convex functions ψ^k and $\phi^k : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ such that $\nabla\psi^k_{\#}\hat{\mu}_0 = \hat{\mu}_k$ and $\nabla\phi^k_{\#}\hat{\nu}_0 = \hat{\nu}_k$. Shifting ψ^k by a constant depending on k allows us to extract a subsequence which converges pointwise a.e. to a convex limit $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ finite at some Lebesgue point of f. It follows that $\nabla\psi^k \to \nabla\psi$ a.e. on $\operatorname{dom}\psi$. Since $\nabla\psi^k \to \infty$ outside $\operatorname{dom}\psi$, tightness of the measures $\hat{\mu}_k \to \hat{\mu}_0$ implies $\hat{\mu}_0[\operatorname{dom}\psi] = 1$, and $\nabla\psi_{\#}\hat{\mu}_0 = \hat{\mu}_0$ by Lebesgue's dominated convergence theorem. The convex gradient mappings of Brenier's theorem are unique, so we conclude $\nabla\psi(\mathbf{x}) = \mathbf{x}$ f-a.e. Similarly, $\nabla\phi^k \to \nabla\phi = \mathbf{id}$ g-a.e. for a further subsequence. Given $\tilde{\gamma} \in \Gamma_{\leq}^{m_0}(f, g)$, observe $\tilde{\gamma}_k := (\nabla\psi^k \times \nabla\phi^k)_{\#}(\tilde{\gamma})$ belongs to $\Gamma_{\leq}^{m_0}(\mu_k, \nu_k)$. Optimality of $\gamma_k \in \Gamma_{\leq}^{m_k}(\mu_k, \nu_k)$ implies $\mathcal{C}_0^k(\gamma_k) \leq (m_k/m_0)\mathcal{C}_0^k(\tilde{\gamma}_k)$ since $m_k \leq m_0$. We plan to take the limit $k \to \infty$.

First observe $\tilde{\gamma}_k \to \tilde{\gamma}$ weak-$*$; indeed $\xi(\mathbf{x}, \mathbf{y})$ bounded and continuous implies

$$\begin{aligned}\int_{\mathbf{R}^n \times \mathbf{R}^n} \xi \, d\tilde{\gamma}_k &= \int_{\mathbf{R}^n \times \mathbf{R}^n} \xi(\nabla\psi^k(\mathbf{x}), \nabla\phi^k(\mathbf{y})) \, d\tilde{\gamma}(\mathbf{x}, \mathbf{y}) \\ &\to \int_{\mathbf{R}^n \times \mathbf{R}^n} \xi \, d\tilde{\gamma}\end{aligned}$$

as $k \to \infty$ by the dominated convergence theorem. Taking $\xi = c_0 \in C_\infty(\mathbf{R}^n \times \mathbf{R}^n)$ yields

$$\begin{aligned}\int_{\mathbf{R}^n \times \mathbf{R}^n} c_0 \, d(\gamma_0 - \tilde{\gamma}) &= \lim_{k\to\infty} \int_{\mathbf{R}^n \times \mathbf{R}^n} c_0 \, d(\gamma_k - m_k\tilde{\gamma}_k/m_0) \\ &= \lim_{k\to\infty} \int_{\mathbf{R}^n \times \mathbf{R}^n} c_k \, d(\gamma_k - m_k\tilde{\gamma}_k/m_0) \\ &\leq 0.\end{aligned}$$

Here the uniform convergence $c_k \to c_0$ has been used, and a bound on the masses $m_k \to m_0$. Thus $\mathcal{C}_0^0(\gamma_0) \leq \mathcal{C}_0^0(\tilde{\gamma})$ as desired. QED.

THEOREM 3.4 (MONOTONE EXPANSION OF ACTIVE REGIONS). *Fix $0 \le f, g \in L^1(\mathbf{R}^n)$ compactly supported and a continuous cost $c(\mathbf{x}, \mathbf{y})$ on $\mathbf{R}^n \times \mathbf{R}^n$. Let Γ^m_{opt} denote the minimizers of $\mathcal{C}_0(\gamma)$ among joint measures $\gamma \in \Gamma^m_{\le}(f, g)$ of mass $m \ge 0$. There is a curve $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}] \longrightarrow \gamma^m \in \Gamma^m_{opt}$ along which the left and right marginals of $\gamma^{m+\epsilon}$ dominate those of γ^m whenever $\epsilon > 0$. Moreover, each measure $\gamma \in \Gamma^m_{opt}$ with $\mathcal{C}_0(\gamma) < \infty$ lies on such a curve.*

Proof: Suppose $\tilde{\gamma} \in \Gamma^{\tilde{m}}_{opt}$ has finite cost for some $\tilde{m} \le m_{max}$. We shall construct a curve $\gamma^m \in \Gamma^m_{opt}$ whose marginals increase with $m \in [0, m_{max}]$ and which passes through $\tilde{\gamma}$. Recall that the convex set $\Gamma^m_{\le}(f, g)$ is weak-$*$ compact, as a consequence of Lemma 3.2. It costs no generality to assume $\tilde{\gamma}$ is an exposed point of $\Gamma^m_{\le}(f, g)$: if $\tilde{\gamma}$ is not an exposed point, it can be weak-$*$ approximated by a linear combination of exposed points using the Krein-Milman theorem. The same linear combination of curves through these exposed points will pass arbitrarily close to $\tilde{\gamma}$. Taking a subsequential limit of these curves on rational points in $[0, m_{max}]$, allows the desired curve to be constructed following the procedure below. We do not claim continuity of this curve.

Since $\tilde{\gamma}$ is an exposed point of $\Gamma^m_{\le}(f, g)$, there is a cost function $\tilde{c} \in C_\infty(\mathbf{R}^n \times \mathbf{R}^n)$ tending to zero at infinity whose integral $\tilde{\mathcal{C}_0}(\gamma)$ against $\gamma \in \Gamma^m_{\le}(f, g)$ is uniquely minimized at $\tilde{\gamma}$. Then $\tilde{\gamma}$ also minimizes $(1-t)\mathcal{C}_0 + t\tilde{\mathcal{C}_0}$ on $\Gamma^m_{\le}(f, g)$ uniquely. Suppose for each $t = 1/k$, we can construct a curve γ^m_k minimizing $(1-t)\mathcal{C}_0 + t\tilde{\mathcal{C}_0}$ on $\Gamma^m_{\le}(f, g)$, with marginals depending monotonically on $m \in [0, m_{max}]$ and passing through $\tilde{\gamma}$. Letting $k \to \infty$, a weak-$*$ subsequential limit of these curves at $\tilde{m}$ and the rational points of $[0, m_{max}]$ allows the desired curve γ^m through $\tilde{\gamma}$ to be constructed.

From the foregoing, it costs no generality to establish the theorem assuming $\tilde{\gamma}$ minimizes $\mathcal{C}_0(\gamma)$ uniquely on $\Gamma^m_{\le}(f, g)$. Let $[[\lambda]]$ denote the integer part of any real number $\lambda \in \mathbf{R}$. As in [56, Lemma 7], it is possible to find sequences

$$\mu_k = \frac{1}{2^k} \sum_{i=1}^{[[2^k \|f\|_1]]} \delta_{\mathbf{x}_i^k} \quad \text{and} \quad \nu_k = \frac{1}{2^k} \sum_{j=1}^{[[2^k \|g\|_1]]} \delta_{\mathbf{y}_j^k}$$

of discrete measures which converge sub-sequentially $\mu_k \to f(\mathbf{x})d\mathbf{x}$ and $\nu_k \to g(\mathbf{y})d\mathbf{y}$ in the weak-$*$ sense as $k \to \infty$. By displacing them slightly, we may take all points $\mathbf{x}_i^k$ and $\mathbf{y}_j^k$ to be distinct for each given k. It costs no generality to suppose they are all contained in a bounded set Ω independent of k; also, we can multiply the cost c by a cutoff function outside of $\overline{\Omega} \times \overline{U}$ so it belongs to $C_\infty(\mathbf{R}^n \times \mathbf{R}^n 0$. Using Proposition 3.1, we find a measure $\gamma_k^{M/2^k}$ which minimizes $\mathcal{C}_0(\gamma)$ on $\Gamma^M(\mu_k, \nu_k)$ and whose marginals satisfy $\mu_k^{M/2^k} \ge \mu_k^{(M-1)/2^k}$ and $\nu_k^{M/2^k} \ge \nu_k^{(M-1)/2^k}$ inductively for each integer $M \in [1, 2^k m_{max}]$. Using a diagonal process, we find a subsequence $k \to \infty$ such that the measures γ^m_k converge weak-$*$ to a limit γ^m on each dyadic rational $m = M/2^j$ in $[0, m_{max}]$.

By Lemmas 3.2 and 3.3, the limit measure $\gamma^m \in \Gamma^m_{opt}$ inherits optimality, and its marginals are dominated by those of $\gamma^{m+\epsilon}$ whenever $0 < m < m+\epsilon \leq m_{max}$ are dyadic rationals. If $m \in [0, m_{max}]$ is not a dyadic rational, we use Lemma 3.2 to find a sequence of dyadic rationals $m(i)$ increasing to m, for which the measures $\gamma^{m(i)}$ converge, and define γ^m as their weak-$*$ limit. Again $\gamma^m \in \Gamma^m_{opt}$ by Lemma 3.3. Since the marginals of $\gamma^{m(i)}$ are dominated by those of $\gamma^{m(i)+\epsilon(i)}$ for $\epsilon(i) \geq 0$ dyadic, Lemma 3.2 implies the same is true in the limit $\epsilon(i) \to \epsilon$. The theorem is now complete since the curve γ^m passes through $\Gamma^{\tilde{m}}_{opt} = \{\tilde{\gamma}\}$. QED.

4. Monge-Ampère double obstacle problem

Given $0 \leq f, g \in L^1(\mathbf{R}^n)$ and obstacle functions h and $k : \mathbf{R}^n \longrightarrow \mathbf{R}$, one may ask whether it is possible to find a convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ such that

$$g(\nabla\psi(\mathbf{x})) \det \left[D^2\psi(\mathbf{x})\right] = f(\mathbf{x}) \quad \text{on} \quad U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi(\mathbf{x}) > h(\mathbf{x})\}. \tag{4.1}$$

Without boundary conditions, this problem is severely under-determined. We claim the following auxiliary conditions on its Legendre transform ψ^* (1.11) resolve the degeneracy:

$$\nabla\psi(U_\psi) \subset V_\psi := \{\mathbf{y} \in \mathbf{R}^n \mid \psi^*(\mathbf{y}) > k(\mathbf{y})\} \quad \text{and} \quad \int_{V_\psi} g(\mathbf{y})\, d\mathbf{y} = \int_{U_\psi} f(\mathbf{x})\, d\mathbf{x}. \tag{4.2}$$

As the obstacles are removed, $h, k \to -\infty$, this problem converges to the familiar Monge-Ampère *second boundary value problem.* To see it remains well-determined with obstacles present, we introduce a notion of weak-$*$ solution motivated by Brenier [12].

Definition 4.1 (Weak-$*$ solutions to Monge-Ampère obstacle). *Fix $0 \leq f, g \in L^1(\mathbf{R}^n)$ and Lipschitz obstacles h and $k : \mathbf{R}^n \longrightarrow \mathbf{R}$. A convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ is a* weak-$*$ solution *to (4.1)–(4.2) if its gradient pushes $f\chi_{U_\psi}$ forward to $g\chi_{V_\psi}$. In other words, each Borel test function $\xi : \mathbf{R}^n \longrightarrow \mathbf{R}$ must verify*

$$\int_{\{\mathbf{x} \mid \psi(\mathbf{x}) > h(\mathbf{x})\}} \xi(\nabla\psi(\mathbf{x})) f(\mathbf{x})\, d\mathbf{x} = \int_{\{\mathbf{y} \mid \psi^*(\mathbf{y}) > k(\mathbf{y})\}} \xi(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}. \tag{4.3}$$

Remark 4.2 (Exchange symmetry). *A convex function ψ is a weak-$*$ solution to the Monge-Ampère obstacle problem (4.1)–(4.2) if and only if its Legendre transform ψ^* solves the corresponding problem with data $f \leftrightarrow g$ and obstacles $h \leftrightarrow k$ interchanged: $\nabla\psi_\#(f\chi_{U_\psi}) = g\chi_{V_\psi}$ implies $\nabla\psi^*_\#(g\chi_{V_\psi}) = f\chi_{U_\psi}$ by [56, Remark 16].*

Theorem 4.3 (Obstacle problem via optimal partial transfer). *Fix $0 \le f, g \in L^1(\mathbf{R}^n)$ compactly supported and two superdifferentiable, locally Lipschitz obstacles h and $k : \mathbf{R}^n \longrightarrow \mathbf{R}$, with the property that f vanishes a.e. on $\nabla h^{-1}(\operatorname{spt} g)$ and g vanishes a.e. on $\nabla k^{-1}(\operatorname{spt} f)$. Then (4.1)–(4.2) admits a weak-$*$ solution ψ. Moreover both U_ψ and $\nabla\psi$ are uniquely determined up to sets of f measure zero, and $\gamma_\psi := (\mathbf{id} \times \nabla\psi)_\#(f\chi_{U_\psi})$ uniquely minimizes (1.8) for the cost $c(\mathbf{x}, \mathbf{y}) := h(\mathbf{x}) - \langle\, \mathbf{x}, \mathbf{y}\rangle + k(\mathbf{y})$ with $\lambda = 0$.*

Proof: Define $c(\mathbf{x}, \mathbf{y}) := h(\mathbf{x}) - \langle\, \mathbf{x}, \mathbf{y}\rangle + k(\mathbf{y})$, and notice that the map $\mathbf{y} \in \operatorname{spt} g \longrightarrow \nabla_{\mathbf{x}} c(\mathbf{x}_0, \mathbf{y}) = \nabla h(\mathbf{x}_0) - \mathbf{y}$ is injective and non-vanishing for $\mathbf{x}_0 \in (\operatorname{dom} \nabla h) \setminus \nabla h^{-1}(\operatorname{spt} g)$. It is now easy to check that all hypotheses of Proposition 2.9 are satisfied — including the existence of integrable bounds (2.5). Thus the partial transfer problem (1.8) has a unique solution for $\lambda = 0$. Let us use this to deduce that the weak-$*$ solution ψ to the Monge-Ampère obstacle problem, if it exists, is unique. We claim $\gamma_\psi := (\mathbf{id} \times \nabla\psi)_\#(f\chi_{U_\psi})$ coincides with the minimizer γ from (2.13). Since this minimizer $\gamma \in \Gamma_\le(f, g)$ is unique, and $\gamma \in \Gamma(f_1, g_1)$ for some $f_1 \le f$ and $g_1 \le g$, the above claim implies $f\chi_{U_\psi} = f_1$ and $g\chi_{V_\psi} = g_1$, whence $U_\psi := \{\psi > h\}$ is unique up to a set where f vanishes, and $\nabla\psi$ is the unique convex gradient pushing f_1 forward to g_1 [56].

To see that γ_ψ minimizes (1.8), start with Young's inequality (6.2)

$$\langle\, \mathbf{x}, \mathbf{y}\rangle - h(\mathbf{x}) - k(\mathbf{y}) \le [\psi(\mathbf{x}) - h(\mathbf{x})] + [\psi^*(\mathbf{y}) - k(\mathbf{y})],$$

noting that equality holds when $\mathbf{y} = \nabla\psi(\mathbf{x})$. Setting $-u = [\psi - h]_+$ and $-v = [\psi^* - k]_+$ yields

$$-c(\mathbf{x}, \mathbf{y}) \le -u(\mathbf{x}) - v(\mathbf{y}), \tag{4.4}$$

and we still have equality when $\mathbf{y} = \nabla\psi(\mathbf{x})$ provided $\mathbf{x} \in U_\psi$ and $\mathbf{y} \in V_\psi$. Since V_ψ has full measure for $g\chi_{V_\psi} = \nabla\psi_\#(f\chi_{U_\psi})$, it follows that $\nabla\psi^{-1}(V_\psi)$ has full measure for $f\chi_{U_\psi}$. Thus integrating $\gamma_\psi = (\mathbf{id} \times \nabla\psi)_\#(f\chi_{U_\psi})$ against (4.4) yields

$$\begin{aligned}\int_{\mathbf{R}^n \times \mathbf{R}^n} c(\mathbf{x}, \mathbf{y}) d\gamma_\psi(\mathbf{x}, \mathbf{y}) &= \int_{U_\psi \cap \nabla\psi^{-1}(V_\psi)} c(\mathbf{x}, \nabla\psi(\mathbf{x})) f(\mathbf{x})\, d\mathbf{x} \\ &= \int_{U_\psi} u(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} + \int_{V_\psi} v(\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}.\end{aligned}$$

Since $u(\mathbf{x}) \le 0$ and $v(\mathbf{x}) \le 0$ become equalities outside $U_\psi \times V_\psi$, the last integrals can be extended to all of $\mathbf{R}^n$. Having found a case of equality in (2.13), we have conclude γ_ψ is the desired minimizer.

On the other hand, to demonstrate existence of a weak-$*$ solution to (4.1-4.2), let us begin with a solution $\gamma \in \Gamma(f_1, g_1)$ to the partial transfer problem (1.8) coupled with the maximimizing pair of non-negative functions $(u, v) =$

$(\hat{u} - \hat{u}(\hat{\infty}), \hat{u}_{\hat{c}} + \hat{u}(\hat{\infty}))|_{\mathbf{R}^n \times \mathbf{R}^n}$ from Corollary 2.7. The constraint

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq [h(\mathbf{x}) - u(\mathbf{x})] + [k(\mathbf{y}) - v(\mathbf{y})]$$

is satisfied for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, with equality on $\operatorname{spt} \gamma \subset \partial_{\hat{c}} \hat{u}$. Furthermore, $u(f - f_1) = 0$ and $v(g - g_1) = 0$. Define the convex function $\psi := (k - v)^* \leq h - u$ using the Legendre transform (1.11), and notice that $\psi^* = (k - v)^{**} \leq k - v$. From (6.2)

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &\leq \psi(\mathbf{x}) + \psi^*(\mathbf{y}) \\ &\leq [h(\mathbf{x}) - u(\mathbf{x})] + [k(\mathbf{y}) - v(\mathbf{y})], \end{aligned} \tag{4.5}$$

and equality holds on $(\mathbf{x}, \mathbf{y}) \in \operatorname{spt} \gamma$. Thus $\operatorname{spt} \gamma \subset \partial \psi$, which implies $\nabla \psi_\# f_1 = g_1$ and similarly $\nabla \psi^*_\# g_1 = f_1$ [56, Proposition 10]. It remains to show $f_1 = f \chi_{U_\psi}$ and $g_1 = g \chi_{V_\psi}$ to complete the proof of (4.3). Letting $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ and $\pi'(\mathbf{x}, \mathbf{y}) = \mathbf{y}$, as in (4.5) we have $\psi(\mathbf{x}) \leq h(\mathbf{x}) - u(\mathbf{x})$ with equality on $\pi(\operatorname{spt} \gamma)$ and $\psi^*(\mathbf{y}) \leq k(\mathbf{y}) - v(\mathbf{y})$ with equality on $\pi'(\operatorname{spt} \gamma)$. Thus $\psi \geq h$ holds f_1-a.e., and $\psi^* \geq k$ holds g_1-a.e., since $u, v \leq 0$. Moreover,

$$\begin{aligned} U_\psi &:= \{\psi \;\; > h\} \subset \{\mathbf{x} \in \mathbf{R}^n \mid u(\mathbf{x}) < 0\} \\ V_\psi &:= \{\psi^* > k\} \subset \{\mathbf{y} \in \mathbf{R}^n \mid v(\mathbf{y}) < 0\} \end{aligned}$$

Since $(f - f_1)u = 0$, we conclude $f - f_1 \leq f \chi_{\mathbf{R}^n \setminus U_\psi}$ and $f \chi_{U_\psi} \leq f_1 \leq f \chi_{\{\psi \geq h\}}$. Finally we claim $\{\mathbf{x} \mid \psi(\mathbf{x}) = h(\mathbf{x})\}$ is a set of f_1 measure zero, so that $f \chi_{U_\psi} = f_1$. At any point where $\{\psi = h\}$ has full Lebesgue density, we have $\nabla \psi = \nabla h$ since ψ is subdifferentiable and h is assumed superdifferentiable. But this can only happen on an f_1 negligible set, since $\nabla \psi(\mathbf{x}) \in \operatorname{spt} g_1$ and $\nabla h(\mathbf{x}) \notin \operatorname{spt} g$ elsewhere. A similar argument starting from $g \chi_{V_\psi} \leq g_1 \leq g \chi_{\{\psi^* \geq k\}}$ shows $g \chi_{V_\psi} = g_1$. QED.

Example 4.4 (Square Distance). *Parabolic obstacles* $h(\mathbf{x}) = (|\mathbf{x}|^2 - \lambda)/2$ *and* $k(\mathbf{y}) = (|\mathbf{y}|^2 - \lambda)/2$ *correspond to the quadratic cost* $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2 - \lambda$.

Corollary 4.5 (Quadratic Obstacles). *Fix* $h(\mathbf{x}) = k(\mathbf{x}) = (|\mathbf{x}|^2 - \lambda)/2$ *and* $\lambda > 0$. *If a convex function* $\psi : \mathbf{R}^n \longrightarrow \mathbf{R}$ *satisfies the constraints* $h \leq \psi \leq h + \lambda$, *so will its Legendre transform. Defining non-contact sets* $U_\psi := \{\mathbf{x} \mid h < \psi\}$, $A_\psi := \{\mathbf{x} \mid \psi < h + \lambda\}$ *and* $W_\psi := U_\psi \cap A_\psi$ *then yields* $U_{\psi^*} = A_\psi$ *and* $W_\psi = W_{\psi^*}$. *If* g *vanishes outside* U_ψ *and* f *vanishes outside* A_ψ, *and* $\nabla \psi_\# (f \chi_{W_\psi}) = g \chi_{W \psi}$, *then* ψ *is a weak-$*$ solution to the Monge-Ampère obstacle problem (4.3). Moreover, the hypotheses of Theorem 4.3 imply at least one weak-$*$ solution satisfies all these additional constraints.*

Proof: To begin, assume a convex function ψ satisfies $h \leq \psi \leq h + \lambda$. Since $h^* = h + \lambda$, taking Legendre transforms yields $h + \lambda \geq \psi^* \geq h$. We also claim that the equations $\psi(\mathbf{x}) = h(\mathbf{x}) + \lambda$ and $\psi^*(\mathbf{x}) = h(\mathbf{x})$ have the same solutions $\mathbf{x} \in \mathbf{R}^n$. If either equation holds, then $\mathbf{x} = \nabla\psi(\mathbf{x}) \in \partial\psi(\mathbf{x})$. The case of equality in (6.2) then yields $|\mathbf{x}|^2 = \psi(\mathbf{x}) + \psi^*(\mathbf{x})$. Substituting $\psi(\mathbf{x}) = h(\mathbf{x}) + \lambda$ gives $\psi^*(\mathbf{x}) = h(\mathbf{x})$ and conversely, completing the proof that $A_\psi = U_{\psi^*}$. Now $W_\psi = U_{\psi^*} \cap U_\psi =: W_{\psi^*}$ is clear.

Assume in addition that $\nabla\psi_\#(f\chi_{W_\psi}) = g\chi_{W_\psi}$, while g vanishes outside U_ψ and f vanishes outside A_ψ. Then $f\chi_{U_\psi} = f\chi_{W_\psi}$ and $g\chi_{A_\psi} = g\chi_{W_\psi}$, so ψ verifies our definition $\nabla\psi_\#(f\chi_{U_\psi}) = g\chi_{U_{\psi^*}}$ of weak-$*$ solution to the Monge-Ampère obstacle problem.

To show a solution exists satisfying these extra constraints, let $\tilde\psi$ be the weak-$*$ solution provided by Theorem 4.3; i.e., a convex funtion — lower semi-continuous after modifying its values on a negligle set — whose gradient pushes $f\chi_{U_{\tilde\psi}}$ forward to $g\chi_{U_{\tilde\psi^*}}$. As before, we use the notation $V_\psi := U_{\psi^*}$ for convenience. Thus $\nabla\tilde\psi(\mathbf{x}) \in V_{\tilde\psi} \cap \operatorname{spt} g$ for f-a.e. $\mathbf{x} \in U_\psi$. Defining the convex function $\phi := \max\{\tilde\psi^*, h\}$, we see $\phi(\mathbf{y}) \geq \tilde\psi^*(\mathbf{y})$ with equality on $U_\phi = V_{\tilde\psi}$. Thus $\phi^*(\mathbf{x}) \leq \tilde\psi^{**}(\mathbf{x}) = \tilde\psi(\mathbf{x})$ with equality if $\partial\tilde\psi(\mathbf{x})$ intersects $V_{\tilde\psi}$. The latter inequality implies $U_{\phi^*} \subset U_{\tilde\psi}$, while the cases of equality gives ϕ^* coincident with $\tilde\psi$ f-a.e. on $U_{\tilde\psi}$. Thus $U_{\tilde\psi} \setminus U_{\phi^*}$ has f measure zero, and $\nabla\phi^* = \nabla\tilde\psi$ holds f-a.e. on $U_{\tilde\psi}$. We conclude ϕ^* is a weak-$*$ solution to the same Monge-Ampère obstacle problem as $\tilde\psi$. By Remark 4.2, this is equivalent to asserting that $\phi^{**} = \max\{\tilde\psi^*, h\}$ solves the same Monge-Ampère obstacle problem as $\tilde\psi^*$. The symmetry $f \leftrightarrow g$ then shows that $\psi := \max\{\phi^*, h\}$ solves the same Monge-Ampère obstacle problem as ϕ^*, and hence as $\tilde\psi$. We claim ψ is the desired solution.

Obviously, $h \leq \psi$ from the definition, while $\psi \leq h + \lambda$ follows from $\phi \geq h$ via $\phi^* \leq h^*$. The corollary will be complete if we can prove f-vanishes outside $A_\psi = V_\psi$ and g vanishes outside U_ψ. Notice $\mathbf{x} \notin A_\psi$ implies $\psi(\mathbf{x}) = h(\mathbf{x}) + \lambda$ hence $\nabla\psi(\mathbf{x}) = \nabla h(\mathbf{x}) = \mathbf{x}$. Thus $\nabla\psi$ coincides with the identity map on $\mathbf{R}^n \setminus A_\psi \subset U_\psi$, the inclusion following from $\lambda > 0$. Now $[\mathbf{R}^n \setminus A_\psi] \setminus \operatorname{spt} g$ carries zero mass for $g\chi_{V_\psi} = \nabla\psi_\#(f\chi_{U_\psi})$, hence zero mass for f. We conclude f vanishes throughout $\mathbf{R}^n \setminus A_\psi$, since it vanishes on $\operatorname{spt} g$ by the hypotheses of the theorem.

By symmetry $\nabla\psi^*$ pushes $f\chi_{U_\psi \cap A_\psi}$ forward to $g\chi_{V_\psi}$, and the preceding paragraph, applied to ψ^* instead of ψ, shows g vanishes outside $U_\psi = A_{\psi^*}$. Thus $\nabla\psi_\#(f\chi_{W\psi}) = g\chi_{A_\psi} = g\chi_{W_\psi}$, concluding the proof of the corollary. QED.

Remark 4.6 (Points outside active region are fixed). *Note $h \leq \psi \leq h + \lambda$, or equivalently $\psi(\mathbf{x}) - |\mathbf{x}|^2/2 \in [-\lambda/2, \lambda/2]$, forces the convex gradient $\nabla\psi(\mathbf{x}) = \mathbf{x}$ to coincide with the identity map for a.e. $\mathbf{x}$ in the closed contact set $\mathbf{R}^n \setminus W_\psi$. Thus $\nabla\psi_\#(f + g(1 - \chi_{W_\psi})) = f(1 - \chi_{W_\psi}) + g$.*

5. Semiconcavity of free boundary

Let us return now to the constrained optimization problem of transporting a fixed fraction $m \leq \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$ of the total available mass, choosing the locations transported and supplied so as to minimize the special cost $c(\mathbf{x}, \mathbf{y}) := |\mathbf{x} - \mathbf{y}|^2/2$. In the preceding §4 this was demonstrated to be equivalent to solving a Monge-Ampère obstacle problem (4.3) with obstacle $h(\mathbf{x}) = k(\mathbf{x}) := (|\mathbf{x}|^2 - \lambda)/2$ for a suitable Lagrange multiplier $\lambda \geq 0$. The solution was unique if f vanishes a.e. on $\operatorname{spt} g$, and takes the form of a convex function ψ sandwiched between the parabolas $h \leq \psi \leq h + \lambda$, whose gradient on the active region $W_\psi := \{h < \psi < h + \lambda\}$ pushes $f\chi_{W_\psi}$ forward to $g\chi_{W_\psi}$. We wish to investigate smoothness of the free boundary of the active region, and of the map $\nabla\psi$. Our first step is to show that ∂W_ψ carries none of the mass of either f or of g, under the simplifying assumption that a hyperplane separates $\operatorname{spt} f$ from $\operatorname{spt} g$. Negligibility of the boundary plays a techical role in our subsequent arguments for differentiability of ψ and ∂W_ψ, and can be summarized philosophically by stating that almost every source and sink must be either unambiguously active or unambiguously inactive, in the sense that it has a whole neighbourhood belonging to one of the open sets W_ψ or $\mathbf{R}^n \setminus \overline{W_\psi}$. Since an interior ball condition was derived in Corollary 2.4–2.5, our starting point will be a familiar lemma about unions of spheres — or equivalently, suprema of half-spherical graphs.

Lemma 5.1 (Semiconvex suprema of hemispherical graphs). *Fix $0 < \delta < R_0$. Extend the hemispherical cap $h_R(\mathbf{X}) = \sqrt{R^2 - |\mathbf{X}|^2}$ to $\mathbf{R}^{n-1}$ by setting $h_R(\mathbf{X}) = -\infty$ if $|\mathbf{X}| \geq R$. If*

$$u(\mathbf{X}) = \sup_{(\mathbf{Y},\lambda,R)\in\mathcal{A}} [h_R(\mathbf{X} - \mathbf{Y}) - \lambda]_+, \tag{5.1}$$

where $\mathcal{A} \subset \mathbf{R}^{n-1} \times [\delta, \infty] \times [0, R_0]$ and $[\lambda]_+ := \max\{\lambda, 0\}$, then $u(\mathbf{X}) + \delta^{-3}R_0^2|\mathbf{X}|^2/2$ is convex on $\mathbf{R}^{n-1}$.

Proof: Define

$$h_R^\delta(r) := \begin{cases} \sqrt{R^2 - r^2} & \text{if } r \leq \sqrt{R^2 - \delta^2} \\ \dfrac{R^2 - r\sqrt{R^2 - \delta^2}}{\delta} & \text{if } r \geq \sqrt{R^2 - \delta^2} \end{cases}$$

and notice that $h_R^\delta(r) + \delta^{-3}R^2r^2/2$ is convex on $r \in \mathbf{R}$, and $h_R^\delta(|\mathbf{X}|) \geq h_R(\mathbf{X})$ with equality if $h_R^\delta(|\mathbf{X}|) \geq \delta$. For $\lambda \geq \delta$ we have $[h_R(\mathbf{X} - \mathbf{Y}) - \lambda]_+ = [h_R^\delta(|\mathbf{X} - \mathbf{Y}|) - \lambda]_+$. From (5.1),

$$u(\mathbf{X}) + \frac{R_0^2|\mathbf{X}|^2}{2\delta^3} = \sup_{(\mathbf{Y},\lambda,R)\in\mathcal{A}} \max\left\{\frac{R_0^2|\mathbf{X}|^2}{2\delta^3}, \frac{R_0^2|\mathbf{X}|^2}{2\delta^3} + h_R^\delta(|\mathbf{X} - \mathbf{Y}|) - \lambda\right\}$$

expresses $u(\mathbf{X}) + \delta^{-3}R_0^2|\mathbf{X}|^2/2$ as a supremum of convex functions. QED.

In the sequel, we denote the distance between two subsets $U, V \subset \mathbf{R}^n$ by

$$\operatorname{dist}(U,V) := \inf_{\mathbf{x}\in U, \mathbf{y}\in V} |\mathbf{x}-\mathbf{y}|.$$

Proposition 5.2 (Semiconcavity of the active domain). *Fix $0 \le f, g \in L^1(\mathbf{R}^n)$ compactly supported, with $\operatorname{spt} f$ in the upper halfspace $H = \mathbf{R}^{n-1}_+ := \{(\mathbf{X}, x_n) \in \mathbf{R}^n \mid x_n > 0\}$ and $\operatorname{spt} g$ in the lower halfspace $\mathbf{R}^n \setminus \overline{H}$. Let ψ be a convex function satisfying the constraints $h \le \psi \le h + \lambda$, where $h(\mathbf{x}) = (|\mathbf{x}|^2 - \lambda)/2$. If $\nabla\psi_\#(f\chi_{W_\psi}) = g\chi_{W_\psi}$ on the noncontact set $W_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid 0 < \psi - h < \lambda\}$, there is a function $u : \mathbf{R}^{n-1} \longrightarrow [0, \infty)$ such that the domain $U_+ := \{(\mathbf{X}, x_n) \in \mathbf{R}^n \mid x_n < u(\mathbf{X})\}$ differs from W_ψ by a set of f measure zero. Moreover, $u(\mathbf{X}) + \delta^{-3}R^2|\mathbf{X}|^2/2$ is convex on $\mathbf{R}^{n-1}$, if $|\mathbf{x}-\mathbf{y}| \le R$ for all $(\mathbf{x}, \mathbf{y}) \in \operatorname{spt} f \times \operatorname{spt} g$ and $\delta = \operatorname{dist}(\operatorname{spt} g, \partial H)$.*

Proof: By Theorem 4.3 and its corollary, $\gamma_\lambda := (\mathbf{id} \times \nabla\psi)_\#(f\chi_{W_\psi})$ is the unique minimizer of (1.8). Define the set U by (2.9). Then U is a union of balls of radius at most R, centered at points $\mathbf{y} = (\mathbf{Y}, \lambda) \in \operatorname{spt} g$ at least distance $\lambda \le -\delta$ into the lower halfspace. Therefore, $U \cup [\mathbf{R}^n \setminus \overline{H}]$ coincides with a domain $U_+ := \{(\mathbf{X}, x_n) \in \mathbf{R}^n \mid x_n < u(\mathbf{X})\}$, whose boundary is the graph of a function $u(\mathbf{X})$ of the form (5.1). Lemma 5.1 asserts convexity of $u(\mathbf{X}) + \delta^{-3}R^2|\mathbf{X}|^2/2$.

It remains to show that U_+ differs from W_ψ by a set of f measure zero. From its definition, $\overline{U}$ contains the projection $\pi(\operatorname{spt}\gamma_\lambda)$ under $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ of $\operatorname{spt}\gamma_\lambda$. Thus $\overline{U_+}$ contains $\operatorname{spt} f\chi_{W_\psi}$. Since we have just shown the boundary of U_+ to be a semiconvex graph, it has Hausdorff dimension $n-1$ and is negligible with respect to f. Thus U_+ contains the full mass of $f\chi_{W_\psi}$. On the other hand, Corollary 2.4 asserts that U carries zero mass for $f(1-\chi_{W_\psi})$. The same is true for U_+, since $\operatorname{spt} f$ lies in the upper halfspace. Thus, apart from an f neglible set, $U_+ = W_\psi$ as desired. QED.

6. Partial transfer regularity and free boundary

Let us now return to our analysis of transporting that portion $m \le \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$ of the total available mass which minimizes the special cost $c(\mathbf{x}, \mathbf{y}) := |\mathbf{x}-\mathbf{y}|^2/2$, we recall its equivalence to the Monge-Ampère obstacle problem (4.3) with obstacles $h(\mathbf{x}) = k(\mathbf{x}) := (|\mathbf{x}|^2 - \lambda)/2$ for a suitable Lagrange multiplier $\lambda > 0$. The solution was essentially unique if f vanishes a.e. on $\operatorname{spt} g$, and for some $\lambda \ge 0$ takes the form of a convex function ψ sandwiched between the parabolas $h \le \psi \le h + \lambda$, whose gradient on the active region $W_\psi := \{h < \psi < h + \lambda\}$ pushes $f\chi_{W_\psi}$ forward to $g\chi_{W_\psi}$. In Section §5 we saw that when a hyperplane separates $\operatorname{spt} f$ from $\operatorname{spt} g$, then W_ψ is semiconcave. Our goal for the next two sections will be to prove Hölder differentiability of

ψ and ψ^* on W_ψ, and of the free boundary ∂W_ψ itself. Differentiability is deduced in the present section relying on [15]; Hölder estimates are postponed to the next section. As a byproduct, we improve our uniqueness result to assert that ψ is uniquely determined, and not merely up to additive constants or almost everywhere; compare Corollary 6.4 to Theorem 4.3. We also deduce that free boundary always maps to fixed boundary, and path-connectedness of the active regions. Of course, all these results require additional convexity and boundedness assumptions concerning the geometry of mass distributions $0 \le f, g \in L^1(\mathbf{R}^n)$:

Definition 6.1 (Data of convex support). *The trio (f, g, h) of functions on $\mathbf{R}^n$ consitute* data of convex support *if $h(\mathbf{x}) = k(\mathbf{x}) := (|\mathbf{x}|^2 - \lambda)/2$ for some $\lambda \in \mathbf{R}$, and there exist bounded, strictly convex domains $\Omega \subset \mathbf{R}^n$ and $\Lambda \subset \mathbf{R}^n$ separated from each other by positive distance, such that $0 \le f = f\chi_\Omega$ and $0 \le g = g\chi_\Lambda$ with $\log f$ and $\log g$ bounded away from $\pm\infty$ on the respective domains Ω and Λ.*

Even with such assumptions, we cannot expect the free regions $W_\psi \cap \Omega$ and $W_{\psi^*} \cap \Lambda$ to be convex. However, since $\nabla\psi(\mathbf{x}) = \mathbf{x}$ coincides with the identity outside W_ψ, it was noted at Remark 4.6 that $\nabla\psi_\#(f + g(1 - \chi_{W_\psi})) = f(1 - \chi_{W_\psi}) + g$. The target measure $g = g\chi_\Lambda$ is assumed to be bounded above and below on the convex set Λ, so we shall presently be able to invoke Caffarelli's interior regularity theory [15] to deduce local regularity on the relevant domain for the complete transfer problem $\nabla\psi_\#(f\chi_{W_\psi} + g(1 - \chi_{W_\psi})) = g$. Recall the results of that theory:

Theorem 6.2 (Map to a convex target is locally smooth [15]). *Fix $f = f\chi_\Omega$ and $g = g\chi_\Lambda$ nonnegative, where $\Omega \subset \mathbf{R}^n$ and $\Lambda \subset \mathbf{R}^n$ are open and Λ is convex. Here $|\log f|$ and $|\log g|$ are assumed to be bounded on Ω and Λ respectively, and to satisfy (1.1). If $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ is convex with gradient pushing f forward to g, then ψ is $C^{1,\alpha}_{loc}(\Omega)$ smooth and strictly convex on Ω [15]. If f and g are Hölder continuous where positive, Caffarelli goes on to assert $\psi \in C^{2,\alpha}_{loc}(\Omega)$ for some $\alpha > 0$. Smoother f and g imply further regularity of u via standard elliptic theory [44].*

In the next theorem we improve this result slightly, by using strict convexity of the domains to deduce continuity of the map up to the boundary, including the free boundary in the partial mass transfer problem, or equivalently the Monge-Ampère double obstacle problem. We derive differentiability of the free boundary at the same time. As corollaries to the proof, we obtain a strengthened uniqueness result, and observe that the free boundary of W_ψ in Ω is a C^1 hypersurface, along which the transportation map displaces only in the perpendicular direction.

Let us first recall the subdifferential of a convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ defined by

$$\partial\psi := \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^n \times \mathbf{R}^n \mid \psi(\mathbf{z}) \geq \psi(\mathbf{x}) + \langle\, \mathbf{y},\, \mathbf{z} - \mathbf{x}\rangle \text{ for all } \mathbf{z} \in \mathbf{R}^n\}.$$

It consists of the (point,slope) pairs which parameterize supporting hyperplanes to graph(ψ). For $\mathbf{x} \in X \subset \mathbf{R}^n$, we also write $\partial\psi(\mathbf{x}) := \{\mathbf{y} \in \mathbf{R}^n \mid (\mathbf{x}, \mathbf{y}) \in \partial\psi\}$ and $\partial\psi(X) := \cup_{\mathbf{x}\in X}\partial\psi(\mathbf{x})$; thus $\partial\psi(\mathbf{x}) = \{\nabla\psi(\mathbf{x})\}$ at a point $\mathbf{x} \in \operatorname{dom}\nabla\psi$ of differentiability. Any pair of points $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \partial\psi$ satisfy the monotonicity condition

$$\langle\, \mathbf{x} - \mathbf{x}',\, \mathbf{y} - \mathbf{y}'\rangle \geq 0, \tag{6.1}$$

a relation which we shall often have use for. An important connection between ψ and its Legendre transform ψ^* (1.11) is given by Young's inequality

$$\psi(\mathbf{x}) + \psi^*(\mathbf{y}) \geq \langle\, \mathbf{x},\, \mathbf{y}\rangle, \tag{6.2}$$

in which equality holds if and only if $(\mathbf{x}, \mathbf{y}) \in \partial\psi$. When ψ is lower semicontinuous as well as convex, then $\psi^{**} = \psi$, and $(\mathbf{x}, \mathbf{y}) \in \partial\psi$ is equivalent to $(\mathbf{y}, \mathbf{x}) \in \partial\psi^*$. We use the notation int $[X]$ and $\overline{X}$ to denote the interior and closure, respectively, of a set $X \subset \mathbf{R}^n$.

Theorem 6.3 (Optimal homeomorphism between active regions). *Fix data of convex support $(f, g, h = k)$, and a weak-$*$ solution ψ to the obstacle problem (4.3). If $U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi > h\}$, then some $\tilde\psi \in C^1(\mathbf{R}^n) \cap C^{1,\alpha}_{loc}(\Omega \cap U_\psi)$ agrees with ψ on $\Omega \cap U_\psi$ and with h^* on $\Lambda \backslash U_{\psi^*}$, and $\nabla\tilde\psi(\mathbf{R}^n) = \overline{\Lambda}$. Furthermore, $\nabla\tilde\psi : \overline{\Omega \cap U_\psi} \longrightarrow \overline{\Lambda \cap U_{\psi^*}}$ is a homeomorphism, where $\Omega := \operatorname{int}[\operatorname{spt} f]$ and $\Lambda := \operatorname{int}[\operatorname{spt} g]$ are the convex domains of f and g. Hölder continuity of f on Ω and g on Λ imply $\psi \in C^{2,\alpha}_{loc}(\Omega \cap U_\psi)$.*

Proof: The theorem is proved in two parts. First we establish the conclusion for a particular weak-$*$ solution ψ to the double obstacle problem. Then in Corollary 6.4, we deduce that no other solutions exist, concluding the theorem as stated. Let us begin with an elementary observation:

Claim #1: Given three domains $\Omega, U, W \subset \mathbf{R}^n$, if $\Omega \cap \partial U$ is a locally Lipschitz submanifold and Ω intersects $U \Delta W := (U \setminus W) \cup (W \setminus U)$ in a set of zero volume, then $W \cap \Omega \subset U$.

Proof of claim: Every ball $B_\epsilon(\mathbf{x}) \subset W \cap \Omega$ intersects U in a set of full volume, so $\mathbf{x} \in \overline{U}$. However, $\mathbf{x} \notin \partial U$ since the Lipschitz boundary of U would divide the ball into two subsets of positive volume one of which lies outside of U. This forces $\mathbf{x} \in U$. *End of claim.*

Now let Ω and $\Lambda \subset \mathbf{R}^n$ denote the bounded strictly convex domains whose closures form disjoint sets spt f and spt g, and note $f(\mathbf{x})\ d\mathbf{x}$ and $g(\mathbf{y})\ d\mathbf{y}$ are mutually absolutely continuous with respect to Lebesgue on the domains Ω and Λ respectively. Use invariance of the problem under rigid motions to choose coordinates so that $\overline{\Omega}$ and $\overline{\Lambda}$ are strictly separated by the boundary of the upper half-space $H = \mathbf{R}^n_+ := \{(\mathbf{X}, x_n) \in \mathbf{R}^n \mid x_n > 0\}$. Corollary 4.5 provides a weak-$*$ solution to the Monge-Ampère obstacle problem (4.3) satisfying the following additional constraints: $h \le \psi \le h + \lambda$ and $f(1 - \chi_{U_{\psi^*}}) = 0 = g(1 - \chi_{U_\psi})$; i.e. a convex funtion ψ whose gradient pushes $f\chi_{W_\psi}$ forward to $g\chi_{W_\psi}$, where $W_\psi := U_\psi \cap U_{\psi^*}$. Remark 4.2 shows the Legendre transform ψ^* of ψ satsifies $\nabla\psi^*_\#(g\chi_{W_\psi}) = f\chi_{W_\psi}$. Notice $\psi^{**} = \psi$ since ψ is convex and continuous. Proposition 5.2 provides a domain $U := \{(\mathbf{X}, x_n) \mid x_n < u(\mathbf{X})\}$ which differs from U_ψ by a set of f measure zero; (the difference is also g negligible, since $U \supset \mathbf{R}^n \setminus \overline{H}$ contains $\operatorname{spt} g$). Moreover, semiconvexity of u guarantees ∂U is a Lipschitz graph over ∂H and hence has measure zero. Since f is mutually continuous with respect to Lebesgue on Ω, we conclude $U_\psi \cap \Omega \subset U \cap \Omega$ from Claim #1. Similarly, there exists a semiconvex function $v : \partial H \longrightarrow \mathbf{R}$ such that $V := \{(\mathbf{X}, x_n) \mid x_n > v(\mathbf{X})\}$ differs from U_{ψ^*} by a set of $f + g$ measure zero, with ∂V having zero volume and $U_{\psi^*} \cap \Lambda \subset V \cap \Lambda$.

By Remark 4.6, we have $h \le \psi^* \le h + \lambda$ with $\nabla\psi^*(\mathbf{y}) = \mathbf{y}$ a.e. outside $W_{\psi^*} = W_\psi$, so $\nabla\psi^*_\#(f(1-\chi_U)+g\chi_V) = f$ also. This represents transportation by convex gradient to a density $f = f\chi_\Omega$ bounded away from zero and infinity on the (strictly) convex domain Ω. The source measure $f(1 - \chi_U) + g\chi_V$ has density bounded away from zero and infinity on the domain $(V \cap \Lambda) \cup (\Omega \setminus \overline{U})$, and this domain has full mass since ∂U is a set of zero volume. The interior results of Caffarelli [15, our Theorem 6.2] then assert that ψ^* is $C^{1,\alpha}_{loc}$ smooth and strictly convex on $V \cap \Lambda$, with $\psi^* \in C^{2,\alpha}_{loc}(\Lambda \cap V)$ if f and g are Hölder continuous on Ω and on Λ. Since $\psi^*(\mathbf{y}) = (|\mathbf{y}|^2 - \lambda)/2$ outside U^*_ψ, strict convexity of ψ^* extends to the full domain Λ. We claim

$$\tilde{\psi} := \phi^* \text{ where } \phi(\mathbf{y}) = \begin{cases} \psi^*(\mathbf{y}) & \text{if } \mathbf{y} \in \overline{\Lambda} \\ +\infty & \text{otherwise,} \end{cases} \tag{6.3}$$

is the desired extension of ψ. Notice $\mathbf{y} \in \partial\phi^*(\mathbf{x})$ implies $\mathbf{y} \in \overline{\Lambda}$ from the equality cases in Young's inequality (6.2). Thus ϕ^* is globally Lipschitz, since $\partial\phi^*(\mathbf{R}^n) \subset \overline{\Lambda}$ and Λ is bounded. We assert $\phi^* \in C^1(\mathbf{R}^n)$. If not, there must be a point $\mathbf{x} \notin \operatorname{dom}\nabla\phi^*$. Then strict convexity of ϕ fails on a segment in $\partial\phi^*(\mathbf{x})$ passing through the interior of the strictly convex domain Λ. The contradicts the strict convexity asserted above, establishing continuous differentiability globally $\phi^* \in C^1(\mathbf{R}^n)$.

It remains to show that $\psi = \phi^*$ throughout Ω. Notice $\phi(\mathbf{y}) \ge \psi^*(\mathbf{y})$ with equality on $\overline{\Lambda}$. Thus $\phi^*(\mathbf{x}) \le \psi(\mathbf{x})$ with equality if $\partial\psi(\mathbf{x})$ intersects $\overline{\Lambda}$, again from the cases of equality in (6.2). From $\nabla\psi_\#(f\chi_U + g(1-\chi_V)) = g$ in Remark 4.6 we have $\nabla\psi(\mathbf{x}) \in \overline{\Lambda}$ a.e. on $(U \cap \Omega) \cup (\Lambda \setminus \overline{V})$, hence everywhere since both convex functions are continuous. Note $\psi = h + \lambda$ on $\mathbf{R}^n \setminus \overline{V}$ from the same remark. We also conclude $\nabla\psi : U \cap \Omega \longrightarrow \overline{\Lambda}$ extends to a continuous map from $\overline{U \cap \Omega}$ into $\overline{V \cap \Lambda}$; it cannot take values outside $\overline{V}$ since f doesn't vanish on U and $\nabla\psi_{\#}(f\chi_U) = g\chi_V$ is supported on $\overline{V}$.

Claim #2: The free boundary $\partial U_\psi \cap \Omega$ is contained in the graph of a C^1 function over ∂H. The direction $\nabla\tilde{\psi}(\mathbf{x}) - \mathbf{x}$ transported gives the inward normal to U_ψ at any point $\mathbf{x} \in \Omega \cap \partial U_\psi$.

Proof of claim: Notice the C^1 function $h - \tilde{\psi}$ increases with x_n in the upper half space. Indeed $\partial(h - \tilde{\psi})/\partial x_n > 0$ throughout $\overline{H}$, since $\nabla h(\mathbf{x}) = \mathbf{x} \in \overline{H}$ and $\nabla\tilde{\psi}(\mathbf{x}) \in \overline{\Lambda}$ lie on opposite sides of the hyperplane $x_n = 0$. Thus the zero set $Z_+ := \{\mathbf{x} \in H \mid h(\mathbf{x}) = \tilde{\psi}(\mathbf{x})\}$ is contained in the graph of a C^1 function $x_n = \tilde{u}(\mathbf{X})$ over ∂H, by the implicit function theorem. Moreover, $Z_+ \cap \Omega = \Omega \cap \partial U_\psi$, since $\psi \geq \tilde{\psi}$ becomes an equality on the closure of $U_\psi \cap \Omega \subset U \cap \Omega$. This proves the first part of the claim. Since $\tilde{\psi} - h$ vanishes along the free boundary $\mathbf{x} \in \Omega \cap \partial U_\psi$, its gradient $\nabla\tilde{\psi}(\mathbf{x}) - \mathbf{x}$ must be orthogonal to ∂U_ψ; this gradient is non-vanishing as noted and must be directed towards U_ψ since $\tilde{\psi} - h > 0$ inside $U_\psi \cap \Omega$. *End of claim.*

Now that $\Omega \cap \partial U_\psi$ is C^1 smooth, Claim #1 yields $U_\psi \cap \Omega = U \cap \Omega$. Since Ω is open, we may henceforth write $\Omega \cap \partial U$ and $\Omega \cap \partial U_\psi$ interchangeably. To see $U \cap \Omega = \operatorname{int}[\overline{U \cap \Omega}]$ as in Remark 6.6, observe $\overline{U \cap \Omega} \subset \{(\mathbf{X}, x_n) \in \overline{\Omega} \mid x_n \leq u(\mathbf{X})\}$, whence

$$\operatorname{int}[\overline{U \cap \Omega}] \subset \{(\mathbf{X}, x_n) \in \Omega \mid x_n < u(\mathbf{X})\} = U \cap \Omega.$$

The reverse inclusion is obvious.

On the other hand, Remark 4.6 also asserts that the gradient of ψ pushes $f\chi_U + g(1 - \chi_V)$ forward to a measure $g\chi_V$ with convex support. Applying the same arguments again, Caffarelli asserts $\psi \in C^{1,\alpha}_{loc}(\Omega \cap U_\psi)$ is strictly convex, with $\psi \in C^{2,\alpha}_{loc}(\Omega \cap U_\psi)$ if the data are Hölder. The exchange symmetry $f \leftrightarrow g$ then shows $\partial V \cap \Lambda$ is a continuously differentiable graph over ∂H and $V \cap \Lambda = U_{\psi^*} \cap \Lambda$. Likewise $\nabla\psi^* : V \cap \Lambda \longrightarrow \overline{U \cap \Omega}$ extends to a continuous map from $\overline{V \cap \Lambda}$ into $\overline{U \cap \Omega}$. Finally, $\nabla\psi^*$ is one-to-one by strict convexity, hence invariance of domain [63] shows $\nabla\psi^*(V \cap \Lambda)$ is an open subset of $\overline{U}$ and of $\overline{\Omega}$. Since Ω is convex and U is semiconcave, we conclude $\nabla\psi^*(V \cap \Lambda) \subset U \cap \Omega$. By exchange symmetry, it must also be true that $\nabla\psi(U \cap \Omega) \subset V \cap \Lambda$.

Now, for each $\mathbf{x} \in U \cap \Omega$ and $\mathbf{y} \in V \cap \Lambda$ the equality conditions in Young's inequality (6.2) imply $\nabla\psi^*(\nabla\psi(\mathbf{x})) = \mathbf{x}$ and $\nabla\psi(\nabla\psi^*(\mathbf{y})) = \mathbf{y}$. Thus $\nabla\psi : U \cap \Omega \longrightarrow V \cap \Lambda$ is a homeomorphism with inverse map $\nabla\psi^*$. Since both maps extend continuously to the boundary, their extensions give a homeomorphism $\nabla\tilde{\psi}$ between $\overline{U \cap \Omega}$ and $\overline{V \cap \Lambda}$. Recalling $\partial\tilde{\psi}(\mathbf{R}^n) \subset \overline{\Lambda}$ with $\tilde{\psi} = h + \lambda = h^*$ on $\Lambda \setminus \overline{V}$ (hence on $\Lambda \setminus U_{\psi^*}$) , $\nabla h^*(\mathbf{y}) = \mathbf{y}$ makes it clear that $\nabla\tilde{\psi}(\mathbf{R}^n) = \overline{\Lambda}$.

The conclusions of the proposition have now been established for the special weak-$*$ solution provided by Corollary 4.5. We show they extend to all weak-$*$ solutions in the following corollary, which will be derived from the results proved so far. QED.

Corollary 6.4 (Uniqueness, not just a.e.). *Given data* $(f, g, h = k)$ *of convex support, any two weak-$*$ solutions* ψ *and* ϕ *to the obstacle problem (4.3) satisfy* $\Omega \cap U_\psi = \Omega \cap U_\phi$*, where* $U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi(\mathbf{x}) > h(\mathbf{x})\}$ *and* $\Omega = \mathrm{int}\,[\mathrm{spt}\, f]$*. Moreover,* $\psi - \phi$ *is constant on* $\Omega \cap U_\psi$*, and the constant vanishes unless transportation is complete (1.1) and both obstacles fail to bind:* $\Omega \subset U_\psi$ *and* $\Lambda := \mathrm{int}\,[\mathrm{spt}\, g] \subset U_{\psi^*}$*.*

Proof: Let ψ be the weak-$*$ solution for which the conclusions of the proposition were derived above, and ϕ any other solution to the Monge-Ampère obstacle problem $\nabla\phi_{\#}(f\chi_{U_\phi}) = g\chi_{U_{\phi^*}}$. The uniqueness assertion of Theorem 4.3 claims $\nabla\phi$ differs from $\nabla\psi$ and U_ϕ differs from U_ψ only on a set of f measure zero. Since f is mutually continuous with respect to Lebesgue measure on Ω, and $\Omega \cap \partial U_\psi$ is C^1, we conclude $\Omega \cap U_\phi \subset U_\psi$ from claims #1–2 of the preceding proof. On each connected component U_i of $\Omega \cap U_\psi$, we have $\psi - \phi$ constant since their gradients agree a.e. If $\Omega \cap \partial U_i$ is non-empty for every connected component, then $\psi(\mathbf{x}) = h(\mathbf{x})$ at some $\mathbf{x} \in \Omega \cap \partial U_i$. Now, every neighbourhood of $\mathbf{x}$ intersects U_ϕ, so $\phi(\mathbf{x}) \geq h(\mathbf{x})$. But strict equality would force $\mathbf{x} \in \Omega \cap U_\phi \subset U_\psi$ contradicting $\mathbf{x} \in \partial U_i$. Since $\psi - \phi$ is constant on one side of the C^1 curve ∂U_i and vanishes at $\mathbf{x}$, continuity implies the constant is zero. (If $\mathbf{x} \not\in \mathrm{int}\,[\mathrm{dom}\,\phi]$, then $\phi = +\infty$ in a half space on the other side of the curve ∂U_i, forcing U_ψ to approach $\mathbf{x}$ from both sides of ∂U_i and contradicting Proposition 5.2.) Thus $\phi = \psi$ on $\Omega \cap U_\psi = \cup_i U_i$ and hence on the subset $\Omega \cap U_\phi$. The definition of U_ψ then forces $\Omega \cap U_\phi = \Omega \cap U_\psi$.

It remains to consider the possibility $\Omega \cap \partial U_i$ empty for some i. In that case U_i contains the connected set Ω, since its construction ensures U_i is not disjoint from Ω unless $\Omega \cap U_\psi$ is empty, in which case the corollary is trivial: $\psi = \phi = h$ on Ω. Therefore, assume $\psi - \phi$ differ by a constant on $\Omega \subset U_i$. Symmetry under the interchange $f \leftrightarrow g$ implies either $U_{\psi^*} \supset \Lambda$ — in which case $\|f\|_{L^1} = \|g\|_{L^1}$ and transportation is complete — or else $\psi^* = \phi^*$ on $U_{\psi^*} \cap \Lambda$ as above. In the latter case, choose any point $\mathbf{x} \in \Omega \subset U_i$; then $\nabla\phi(\mathbf{x}) = \nabla\psi(\mathbf{x}) = \mathbf{y} \in \Lambda$ by Theorem 6.3 and the equality $\psi(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle - \psi^*(\mathbf{y}) = \phi(\mathbf{x})$ forces $\psi = \phi$ throughout Ω to conclude the argument. On the other hand, if transportation is complete, it remains to show $\Omega \subset U_\phi$. If not, take $\mathbf{x} \in \Omega \backslash U_\phi$. There $\nabla\psi(\mathbf{x}) = \nabla\phi(\mathbf{x}) = \nabla h(\mathbf{x}) = \mathbf{x} \not\in \Lambda$, contradicting the proposition and concluding the corollary. QED.

Remark 6.5 (Map is continuous and normal to free boundary). *If a hyperplane* ∂H *separates* spt f *from* spt g *strictly, the free boundary* $\partial U_\psi \cap \Omega$ *is contained in the graph of a* C^1 *function over* ∂H*. Morever,* $\nabla\tilde{\psi}(\mathbf{x}) - \mathbf{x}$ *gives the inward normal to* U_ψ *at* $\mathbf{x} \in \Omega \cap \partial U_\psi$*. This is Claim #2 of Theorem 6.3's proof.*

Remark 6.6 (Homeomorphism of active interiors). *From the end of the preceding proof, it is also worth noting* $U_\psi \cap \Omega = \operatorname{int}\left[\overline{U_\psi \cap \Omega}\right]$*, and* $\nabla\psi : U_\psi \cap \Omega \longrightarrow U_{\psi^*} \cap V$ *is a homeomorphism.*

Corollary 6.7 (Path-connectedness of active region). *If* $\tilde{\psi}$, U_ψ, Ω *and* Λ *are from Theorem 6.3, then* $\overline{\Omega \cap U_\psi}$ *is path connected.*

Proof: Fix $\mathbf{x}_0, \mathbf{x}_1 \in \overline{\Omega \cap U_\psi}$ and consider the segment $\mathbf{x}_t := (1-t)\mathbf{x}_0 + t\mathbf{x}_1$ joining them. Let $]t', t''[$ be a maximal subinterval of $[0,1]$ corresponding to a piece of the segment $[\mathbf{x}_0, \mathbf{x}_1]$ not contained in $\overline{\Omega \cap U_\psi}$, if any such piece exists. We shall prove the corollary by constructing a path connecting $\mathbf{x}_{t'}$ to $\mathbf{x}_{t''}$ in $\overline{\Omega \cap U_\psi}$ for each such maximal open subinterval. Since the entire segment $[\mathbf{x}_0, \mathbf{x}_1]$ lies in the strictly convex set $\overline{\Omega}$, both $\mathbf{x}_{t'}$ and $\mathbf{x}_{t''}$ lie on the free boundary $\Omega \cap \partial U_\psi$. We will prove the segment $\mathbf{y}_s := (1-s)\nabla\tilde{\psi}(\mathbf{x}_{t'}) + s\nabla\tilde{\psi}(\mathbf{x}_{t''})$ parameterized by $s \in [0,1]$ lies in $\overline{\Lambda \cap U_{\psi^*}}$. Then the homeomorphism $\nabla\tilde{\psi}$ of the theorem gives the desired path $t \in [t', t''] \longrightarrow \nabla\tilde{\psi}^{-1}(\mathbf{y}_{(t-t')/(t''-t')})$ in $\overline{\Omega \cap U_\psi}$ linking $\mathbf{x}_{t'}$ to $\mathbf{x}_{t''}$.

Let $\mathbf{z}_s := (1-s)\mathbf{x}_{t'} + s\mathbf{x}_{t''}$ reparameterize the segment $[\mathbf{x}_{t'}, \mathbf{x}_{t''}]$. At the endpoints $s = 0, 1$, we have $(\mathbf{z}_s, \mathbf{y}_s)$ in the support of the optimal joint measure $\gamma_\lambda := (\mathbf{id} \times \nabla\tilde{\psi})_\#(f\chi_{U_\psi})$, hence $|\mathbf{z}_s - \mathbf{y}_s|^2/2 \le \lambda$ from Corollary 2.4 (and Theorem 4.3). The same inequality holds on the interior $s \in]0,1[$ of the segments due to convexity of the cost. There $\mathbf{z}_s \in \Omega \backslash \overline{U_\psi}$ by construction. Since we are dealing with data (f, g, h) of convex support, $\mathbf{y}_s \in \Lambda \setminus \overline{U_{\psi^*}}$ would force $|\mathbf{x} - \mathbf{y}|^2/2 \ge \lambda$ in a whole neighbourhood of $(\mathbf{z}_s, \mathbf{y}_s)$ by Corollary 2.4, producing the contradiction $|\mathbf{z}_s - \mathbf{y}_s|^2/2 > \lambda$. We can only conclude $\mathbf{y}_s \in \overline{\Lambda \cap U_{\psi^*}}$ for all $s \in [0,1]$, which completes the corollary. QED.

Lemma 6.8 (Ball condition; free boundary never maps to free boundary). *Take* $\tilde{\psi}$, U_ψ, Ω *and* Λ *from Theorem 6.3. If* $\mathbf{x} \in \overline{U_\psi \cap \Omega}$ *and* $\mathbf{y} := \nabla\tilde{\psi}(\mathbf{x})$ *then* $\Omega \cap B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \subset U_\psi$*. Likewise* $\Lambda \cap B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{x}) \subset U_{\psi^*}$*. If* $\mathbf{x} \in \Omega \cap \partial U_\psi$ *then* $\mathbf{y} \notin \overline{\Lambda \cap \partial U_{\psi^*}}$*.*

Proof: Recall that ψ is a weak-$*$ solution to the Monge-Ampère double obstacle problem with data $(f, g, h = k)$ of convex support. Thus $\gamma := (\mathbf{id} \times \nabla\psi)_\#(f\chi_U)$ is the measure minimizing (1.8) according to Theorem 4.3. The continuous dependence of $\mathbf{y} = \nabla\tilde{\psi}(\mathbf{x})$ on $\mathbf{x} \in \overline{U_\psi \cap \Omega}$ proved in Theorem 6.3 combines with positivity of f on the domain $U_\psi \cap \Omega$ to yield $(\mathbf{x}, \mathbf{y}) \in \operatorname{spt}\gamma$. Thus the ball $B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{x})$ is disjoint from $\operatorname{spt}[g(1-\chi_V)] = \overline{\Lambda} \setminus V$, according to Example 2.5; here $V := U_{\psi^*}$. We conclude $\Lambda \cap B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{x})$ is contained in V, and $\Omega \cap B_{|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \subset U_\psi$ follows from the usual exchange symmetry $f \leftrightarrow g$.

For the second part of the lemma, we'll assume $\mathbf{x} \in \Omega \cap \partial U_\psi$ and $\mathbf{y} \in \overline{\Lambda \cap \partial V}$ to derive a contradiction. The free boundary $\overline{\Lambda \cap \partial V}$ is C^1 by Remark

6.5, and $\mathbf{y}-\mathbf{x}$ is a positive multiple of the outward normal vector $\hat{\mathbf{n}}_V(\mathbf{y})$ to the active region V at $\mathbf{y}$. Now $\mathbf{x}_r := \mathbf{x} + r\hat{\mathbf{n}}_V(\mathbf{y})$ parameterizes the line segment from $\mathbf{x}$ to $\mathbf{y}$. Setting $\mathbf{y}_r := \nabla\tilde{\psi}(\mathbf{x}_r)$ we have $\Omega \cap B_{|\mathbf{x}_r-\mathbf{y}_r|}(\mathbf{y}_r) \subset U_\psi$ for each $r \geq 0$ small enough that $\mathbf{x}_r \in \overline{U_\psi \cap \Omega}$. Since $r = 0$ is certainly small enough and $B_\epsilon(\mathbf{x}_0) \subset \Omega$ for some $\epsilon > 0$, we conclude $r = \epsilon$ is also small enough because $\mathbf{x}_\epsilon \in \Omega \cap B_{|\mathbf{x}_0-\mathbf{y}_0|}(\mathbf{y}_0) \subset U_\psi$. On the other hand, monotonicity of $\nabla\tilde{\psi}$ implies $0 \leq \langle\, \mathbf{x}_\epsilon - \mathbf{x}_0,\, \mathbf{y}_\epsilon - \mathbf{y}_0\rangle = \epsilon\langle\, \hat{\mathbf{n}}_V(\mathbf{y}_0),\, \mathbf{y}_\epsilon - \mathbf{y}_0\rangle$. Now $\mathbf{x}_\epsilon$ lies on the line segment joining $\mathbf{x}_0$ to $\mathbf{y}_0$, and $\mathbf{y}_\epsilon \neq \mathbf{y}_0$ because $\nabla\tilde{\psi}$ is a homeomorphism of $\overline{U_\psi \cap \Omega}$ onto $\overline{V \cap \Lambda}$. In the halfspace $\{\mathbf{y} \in \mathbf{R}^n \mid \langle\, \hat{\mathbf{n}}_V(\mathbf{y}_0),\, \mathbf{y} - \mathbf{y}_0\rangle \geq 0\}$ containing $\mathbf{y}_\epsilon \neq \mathbf{y}_0$, we see $\mathbf{y}_0$ is the point closest to $\mathbf{x}_\epsilon$. Thus $\mathbf{y}_0 \in B_{|\mathbf{x}_\epsilon-\mathbf{y}_\epsilon|}(\mathbf{x}_\epsilon)$. Any point $\tilde{\mathbf{y}} \in \Lambda \cap V$ sufficiently close to $\mathbf{y}_0$ must also belong to $\Lambda \cap B_{|\mathbf{x}_\epsilon-\mathbf{y}_\epsilon|}(\mathbf{x}_\epsilon) \subset V$. But this contradicts $\mathbf{y}_0 \in \overline{\Lambda \cap \partial V}$ as desired. QED.

Corollary 6.9 (Inactive region maps to target boundary). *Take $\tilde{\psi}$, U_ψ, Ω and Λ from Theorem 6.3. If $\mathbf{z} \in \overline{\Omega} \setminus \overline{\Omega \cap U_\psi}$ then $\nabla\tilde{\psi}(\mathbf{z}) \in \partial\Lambda$.*

Proof: Theorem 6.3 asserts that $\nabla\tilde{\psi}(\mathbf{R}^n) = \overline{\Lambda}$, with $\nabla\tilde{\psi}(\mathbf{y}) = \mathbf{y}$ on $\Lambda \setminus U_{\psi^*}$ and $\nabla\tilde{\psi} : \overline{\Omega \cap U_\psi} \longrightarrow \overline{\Lambda \cap U_{\psi^*}}$ a homeomorphism. If the corollary failed to be true, some $\mathbf{z} \in \overline{\Omega} \setminus \overline{\Omega \cap U_\psi}$ would map to $\mathbf{y} := \nabla\tilde{\psi}(\mathbf{z}) \in \Lambda$. We consider the three possibilities (i) $\mathbf{y} \in \Lambda \setminus \overline{U_{\psi^*}}$, (ii) $\mathbf{y} \in \Lambda \cap U_{\psi^*}$, and (iii) $\mathbf{y} \in \Lambda \cap \partial U_{\psi^*}$, one at a time. In case (i) convexity of $\tilde{\psi}$ implies $\nabla\tilde{\psi}$ is constant on the line segment $]\mathbf{z}, \mathbf{y}[$, which cannot happen since $\nabla\tilde{\psi}$ is the identity map in a neighbourhood of $\mathbf{y}$. In case (ii), $\mathbf{y} = \nabla\tilde{\psi}(\mathbf{x})$ for some $\mathbf{x} \in \Omega \cap U_\psi$; again $\nabla\tilde{\psi}$ would be constant on the segment $]\mathbf{z}, \mathbf{x}[$, which contradicts $\nabla\tilde{\psi}$ being a homeomorphism near $\mathbf{x}$. Finally, in case (iii) $\mathbf{y} = \nabla\tilde{\psi}(\mathbf{x})$ for some $\mathbf{x}$ on the boundary of $\overline{\Omega \cap U_\psi}$, and $\mathbf{x} \notin \overline{\Omega \cap \partial U_\psi}$ by Lemma 6.8 and exchange symmetry $\mathbf{x} \leftrightarrow \mathbf{y}$. Then $\Omega \cap B_r(\mathbf{x}) \subset U_\psi$ for $r > 0$ sufficiently small, and $\nabla\tilde{\psi}$ would be constant on the segment $]\mathbf{z}, \mathbf{x}[$. Strict convexity of Ω forces this segment into Ω hence into U_ψ, contradicting that $\nabla\tilde{\psi}$ is a homeomorphism on $\Omega \cap U_\psi$. The only conclusion can be that the corollary is true: $\nabla\tilde{\psi}(\mathbf{z}) \in \partial\Lambda$. QED.

7. Bi-Hölder estimates for maps and free normals

To quantify continuous differentiability of the potential ψ and the free boundary through Hölder derivative estimates, we exploit the renormalization methods developed by Caffarelli to treat boundary regularity for data of convex support in the complete transfer case $m = \|f\|_{L^1} = \|g\|_{L^1}$ [16] [17]. However, even for data of convex support, we cannot expect the free regions $U_\psi \cap \Omega$ and $U_{\psi^*} \cap \Lambda$ to be convex, which means new arguments are needed to localize the situation before this method applies. In Caffarelli's approach, convexity of the domain and range play different roles. Domain convexity is used locally, to ensures the Monge-Ampère measure has a doubling property, while convexity of the range is used globally, to ensure all Aleksandrov mass is accounted for, so in the entire space the Monge-Ampère measure has no singular part. In the discussion below, convexity of Λ proves sufficient for the latter purpose. By localizing the former property, we shall be able to quantify strict convexity of ψ up to the fixed boundary $\partial\Omega$, and to any parts of the free boundary which happen to be convex locally. Now recall Lemma 6.8, which states that free boundary never maps to free boundary; in other words, the free boundary of $U_{\psi^*} \cap \Lambda$ is parameterized by part of the fixed boundary of $U_\psi \cap \Omega$. Thus showing ψ is p-uniformly convex (7.14) *away from* the free boundary of $U_\psi \cap \Omega$ implies $\nabla\psi^*$ is Hölder continuous *up to* the free boundary $\partial U_{\psi^*} \cap \Lambda$, where it gives the inward normal $\mathbf{n}_{U_{\psi^*}}(\mathbf{y}) = \nabla\psi^*(\mathbf{y}) - \mathbf{y}$. The usual exchange symmetry $f \leftrightarrow g$ implies $\nabla\psi$ too is Hölder continuous up to the free boundary of its domain $U_\psi \cap \Omega$, and bi-Hölder up to those parts of the fixed boundary $\partial\Omega$ which map to points where the target set $U_{\psi^*} \cap \Lambda$ is convex.

To make this result precise, it is useful to distinguish certain types of boundary points. A domain $U \subset \mathbf{R}^n$ is called *locally convex* at $\mathbf{x} \in \mathbf{R}^n$ if $U \cap B_R(\mathbf{x})$ is a convex set for some ball of radius $R > 0$ around $\mathbf{x}$. The non-convex part of the free boundary is then a closed set denoted by

$$\partial_{nc} U_\psi := \{\mathbf{x} \in \overline{\Omega \cap U_\psi} \mid \Omega \cap U_\psi \text{ fails to be locally convex at } \mathbf{x}\}. \tag{7.1}$$

We have not been able to quantify uniform convexity of the solution ψ at such points. Also, we are not able to rule out the possibility of a tangential intersection of the free with the fixed boundary, nor to prove Hölder continuity of the map or free normal at such intersections. Since they are distinguished by method, we denote these non-transverse intersection points by

$$\partial_{nt}\Omega := \{\mathbf{x} \in \partial\Omega \cap \overline{\Omega \cap \partial U_\psi} \mid \langle \nabla\tilde{\psi}(\mathbf{x}) - \mathbf{x}, \mathbf{z}\rangle \le 0 \text{ for all } \mathbf{z} \in \Omega\}. \tag{7.2}$$

Here $\tilde{\psi}$ is the extension of ψ from Theorem 6.3. Notice that when $\partial\Omega$ is differentiable at $\mathbf{x} \in \partial_{nt}\Omega$, then $\nabla\tilde{\psi}(\mathbf{x}) - \mathbf{x}$ gives the outward normal to Ω by (7.2); it gives the inward normal to U_ψ by Remark 6.5. We define the non-convex points $\partial_{nc}U_{\psi^*}$ and non-transverse intersections $\partial_{nt}\Lambda$ in the target domain analogously.

Let us briefly review Caffarelli's method, which develops dramatically from ideas going back to Aleksandrov [4]. Given a convex function $\psi :\longrightarrow]-\infty, \infty]$, we associate to it a measure M_ψ on $\mathbf{R}^n$ — called the *Monge-Ampère measure* — given by

$$\mathrm{M}_\psi(B) := \mathrm{vol}\,[\partial\psi(B)] \tag{7.3}$$

for every Borel set $B \subset \mathbf{R}^n$. If ψ is smooth and strictly convex, then

$$\mathrm{M}_\psi(B) = \int_B \det\left[D^2\psi(\mathbf{x})\right]\, d\mathbf{x},$$

which motivates the name, but M_ψ is a Radon measure on the interior of $\mathrm{dom}\,\psi := \{\psi < +\infty\}$ in any case; see e.g. Gutiérrez [45] or McCann [57, Lemma 4.1].

Definition 7.1 (Universal constant). *For the purpose of this paper, a* universal constant *is one which depends only on dimension n, and $\|\log(f(\mathbf{x})/g(\mathbf{y}))\|_{L^\infty(\Omega\times\Lambda)}$.*

This unorthodox terminology is employed to highlight independence of such constants on choice of solution ψ, data f and g, or domains $\Omega, \Lambda \subset \mathbf{R}^n$, except through

$$-\log\delta_0 := \|\log(f(\mathbf{x})/g(\mathbf{y}))\|_{L^\infty(\Omega\times\Lambda)}. \tag{7.4}$$

When f and g are both characteristic functions, then $\delta_0 = 1$; if they are merely continuous on there respective domains, δ_0 is the lesser of $\inf_\Omega f/\sup_\Lambda g$ and $\inf_\Lambda g/\sup_\Omega f$.

Lemma 7.2 (Mass lives in active domain and inactive target). *Fix data of convex support $(f, g, h = k)$, and a weak-$*$ solution ψ to the obstacle problem (4.3). Set $U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi > h\}$. The extension $\tilde\psi \in C^1(\mathbf{R}^n)$ of ψ from $U_\psi \cap \Omega$ to $\mathbf{R}^n$ given by Theorem 6.3 has a Monge-Ampère measure absolutely continuous with respect to Lebesgue, and whose Radon-Nikodym derivative $d\mathrm{M}_{\tilde\psi}/d\mathrm{vol}$ satisfies*

$$\chi_{\Lambda\setminus U_{\psi^*}} + \delta_0\chi_{\Omega\cap U_\psi} \le d\mathrm{M}_{\tilde\psi}/d\mathrm{vol} \le \chi_{\Lambda\setminus U_{\psi^*}} + \frac{1}{\delta_0}\chi_{\Omega\cap U_\psi} \tag{7.5}$$

for the universal constant $\delta_0 > 0$ of (7.4).

Proof: Theorem 6.3 extends the homeomorphism $\nabla\tilde\psi : \overline{\Omega\cap U_\psi} \longrightarrow \overline{\Lambda\cap U_{\psi^*}}$ to the identity map on $\Lambda\setminus\overline{U_{\psi^*}}$, with $\nabla\tilde\psi(\mathbf{R}^n) = \overline{\Lambda}$ and $\nabla\tilde\psi_\#(f\chi_{U_\psi} + g(1-\chi_{U_{\psi^*}})) = g$. Since $\partial\tilde\psi(\mathbf{R}^n) \subset \overline{\Lambda}$ we have $\mathrm{M}_{\tilde\psi}(B) = \rho[\partial\tilde\psi(B)]$ with $\rho = \mathrm{vol}|_\Lambda$. This means $\mathrm{M}_{\tilde\psi} = \nabla\phi_\#\rho$ is the push-forward of ρ through the gradient of the Legendre transform ϕ of $\tilde\psi$ [57, Lemma 4.1]. From

$$\begin{aligned} \|\frac{1}{g}\|^{-1}_{L^\infty(\Lambda)}\rho \quad &\le g \le \|g\|_{L^\infty(\Lambda)}\rho \\ \|\frac{1}{f}\|^{-1}_{L^\infty(\Omega)}\chi_\Omega &\le f \le \|f\|_{L^\infty(\Omega)}\chi_\Omega \end{aligned}$$

we find

$$\begin{aligned} \|g\|^{-1}_{L^\infty(\Lambda)}\nabla\phi_\# g &\le \nabla\phi_\#\rho \\ &= \mathrm{M}_\psi \quad \le \quad \|1/g\|_{L^\infty(\Lambda)}\nabla\phi_\# g, \end{aligned}$$

whence $\nabla\phi_\# g = f\chi_{U_\psi} + g(1-\chi_{U_{\psi^*}})$ implies (7.5), but with the universal constant δ_0 multiplying both terms on the left and dividing both terms on the right. Since $\nabla\phi(\mathbf{y}) = \mathbf{y}$ on $\Lambda\setminus U_{\psi^*}$ we conclude that the coefficients multiplying $\chi_{\Lambda\setminus U_{\psi^*}}$ are unnecessary. QED.

DEFINITION 7.3 (CONVEX BODIES AND HOMOTHETY). *A bounded convex set $Z \subset \mathbf{R}^n$ with non-empty interior is called a convex body. The* barycenter $\mathbf{z}$ *of Z always refers to its center of mass with respect to Lebesgue volume. For $t > 0$, $t{\cdot}Z := (1-t)\mathbf{z} + tZ = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x} - \mathbf{z} = t(\mathbf{y} - \mathbf{z})$ for some $\mathbf{y} \in Z\}$ denotes the dilation of Z by a factor of t around its center of mass $\mathbf{z}$.*

DEFINITION 7.4 (AFFINE DOUBLING). *A Borel measure μ on $\mathbf{R}^n$* doubles affinely *on $X \subset \mathbf{R}^n$ if there exists $\delta > 0$ such that each point $\mathbf{x} \in X$ has a neighbourhood $N_{\mathbf{x}} \subset \mathbf{R}^n$ on which μ is a Radon measure, and each convex body $Z \subset N_{\mathbf{x}}$ with barycenter in X satisfies $\mu[\frac{1}{2}{\cdot}Z] \geq \delta^2\mu[Z]$. We call δ the* doubling constant *of μ on X, and $N_{\mathbf{x}}$ the doubling neighbourhood of μ around $\mathbf{x}$.*

LEMMA 7.5 (MA MEASURE DOUBLES AWAY FROM NON-CONVEXITIES). *Fix data of convex support $(f, g, h = k)$, and a weak-$*$ solution ψ to the obstacle problem (4.3). The extension $\tilde\psi \in C^1(\mathbf{R}^n)$ of ψ in Theorem 6.3 has a Monge-Ampère measure $\mathrm{M}_{\tilde\psi}$ which doubles affinely on $\overline{\Omega \cap U_\psi} \setminus \partial_{nc}U_\psi$ with $U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi > h\}$ and $\partial_{nc}U_\psi$ as in (7.1). The doubling constant $\delta = 2^{-n/2}\delta_0$ is universal (7.4), while any ball $N_{\mathbf{x}} = B_R(\mathbf{x})$ whose intersection with $(\Omega \cap U_\psi) \cup (\Lambda \setminus U_{\psi^*})$ is convex forms a doubling neighbourhood around $\mathbf{x}$.*

Proof: For $\mathbf{x} \in X := \overline{\Omega \cap U_\psi} \setminus \partial_{nc}U_\psi$ there exists $R > 0$ small enough that $B_R(\mathbf{x})$ is disjoint from $\Lambda \setminus U_{\psi^*}$ and the intersection $W = \Omega \cap U_\psi \cap B_R(\mathbf{x})$ is convex. For any convex body $Z \subset B_R(\mathbf{x})$ we therefore have

$$\delta_0 \mathrm{vol}\,[W \cap Z] \leq \mathrm{M}_{\tilde\psi}[Z] \tag{7.6}$$

$$\leq \frac{1}{\delta_0}\mathrm{vol}\,[W \cap Z] \tag{7.7}$$

from (7.5). Now the remainder of the argument follows [17, Lemma 2.3]. Indeed, suppose the barycenter $\mathbf{z}$ of Z lies in X, a fortiori in $\overline{W}$, and let $\mathbf{z} = \mathbf{0}$ without loss of generality. Define the (not necessarily convex) cone $K := \{\lambda\mathbf{x} \in \mathbf{R}^n \mid \lambda > 0,\ \mathbf{x} \in W \cap \partial[\frac{1}{2}{\cdot}Z]\}$ with vertex at $\mathbf{z} = \mathbf{0}$. Convexity of W and Z imply $K \cap \frac{1}{2}{\cdot}Z \subset W \cap \frac{1}{2}{\cdot}Z$ whereas $W \cap (Z \setminus \frac{1}{2}{\cdot}Z) \subset K \cap (Z \setminus \frac{1}{2}{\cdot}Z)$. These two inclusions combine with (7.7) and (7.6) to imply the doubling property

$$\begin{aligned}
\delta_0 \mathrm{M}_{\tilde\psi}[Z] &\leq \mathrm{vol}\,[W \cap \tfrac{1}{2}{\cdot}Z] + \mathrm{vol}\,[K \cap (Z \setminus \tfrac{1}{2}{\cdot}Z)] \\
&= \mathrm{vol}\,[W \cap \tfrac{1}{2}{\cdot}Z] + (2^n - 1)\mathrm{vol}\,[K \cap \tfrac{1}{2}{\cdot}Z] \\
&\leq 2^n \mathrm{vol}\,[W \cap \tfrac{1}{2}{\cdot}Z] \\
&\leq \frac{2^n}{\delta_0}\mathrm{M}_{\tilde\psi}[\tfrac{1}{2}{\cdot}Z]
\end{aligned}$$

of $\mathrm{M}_{\tilde\psi}$ on X. QED.

As a final ingredient before the method applies, we must be able to isolate the behaviour of convex function $\tilde\psi$ near any locally convex point $\mathbf{z} \in \overline{\Omega \cap U_\psi} \setminus \partial_{nc} U_\psi$ by choosing a *centered affine section*

$$Z_\epsilon(\mathbf{z}) = Z_\epsilon^{\tilde\psi}(\mathbf{z}) := \{\mathbf{x} \in \mathbf{R}^n \mid \tilde\psi(\mathbf{x}) < \epsilon + \tilde\psi(\mathbf{z}) + \langle \mathbf{v}_\epsilon, \mathbf{x} - \mathbf{z}\rangle\}. \tag{7.8}$$

Here $\epsilon > 0$, and $\mathbf{v}_\epsilon \in \mathbf{R}^n$ is chosen to ensure that $\mathbf{z}$ is the barycenter of the bounded set $Z_\epsilon(\mathbf{z})$; such a choice is unique according to Theorems A.7–A.8; see also [16] [17]. The superscript $\tilde\psi$ in $Z_\epsilon^{\tilde\psi}(\mathbf{z})$ is often omitted, and used only to clarify ambiguities when several convex functions are being discussed.

The basic result we will use to quantify strict convexity is a local version of Caffarelli's fundamental lemma [16, Lemma 4] [18, Lemma 2.2], adapted to the situation at hand. For completeness, we recall the proof; the figures from [18] may be helpful. An alternate approach to Caffarelli's result may be found in Gutiérrez and Huang [46].

Lemma 7.6 (Geometric decay of sections). *Given $0 \le t < \bar t \le 1$ and $\delta > 0$, there exists $s_0(t,\bar t) \in]0,1[$ (depending only on $t, \bar t, \delta$, and dimension n), such that whenever $Z_\epsilon(\mathbf{x})$ is a fixed section centered at $\mathbf{x} \in X := \operatorname{spt} M_\psi$ of a convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ whose Monge-Ampère measure satisfies the doubling condition*

$$\mathrm{M}_\psi[\tfrac{1}{2} \cdot Z_{s\epsilon}(\mathbf{z})] \ge \delta^2 \mathrm{M}_\psi[Z_{s\epsilon}(\mathbf{z})] \tag{7.9}$$

for all $s \in [0,1]$ and all $\mathbf{z}$ in the convex set $X \cap Z_\epsilon(\mathbf{x})$, then $\mathbf{z} \in X \cap t \cdot Z_\epsilon(\mathbf{x})$ implies $Z_{s\epsilon}(\mathbf{z}) \subset \bar t \cdot Z_\epsilon(\mathbf{x})$ for all $s \le s_0(t,\bar t)$.

Proof: Fix $0 \le t < 1$ and $\delta > 0$. It suffices to show there exists $s_0(t) > 0$ and $t_0 < 1$, both depending only on n, δ and t, such that: whenever a convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ admits a section $Z_\epsilon(\mathbf{x})$ centered at $\mathbf{x} \in X := \operatorname{spt} M_\psi$ such that $X \cap Z_\epsilon(\mathbf{x})$ is convex, and the doubling condition (7.9) holds for all $s \in [0,1]$ and $\mathbf{z} \in X \cap Z_\epsilon(\mathbf{x})$, then $\mathbf{z} \in X \cap t \cdot Z_\epsilon(\mathbf{x})$ implies $Z_{s\epsilon}(\mathbf{z}) \subset t_0 \cdot Z_\epsilon(\mathbf{x})$ for all $s \le s_0(t)$. Once this intermediate claim is proved, we see convexity of $X \cap Z_{s\epsilon}(\mathbf{x}_0)$ follows from that of $X \cap Z_\epsilon(\mathbf{x})$ for any new center $\mathbf{x}_0 \in X \cap t \cdot Z_\epsilon(\mathbf{x})$ and height $s\epsilon$ with $s \le s_0(t)$, as does the doubling property (7.9) for all $\mathbf{z} \in X \cap Z_{s\epsilon}(\mathbf{x}_0)$. Thus the intermediate claim implies $Z_{s^2\epsilon}(\mathbf{x}_0) \subset t_0 \cdot Z_{s\epsilon}(\mathbf{x}_0)$, and similarly, $Z_{s^k\epsilon}(\mathbf{x}_0) \subset t_0 \cdot Z_{s^{k-1}\epsilon}(\mathbf{x}_0) \subset t_0^{k-1} \cdot Z_{s\epsilon}(\mathbf{x}_0)$ for each integer $k \ge 1$ by induction. Since $t_0 < 1$, the sections $Z_{s^k\epsilon}(\mathbf{x}_0)$ shrink to their common center $\mathbf{x}_0 \in t \cdot Z_\epsilon(\mathbf{x})$, so for any $\bar t > t$ taking $k = k(t,\bar t)$ large enough ensures $t_0^{k-1} \cdot Z_{s\epsilon}(\mathbf{x}_0) \subset \bar t \cdot Z_\epsilon(\mathbf{x})$. Choosing $s_0(t,\bar t) := s_0(t)^{k(t,\bar t)}$ then completes the lemma. We can take $k(t,\bar t)$ to depend on $t, \bar t$ and t_0, but not on the geometry of the sections, by using affine invariance of the geometry and Lemma A.3 to assume $Z = Z_\epsilon(\mathbf{x})$ has center of mass at the origin, and

$B_1(\mathbf{0}) \subset Z \subset \overline{B_{n^{3/2}}(\mathbf{0})}$ without loss of generality. This yields $k(t, \bar{t})$ via an upper bound on the distance separating tZ from $\mathbf{R}^n \setminus (t_0 Z)$, and a lower bound on the distance separating tZ from $\mathbf{R}^n \setminus (\bar{t}Z)$.

The intermediate claim will be argued by contradiction. If it fails to be true, then for some fixed $\delta > 0$ and $t \in [0, 1[$, there is a fortiori a sequence of convex functions $\psi_k : \mathbf{R}^n \longrightarrow]-\infty, \infty]$, and centered affine sections $Z_{\epsilon(k)}(\mathbf{0})$ with $X_k \cap Z_{\epsilon(k)}(\mathbf{0})$ convex, on which the doubling condition (7.9) is satisfied with $X_k := \operatorname{spt} \mathrm{M}_{\psi_k}$, and points $\mathbf{z}_k \in X_k \cap t \cdot Z_{\epsilon(k)}(\mathbf{0})$ such that $Z_{\epsilon(k)/k}(\mathbf{z}_k)$ is not contained in $(1 - 1/k) \cdot Z_{\epsilon(k)}(\mathbf{0})$. Since the hypothesis and conclusion of the lemma are invariant under pre-composition of ψ_k with an affine function, and post-multiplication of ψ_k by a positive scalar, it costs no loss of generality to assume the normalizations $B_1(\mathbf{0}) \subset Z_{\epsilon(k)}(\mathbf{0}) \subset \overline{B_{n^{3/2}}(\mathbf{0})}$ for each k, and $\mathrm{M}_{\psi_k}[Z_{\epsilon(k)}(\mathbf{0})] = 1$. Furthermore, subtracting an affine function allows us to assume $Z_{\epsilon(k)}(\mathbf{0}) = \{\mathbf{x} \mid \psi_k(\mathbf{x}) < 0\}$.

Using Blaschke's theorem together with Lemma A.4, we extract a limiting convex function $\psi_\infty : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ from a subsequence, and a section $S_\infty = \{\mathbf{x} \mid \psi_\infty < 0\}$ containing $B_1(\mathbf{0})$ and contained in $\overline{B_{n^{3/2}}(\mathbf{0})}$, with center of mass at the origin. We have $\psi_\infty(\mathbf{0}) = \lim_{k\to\infty} \epsilon(k)$ bounded above and below by constants depending only on n and $\delta > 0$, while Lemma A.1 implies ψ_∞ vanishes uniformly as the boundary of S_∞ is approached from the interior. Now the height of the sections $Z_{\epsilon(k)/k}(\mathbf{z}_k)$ tends to zero like $1/k$. Since these sections extend beyond $(1 - 1/k) \cdot Z_{\epsilon(k)}(\mathbf{0})$ from their centers in $tZ_{\epsilon(k)}(\mathbf{0})$, the graph of ψ_∞ must coincide with an affine function L_∞ on a set K which extends from $\mathbf{z}_\infty \in tZ_\infty$ to $\mathbf{y}_\infty \in \partial Z_\infty$, where $\mathbf{z}_\infty$ and $\mathbf{y}_\infty$ are subsequential limits of the $\mathbf{z}_k$ and points $\mathbf{y}_k \in Z_{\epsilon(k)/k}(\mathbf{z}_k) \cap \partial(1 - 1/k) \cdot Z_{\epsilon(k)}(\mathbf{0})$ respectively. Since the sections are roughly balanced around their barycenters $\mathbf{z}_k$, there is also a point $\tilde{\mathbf{y}}_\infty \in K$ with $\mathbf{z}_\infty = a\tilde{\mathbf{y}}_\infty + (1 - a)\mathbf{y}_\infty$ and $a^{-1} \in [1 + n^{-3/2}, n^{3/2} + 1]$.

Now pick a point $\mathbf{x}_\infty$ which minimizes ψ_∞ on K,

$$\psi(\mathbf{x}_\infty) = \min_{\mathbf{x}\in K} \psi_\infty(\mathbf{x}) < 0, \tag{7.10}$$

and which is extremal in the convex set $K \cap \{\mathbf{x} \mid \psi(\mathbf{x}) = \psi(\mathbf{x}_\infty)\}$ of such minimizers. It follows that $\mathbf{x}_\infty \in Z_\infty$ from (7.10) and is extremal in the graph of ψ_∞. Corollary A.2 then implies $\mathbf{x}_\infty$ lies in $X_\infty := \operatorname{spt} \mathrm{M}_{\psi_\infty}$. Since M_{ψ_∞} coincides with the weak-$*$ limit of the Monge-Ampère measures M_{ψ_k} as in Gutiérrez [45], we can find $\mathbf{x_k} \in X_k \cap Z_{\epsilon(k)}(\mathbf{0})$ converging to $\mathbf{x}_\infty$. We also note

$$\psi(\mathbf{x}_\infty) \leq (1 + h)\psi_\infty(\mathbf{z}_\infty) \tag{7.11}$$

$$\leq -h \tag{7.12}$$

for some $h > 0$ depending on $1 - t$, which keeps $\mathbf{x}_\infty$ and $\mathbf{z}_\infty$ separated.

Fix $\eta, \sigma > 0$ small, to be specified later, independent of k. Consider the affine sections $Z_{\sigma\epsilon(k)}(\mathbf{w}_k) = \{\mathbf{x} \mid \psi_k \leq L_k\}$ centered at the convex combination $\mathbf{w}_k = (1-\eta)\mathbf{x}_k + \eta\mathbf{z}_k$. Their centers $\mathbf{w}_k$ lie near $\mathbf{x}_k$ since η is small, belong to the convex set $X_k \cap Z_{\epsilon(k)}(\mathbf{0})$, and approach $\mathbf{w}_\infty = (1-\eta)\mathbf{x}_\infty + \eta \mathbf{z}_\infty$ as $k \to \infty$. Thus $\max_{\mathbf{x} \in Z_{\sigma\epsilon(k)}(\mathbf{w}_k)}(L_k - \psi_k)(\mathbf{x})$ remains comparable to σ as $k \to \infty$, according to Lemma A.4. Consider the segment $I_k = [\mathbf{p}_k, \mathbf{q}_k]$ obtained by intersecting $Z_{\sigma\epsilon(k)}(\mathbf{w}_k)$ with the line through $\mathbf{z}_k$ and $\mathbf{x}_k$. Assume $\psi_k(\mathbf{p}_k) \geq \psi_k(\mathbf{q}_k)$, so that $\mathbf{p}_k$ denotes the upper and $\mathbf{q}_k$ the lower end of this segment. The segment must be roughly balanced around the section's center of mass $\mathbf{w}_k = a_k \mathbf{p}_k + (1-a_k)\mathbf{q}_k$, with $a_k^{-1} \in [1 + n^{-3/2}, n^{3/2} + 1]$. Extracting subsequences as usual, the limit $k \to \infty$ yields a segment $I^\sigma = [\mathbf{p}^\sigma, \mathbf{q}^\sigma]$, possibly (semi-)infinite, with $\mathbf{w}_\infty = a_\infty \mathbf{p}^\sigma + (1-a_\infty)\mathbf{q}^\sigma$, and an affine function $L^\sigma = \lim_k L_k$ such that $\psi_\infty < L^\sigma < \psi_\infty + C\sigma$ on $I^\sigma \cap Z_\infty$ and

$$(L^\sigma - \psi_\infty)(\mathbf{w}_\infty) \sim \sigma \tag{7.13}$$

as $\sigma \to 0$. Observe that for $\sigma > 0$ small, the lower endpoint $\mathbf{q}^\sigma$ must lie in Z_∞. Indeed, ψ_k differs from L_k by a quantity of order σ along I_k; if I crosses Z_∞ completely then L^σ must be non-negative along I, and $\psi_\infty > -C\sigma$, which contradicts (7.12) when σ is small, since $\mathbf{x}_\infty \in I$ in this case. Since I_k is roughly balanced, $\mathbf{p}_\sigma$ remains bounded, but can be outside Z_∞. Also, $(L^\sigma - \psi_\infty)((1-t)\mathbf{w}_\infty + t\mathbf{q}^\sigma) \leq C(1-t)^{1/n}\sigma$ tends to zero as $t \to 1$, according to Lemma A.1.

We claim $\mathbf{q}^\sigma \to \mathbf{x}_\infty$ as $\sigma \downarrow 0$, and $\mathbf{p}^\sigma \in K$ for small enough $\sigma > 0$. Let $\mathbf{q}^0$ and $\mathbf{p}^0$ denote accumulation points of $\mathbf{q}^\sigma$ and of $\mathbf{p}^\sigma$ in this limit. Clearly the interior of the segment $[\mathbf{q}^0, \mathbf{p}^0]$ around $\mathbf{w}_\infty$, where ψ_∞ is affine, cannot contain the extremal point $\mathbf{x}_\infty$. Since $I^\sigma = [\mathbf{q}^\sigma, \mathbf{p}^\sigma]$ and $[\mathbf{x}_\infty, \mathbf{z}_\infty]$ are collinear and parallel, $\mathbf{q}^0 \in [\mathbf{x}_\infty, \mathbf{w}_\infty]$. For $\eta > 0$ sufficiently small, rough balancing of I^σ around $\mathbf{w}_\infty = (1-\eta)\mathbf{x}_\infty + \eta\mathbf{z}_\infty$ then forces the upper end $\mathbf{p}^\sigma \in [\mathbf{w}_\infty, \mathbf{z}_\infty]$ when σ is small. In that case $\mathbf{p}^\sigma \in K$ and $L^\sigma = L_\infty = \psi_\infty$ agree at $\mathbf{p}^\sigma$. Furthermore, $\mathbf{q}^\sigma \notin [\mathbf{x}_\infty, \mathbf{z}_\infty]$, since otherwise $I^\sigma \subset K$ would force $L^\sigma = L_\infty = \psi_\infty$ and violate (7.13). Therefore $\mathbf{x}_\infty = \mathbf{q}^0 = \lim_{\sigma \downarrow 0} \mathbf{q}^\sigma$. Since the difference $L^\sigma - L_\infty \geq 0$ dwindles to zero along the segment I^σ, as we move from $\mathbf{q}^\sigma$ through $\mathbf{x}_\infty$ and then $\mathbf{z}_\infty$ to $\mathbf{p}^\sigma$, we find

$$\begin{aligned}(L^\sigma - \psi_\infty)(\mathbf{x}_\infty) &= (L^\sigma - L_\infty)(\mathbf{x}_\infty)\\ &\geq (L^\sigma - L_\infty)(\mathbf{w}_\infty)\\ &= (L^\sigma - \psi_\infty)(\mathbf{w}_\infty) \sim \sigma.\end{aligned}$$

On the other hand, $\mathbf{x}_\infty = (1-t)\mathbf{w}_\infty + t\mathbf{q}^\sigma$ with $t = t(\sigma) \to 1$ as $\sigma \to 0$. Thus

$$(L^\sigma - \psi_\infty)(\mathbf{x}_\infty) \leq C(1-t(\sigma))^{1/n}\sigma = o(\sigma),$$

a contradiction which completes the proof. QED.

Corollary 7.7. *Under the hypotheses of Lemma 7.6, we find* $Z_{s^k\epsilon}(\mathbf{x}) \subset \bar{t}^k \cdot Z_\epsilon(\mathbf{x})$ *for all* $s < s_0(0, \bar{t})$, $\bar{t} \in]0, 1[$, *and integers* $k \geq 0$.

Note that we can recover the central results of the preceding section directly from the conclusions of these lemmas in the complete transfer case $m = \|f\|_1 = \|g\|_1$, thereby avoiding the main argument of [15].

Corollary 7.8 (Complete transfer injects). *Fix data (f, g, h) of convex support $\Omega = \operatorname{int}[\operatorname{spt} f]$ and $\Lambda = \operatorname{int}[\operatorname{spt} g]$. If ψ is a weak-$*$ solution to the obstacle problem (4.1) with $\Omega \subset U_\psi := \{\mathbf{x} \mid \psi > h\}$ and $\Lambda \subset U_{\psi^*}$, then $\psi : \Omega \longrightarrow \mathbf{R}$ is strictly convex.*

Proof: Let $\tilde\psi : \mathbf{R}^n \longrightarrow \mathbf{R}$ extend ψ from Ω to all of $\mathbf{R}^n$, with $\partial\tilde\psi(\mathbf{R}^n) \subset \overline{\Lambda}$. To produce a contradiction, suppose $\tilde\psi$ is affine along a segment containing $\mathbf{z}_0 \in \Omega$ in its relative interior. The Monge-Ampère measure $M_{\tilde\psi}$ is doubling on $\overline{\Omega}$, with doubling neighbourhood $\mathbf{R}^n$, by the conclusion of Lemma 7.5. Thus Lemma 7.6 applies to the section $Z_\epsilon(\mathbf{z}_0)$ centered at $\mathbf{z}_0$ for each $\epsilon > 0$. The section $Z_\epsilon(\mathbf{z}_0)$ contains a segment $[\mathbf{z}_{-1}, \mathbf{z}_1]$ around $\mathbf{z}_0 = (\mathbf{z}_{+1} + \mathbf{z}_{-1})/2$ along which $\tilde\psi$ is affine. Choosing $\bar t \in]0,1[$ small enough, ensures $\mathbf{z}_{\pm1} \not\in \bar t \cdot Z_\epsilon(\mathbf{z}_0)$. Thus $s \le s_0(0, \bar t)$ from the preceding lemma yields $\mathbf{z}_{\pm1} \not\in Z_{s\epsilon}(\mathbf{z}_0)$. But then the affine function $L : \mathbf{R}^n \longrightarrow \mathbf{R}$ defining $Z_{s\epsilon}(\mathbf{z}_0) := \{\mathbf{x} \mid \psi < L\}$ must be less than the affine restriction of $\tilde\psi$ to the entire segment $[\mathbf{z}_{-1}, \mathbf{z}_1]$, contradicting $\mathbf{z}_0 \in Z_{s\epsilon}(\mathbf{z}_0)$. We conclude strict convexity of $\tilde\psi$ on Ω. QED.

This fundamental lemma will shortly be used to show that solutions to our Monge-Ampère obstacle problem are p-uniformly convex, up to the fixed boundary.

Definition 7.9 (p-uniform convexity). *Fix $p \ge 2$ and a domain $\Omega \subset \mathbf{R}^n$. A locally Lipschitz function $\psi : \Omega \longrightarrow \mathbf{R}$, is p-uniformly convex on Ω if there exists $k < \infty$ such that all points of differentiability $\mathbf{x}, \mathbf{x}' \in \Omega \cap \operatorname{dom} \nabla\psi$ satisfy*

$$\langle \nabla\psi(\mathbf{x}) - \nabla\psi(\mathbf{x}'), \mathbf{x} - \mathbf{x}' \rangle \ge k^{1-p} |\mathbf{x} - \mathbf{x}'|^p. \tag{7.14}$$

Since p-uniform convexity quantifies injectivity of the map $\mathbf{y} = \nabla\psi(\mathbf{x})$, the following standard result provides the desired modulus of continuity for the inverse map $\mathbf{x} = \nabla\psi^{-1}(\mathbf{y})$.

Remark 7.10 (Hölder inverse map). *If a convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ is p-uniformly convex on the domain $\Omega \subset \operatorname{dom}\psi$ for some $p \ge 2$, its Legendre transform $\psi^* \in C^{1,\alpha}(\Omega^*)$ has a Hölder continuous gradient on $\Omega^* = \partial\psi(\Omega)$, with Hölder exponent $\alpha = 1/(p-1)$ and Hölder constant k given by (7.14).*

To prove our main theorem requires one more lemma, asserting that the lowest sections converge to a point $\{\mathbf{z}\} = \lim_{\epsilon \to 0} Z_\epsilon(\mathbf{z})$ in Hausdorff distance.

LEMMA 7.11 (UNIFORM LOCALIZATION). *Fix data $(f, g, h = k)$ of convex support $\Omega := \operatorname{int}[\operatorname{spt} f]$ and $\Lambda := \operatorname{int}[\operatorname{spt} g]$. Let $\tilde\psi \in C^1(\mathbf{R}^n)$ be the differentiable extension from Theorem 6.3 of a weak-$*$ solution ψ to the obstacle problem (4.1). Define its centered affine sections $Z_\epsilon(\mathbf{z})$ as in (7.8). Set $U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi > h\}$. For $R > 0$ taking $\epsilon_0 > 0$ small enough implies $Z_{\epsilon_0}(\mathbf{z}) \subset B_R(\mathbf{z})$ for all $\mathbf{z} \in \overline{\Omega \cap U_\psi}$ such that $B_R(\mathbf{z})$ contains no preimages $X_{nt} := \overline{\Omega \cap U_\psi} \cap \nabla\tilde\psi^{-1}(\partial_{nt}\Lambda)$ of tangential intersections of free with fixed boundary (7.2).*

Proof: To produce a contradiction, suppose for some $R > 0$ there exists a sequence $\mathbf{z}^k \in \overline{\Omega \cap U_\psi}$ with $B_R(\mathbf{z}^k)$ disjoint from X_{nt} and $\epsilon(k) \to 0$ such that $Z_{\epsilon(k)}(\mathbf{z}^k) \not\subset B_R(\mathbf{z}^k)$. Extracting a subsequence if necessary yields a limit $\mathbf{z}^k \to \mathbf{z}^\infty$ with the open ball $B_R(\mathbf{z}^\infty)$ still disjoint from X_{nt}. Translating all of the data by $\nabla\tilde\psi(\mathbf{z}^\infty)$, it costs no generality to assume $\nabla\tilde\psi(\mathbf{z}^\infty) = \mathbf{0}$, so that $\tilde\psi$ achieves its minimum value at $\mathbf{z}^\infty$. Since $\nabla\tilde\psi(\mathbf{R}^n) = \overline{\Lambda}$ is bounded in Theorem 6.3, and each section (7.8) is bounded, the slope $\mathbf{v}_{\epsilon(k)}(\mathbf{z}^k)$ of the affine function defining $Z_{\epsilon(k)}(\mathbf{z}^k)$ must lie in Λ. Extracting another subsequence ensures that these slopes converge to a limit $\mathbf{v}_{\epsilon(k)}(\mathbf{z}^k) \to \mathbf{v}_\infty \in \overline{\Lambda}$, while the sections $Z_{\epsilon(k)}(\mathbf{z}^k)$ converge locally in Hausdorff distance to a closed convex set $Z_\infty \subset \mathbf{R}^n$ (by the Blaschke selection theorem [73]). Define $Z_{min} = \arg\min\tilde\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \tilde\psi(\mathbf{x}) = \tilde\psi(\mathbf{z}^\infty)\}$.

Claim #1: $Z_\infty \subset Z_{min}$, and contains a segment L of length $2R/\alpha$ centered at $\mathbf{z}^\infty$.

Proof of claim: Setting $Z_0 := \{\mathbf{x} \mid \tilde\psi(\mathbf{x}) \le \tilde\psi(\mathbf{z}^\infty) + \langle \mathbf{v}_\infty, \mathbf{x} - \mathbf{z}^\infty\rangle\}$ and taking the limit $k \to \infty$ in the definition (7.8) of the k-th section yields $Z_\infty \subset Z_0$. Since centered affine sections are convex bodies, John's Lemma A.3 implies $\mathbf{z}^\infty - \mathbf{x}/\alpha \in Z_\infty$ if $\mathbf{z}^\infty + \mathbf{x} \in Z_\infty$. Now $\tilde\psi(\mathbf{x}) \ge \tilde\psi(\mathbf{z}^\infty)$ implies Z_0 lies on one side $0 \le \langle \mathbf{v}_\infty, \mathbf{x} - \mathbf{z}^\infty\rangle$ of a hyperplane through $\mathbf{z}^\infty$ unless $\mathbf{v}_\infty = \mathbf{0}$. In either case $Z_\infty \subset Z_0$ — being roughly balanced around $\mathbf{z}^\infty$ — lies in $\mathbf{z}^\infty + \{\mathbf{v}_\infty\}^\perp$, the subspace orthogonal to $\mathbf{v}_\infty$. The inequality $\tilde\psi(\mathbf{x}) \ge \tilde\psi(\mathbf{z}^\infty)$ becomes an equality on $Z_0 \cap \mathbf{z}^\infty + \{\mathbf{v}_\infty\}^\perp$, thus $Z_\infty \subset Z_0 \cap \mathbf{z}^\infty + \{\mathbf{v}_\infty\}^\perp \subset Z_{min}$ as desired. Finally, since the convex set $Z_\infty \not\subset B_R(\mathbf{z}^\infty)$, it must contain a segment of length $2R/\alpha$ centered at $\mathbf{z}^\infty$. *End of claim.*

Clearly $\nabla\tilde\psi(\mathbf{x}) = \mathbf{0}$ throughout the set Z_{min} where $\tilde\psi$ is minimized. Theorem 6.3 asserts $\nabla\tilde\psi : \overline{\Omega \cap U_\psi} \longrightarrow \overline{\Lambda \cap U_{\psi^*}}$ is homeomorphic, so Z_{min} cannot intersect the active domain except at the single point $\mathbf{z}^\infty$. Thus $\mathbf{z}^\infty$ must lie on the boundary of $\overline{\Omega \cap U_\psi}$, with the rest of the segment L (and indeed all of Z_{min}) outside it. We shall need to locate the exposed points of the closed convex set Z_{min}. Recall a point $\mathbf{p} \in Z_{min}$ is *exposed* if some hyperplane touches Z_{min} only at $\mathbf{p}$.

Claim #2: The exposed points of Z_{min} lie in the support of the Monge-Ampère measure $\mathrm{M}_{\tilde\psi}$.

Proof of claim: Let $\mathbf{p}$ be an exposed point of Z_{min}. Then some affine function $A(\mathbf{x}) = \langle \hat{\mathbf{n}}, \mathbf{x} - \mathbf{p}\rangle$ takes negative values on $Z_{min} \setminus \{\mathbf{p}\}$. Given $r > 0$, we claim $\delta > 0$ sufficiently small ensures the convex section

$$S_\delta := \{\mathbf{x} \in \mathbf{R}^n \mid \tilde{\psi}(\mathbf{x}) < \tilde{\psi}(\mathbf{p}) + \delta A(\mathbf{x}) + \delta^2\} \tag{7.15}$$

lies in $B_r(\mathbf{p})$. If not, there exists a positive sequence $\delta(j) \to 0$ with $\mathbf{x}_j \in S_{\delta(j)} \setminus B_r(\mathbf{p})$. Since $S_{\delta(j)}$ forms a convex neighbourhood of $\mathbf{p}$, it costs no generality to take $\mathbf{x}_j \in \partial B_r(\mathbf{p})$, whence a subsequence — also denoted $\mathbf{x}_j$ — converges to some $\mathbf{x}_\infty \in B_r(\mathbf{p})$. Taking a limit in the equality (7.15) which defines $\mathbf{x}_j \in S_{\delta(j)}$ shows $\tilde{\psi}(\mathbf{x}_\infty) \le \tilde{\psi}(\mathbf{p})$. Thus $\mathbf{x}_\infty \in Z_{min} \cap \partial B_r(\mathbf{p})$ and $A(\mathbf{x}_\infty) < 0$. Taking j large enough yields $A(\mathbf{x}_j) < A(\mathbf{x}_\infty)/2 < 0$ and $\delta(j) < -A(\mathbf{x}_\infty)/2$. Inequality (7.15) then asserts $\tilde{\psi}(\mathbf{x}_j) < \tilde{\psi}(\mathbf{p})$, contradicting $\mathbf{p} \in Z_{min}$. The only logical escape is $S_\delta \subset B_r(\mathbf{p})$ for δ sufficiently small. Corollary A.2 shows $\mathrm{M}_{\tilde{\psi}}[S_\delta] > 0$. Since $B_r(\mathbf{p})$ can be taken arbitrarily small, we conclude $\mathbf{p} \in \operatorname{spt} \mathrm{M}_{\tilde{\psi}}$ as desired. *End of claim.*

Claim #2 combines with (7.5) to show all exposed points of Z_{min} lie in $\overline{\Omega \cap U_\psi}$ or $\overline{\Lambda \cap U_{\psi^*}}$. The segment L of Claim #1 shows the sole point $\mathbf{z}^\infty \in Z_{min} \cap \overline{\Omega \cap U_\psi}$ cannot be exposed in Z_{min}, so all exposed points of Z_{min} are contained in the compact convex set $\overline{\Lambda}$. Being limits of exposed points, the extreme points of Z_{min} also lie in $\overline{\Lambda}$ according to Straszewicz' theorem [68, §18.6]. Here *extreme* means $\mathbf{p} \in Z_{min}$ cannot be expressed as a convex combination $\mathbf{p} = (1-\lambda)\mathbf{p}_0 + \lambda \mathbf{p}_1$ of points $\mathbf{p}_0, \mathbf{p}_1 \in Z_{min}$ with $\lambda \in]0,1[$ unless $\mathbf{p}_0 = \mathbf{p}_1$. Similarly, a *direction* $\mathbf{q}$ in the *recession cone* $\mathrm{rc}[Z_{min}] := \lim_{\lambda\downarrow 0} \lambda Z_{min}$ is *extreme* if $\mathbf{q} = (1-\lambda)\mathbf{q}_0 + \lambda\mathbf{q}_1$ with $\lambda \in]0,1[$ forces $\mathbf{q}_0$ to be non-negative scalar multiple of $\mathbf{q}_1$ or vice versa.

A variation of Minkowksi's theorem given by Rockafellar [68, §18.5] asserts that any closed convex set which does not contain a full line, can be expressed as the convex hull of its extreme points plus its extreme directions: $Z_{min} = \operatorname{conv}\operatorname{ext}[Z_{min}] + \mathrm{rc}[Z_{min}]$. Note that Z_{min} does not contain a full line, since this would limit the dimension of $\{\tilde{\psi}^* < \infty\}$ to $n-1$, violating $\nabla\tilde{\psi}(\mathbf{R}^n) = \overline{\Lambda}$. Thus $\mathbf{z}^\infty = \mathbf{p} + \mathbf{q}$ where $\mathbf{p} \in \overline{\Lambda}$ and $\mathbf{q} \in \mathrm{rc}(Z_{min})$. Observe that $\mathbf{q} \neq \mathbf{0}$ since $\mathbf{z}^\infty \in \overline{\Omega}$ lies a positive distance from $\overline{\Lambda}$ by hypothesis. Thus Z_{min} contains a half-line in direction $\mathbf{q}$; let us choose coordinates in which $\mathbf{q}$ parallels, say, the negative x_n-axis. Gradient monotonicity then forces $\partial\tilde{\psi}/\partial x_n \ge 0$ throughout $\mathbf{R}^n$, so that $\overline{\Lambda} = \nabla\tilde{\psi}(\mathbf{R}^n)$ must lie in the upper halfspace $x_n \ge 0$. Thus Λ has $\hat{\mathbf{n}}_V(\mathbf{0}) = -\mathbf{e}_n$ as an outer normal at $\nabla\tilde{\psi}(\mathbf{z}^\infty) = \mathbf{0} \in \overline{\Lambda \cap U_{\psi^*}}$. Two cases remain to consider: either (a) $\nabla\tilde{\psi}(\mathbf{z}^\infty) = \mathbf{0}$ lies on the free boundary $\overline{\Lambda \cap \partial U_{\psi^*}}$, (b) or not. We address the second case first.

Case (b): $\mathbf{0} \notin \overline{\Lambda \cap \partial U_{\psi^*}}$ (not a free boundary point). In this case there is a small ball $B_\delta(\mathbf{0})$ which does not intersect the free boundary $\Lambda \cap \partial U_{\psi^*}$. For any $\mathbf{y} \in \Lambda \cap B_\delta(\mathbf{0})$, the segment joining $\mathbf{0}$ to $\mathbf{y}$ will lie in the non-empty convex set $\Lambda \cap B_\delta(\mathbf{0}) \subset \Lambda \cap U_{\psi*}$. For $\lambda \in]0,1[$, monotonicity then yields

$$\lambda \langle \nabla \tilde{\psi}^{-1}(\lambda \mathbf{y}) - \mathbf{p},\, \mathbf{y} - \mathbf{0} \rangle \geq 0,$$

from which we recover $\langle \mathbf{z}^\infty - \mathbf{p},\, \mathbf{y} \rangle \geq 0$ in the limit $\lambda \downarrow 0$. Since $\mathbf{q} = \mathbf{z}^\infty - \mathbf{p}$ parallels the negative x_n-axis, this contradicts the fact that $\mathbf{y} \in \Lambda \subset \{x_n > 0\}$ lies in the upper halfspace.

Case (a): $\mathbf{0} \in \overline{\Lambda \cap \partial U_{\psi^*}}$ (at the intersection of the fixed with the free boundary). Since $\mathbf{z}^\infty \notin X_{nt}$, (7.2) asserts $\mathbf{z}^\infty = \mathbf{z}^\infty - \nabla\psi(\mathbf{z}^\infty)$ is not an outward normal to Λ at $\mathbf{0}$. It follows that the intersection $\Lambda \cap B_{|\mathbf{z}^\infty|}(\mathbf{z}^\infty)$ is non-empty; it is manifestly convex, and contained in $\Lambda \cap U_{\psi^*}$ according to Lemma 6.8. For a line segment joining $\mathbf{0}$ to $\mathbf{y} \in \Lambda \cap B_{|\mathbf{z}^\infty|}(\mathbf{z}^\infty)$, the argument of case (b) now yields the same contradiction. The conclusion must be that for $\epsilon_0 > 0$ small enough, $Z_{\epsilon_0}(\mathbf{z}) \subset B_R(\mathbf{z})$ as desired. QED.

Remark 7.12. *If $\partial_{nt}\Lambda$ is empty, the constant $\epsilon_0(\psi, R)$ of the preceding lemma can be shown to depend on ψ only through the coarse geometrical parameters of the problem: the distance separating the convex domains Ω and Λ and their inner and outer radii, the universal constant δ_0 of (7.4), and the minimal angle separating the free inward normal from the fixed outward normal among points of intersection $\partial\Lambda \cap \overline{\Lambda \cap \partial U_{\psi^*}}$ between the target's fixed and the free boundaries. This is established by repeating the proof given above, for any sequence of data (f_k, g_k, h) and solutions ψ_k sharing the same coarse parameters, thus permitting extraction of a subsequential limit.*

Theorem 7.13 (p-uniform convexity along convex boudaries). *Fix data (f, g, h) of convex support $\Omega := \mathrm{int}\,[\mathrm{spt}\, f]$ and $\Lambda := \mathrm{int}\,[\mathrm{spt}\, g]$. For a weak-$*$ solution ψ to the obstacle problem (4.1), let $U_\psi = \{\mathbf{x} \in \mathbf{R}^n \mid \psi > h\}$. Given $R > 0$ and $\mathbf{x} \in \overline{\Omega \cap U_\psi}$, ψ will be p-uniformly convex (7.17) on $\Omega \cap U_\psi \cap B_{r/2}(\mathbf{x})$ if $B_{2R}(\mathbf{x})$ is disjoint from $\Lambda \cup X_{nt}$ and has convex intersection with $\Omega \cap U_\psi$, where $\epsilon_0 = \epsilon_0(\psi, R)$, X_{nt}, and $r = \beta {\epsilon_0}^{n/2}/R^{n-1}$ are from Lemmas 7.11 and A.5. The convexity exponent $p = \log s_0(0, \bar{t})/\log \bar{t}$ from Lemma 7.6 is universal, as is the constant k times $(\epsilon_0/R^p)^{1/(p-1)}$.*

Proof: Fix $R > 0$, and $\mathbf{x} \in \overline{\Omega \cap U_\psi}$ such that $B_{2R}(\mathbf{x})$ is disjoint from $\Lambda \cup X_{nt}$ and has convex intersection with $\Omega \cap U_\psi$. Extend ψ to $\tilde{\psi} \in C^1(\mathbf{R}^n)$ as in Theorem 6.3. According to Lemma 7.5, the Monge-Ampère measure $M_{\tilde{\psi}}$ has a doubling neighbourhood $B_R(\mathbf{z})$ around each $\mathbf{z} \in \overline{\Omega \cap U_\psi} \cap B_R(\mathbf{x})$, where it doubles affinely with a universal constant. Set $X = \mathrm{spt}\, M_{\tilde{\psi}} = \overline{\Omega \cap U_\psi} \cup \overline{\Lambda \setminus U_{\psi^*}}$, from Lemma 7.2. Since $\Omega \cap U_\psi \cap B_{2R}(\mathbf{x})$ is hypothesized to be convex, so is $X \cap B_R(\mathbf{z}) = \overline{\Omega \cap U_\psi} \cap B_R(\mathbf{z})$.

Choose $\epsilon_0 > 0$ and $r = \beta {\epsilon_0}^{n/2}/R^{n-1}$ from Lemmas 7.11 and A.5 to ensure $B_r(\mathbf{z}) \subset Z_{\epsilon_0}(\mathbf{z})$ and $Z_{s\epsilon_0}(\mathbf{z}) \subset B_R(\mathbf{z})$ for all $\mathbf{z} \in X \cap B_R(\mathbf{x})$ and $s \in]0,1]$. Since

$B_R(\mathbf{z})$ is an affine doubling neighbourhood for $M_{\tilde\psi}$ around $\mathbf{z} \in X \cap B_R(\mathbf{x})$, and the sets $X \cap Z_{s\epsilon_0}(\mathbf{z})$ are all convex, we are in a position to apply Lemma 7.6. As in [18, Corollary 2.3], our first goal is to deduce the following expression of strict convexity.

Claim #1: Fix $t \in]0,1[$ so that $t/(1-t) \geq n^{3/2} =: \alpha$ and the corresponding $s_0(t,1)$ from Lemma 7.6; here α denotes F. John's (universal) balancing constant (Lemma A.3). Every $\epsilon \in]0,\epsilon_0]$, $\mathbf{z}_0 \in X \cap B_R(\mathbf{x})$ and $\mathbf{z}_1 \in X \cap \partial Z_\epsilon(\mathbf{z}_0)$ satisfy

$$\psi(\mathbf{z}_1) \geq \psi(\mathbf{z}_0) + \langle \nabla\psi(\mathbf{z}_0), \mathbf{z}_1 - \mathbf{z}_0 \rangle + \epsilon s_0(t,1)/t \tag{7.16}$$

Proof of claim: Translating the data (f,g), it costs no loss of generality to assume $\nabla\psi(\mathbf{z}_0) = \mathbf{0}$. Now $\mathbf{z}_t := (1-t)\mathbf{z}_0 + t\mathbf{z}_1 \in t \cdot \overline{Z_\epsilon(\mathbf{z}_0)}$, where $t/(1-t) \geq \alpha$ is still fixed as above. Thus $Z_{s_0\epsilon}(\mathbf{z}_t) \subset Z_\epsilon(\mathbf{z}_0)$ by Lemma 7.6 (and A.8), with $s_0 = s_0(t,1)$. In particular, $\mathbf{z}_1 \in \partial Z_\epsilon(\mathbf{z}_0)$ cannot be an interior point of $Z_{s_0\epsilon}(\mathbf{z}_t)$; nor can $\mathbf{z}_0$, for our choice of t would then force $\mathbf{z}_1$ also to be an interior point (since the affine section $Z_{s_0\epsilon}(\mathbf{z}_t) = \{\psi < L\}$ is roughly balanced around its center $\mathbf{z}_t$). If $L(\mathbf{y})$ is the affine function defining this section, we know $L(\mathbf{z}_0) \leq \psi(\mathbf{z}_0)$ and $L(\mathbf{z}_1) \leq \psi(\mathbf{z}_1)$, but $L(\mathbf{z}_t) = \psi(\mathbf{z}_t) + s_0\epsilon \geq \psi(\mathbf{z}_0) + s_0\epsilon$ since $\nabla\psi(\mathbf{z}_0) = \mathbf{0}$. Along the segment joining $\mathbf{z}_0$ to $\mathbf{z}_t$, the slope of L is at least $s_0\epsilon/|\mathbf{z}_t - \mathbf{z}_0|$; by the time it reaches $\mathbf{z}_1$, this linear function will have attained a value $L(\mathbf{z}_1) \geq L(\mathbf{z}_t) + |\mathbf{z}_1 - \mathbf{z}_t| s_0\epsilon/|\mathbf{z}_t - \mathbf{z}_0|$. The desired estimate follows:

$$\psi(\mathbf{z}_1) \geq L(\mathbf{z}_1) \geq L(\mathbf{z}_t) + s_0\epsilon\frac{1-t}{t} \geq \psi(\mathbf{z}_0) + s_0\epsilon/t.$$

End of claim.

Our next claim completes the proof of the theorem:

Claim #2: Given $\bar t \in]0,1[$, take $s_0(0,\bar t)$ from Lemma 7.6. Then every $\mathbf{z}_0, \mathbf{z}_1 \in X \cap B_{r/2}(\mathbf{x})$ satisfy

$$\langle \nabla\psi(\mathbf{z}_1) - \nabla\psi(\mathbf{z}_0), \mathbf{z}_1 - \mathbf{z}_0 \rangle \geq \epsilon_0 \frac{s_0((\alpha^{-1}+1)^{-1},1)}{(\alpha^{-1}+1)^{-1}} \left(\frac{\bar t|\mathbf{z}_1 - \mathbf{z}_0|}{R}\right)^{\frac{\log s_0(0,\bar t)}{\log \bar t}}. \tag{7.17}$$

Proof of claim: Given $\mathbf{z}_0, \mathbf{z}_1 \in X \cap B_{r/2}(\mathbf{x})$, we have $\mathbf{z}_1 \in B_r(\mathbf{z}_0) \subset Z_{\epsilon_0}(\mathbf{z}_0)$. We assume $\mathbf{z}_1 \neq \mathbf{z}_0$, since otherwise there is nothing to prove. The centered affine sections $Z_{\epsilon_0}(\mathbf{z}_0)$ vary continuously with ϵ_0, and they tend to $\mathbf{z}_0$ in the limit $\epsilon_0 \to 0$, according to Lemmas 7.11 and A.8. Thus there exists some $\epsilon \leq \epsilon_0$ such that $\mathbf{z}_1 \in \partial Z_\epsilon(\mathbf{z}_0)$. Summing $\psi(\mathbf{z}_0) \geq \psi(\mathbf{z}_1) + \langle \nabla\psi(\mathbf{z}_1), \mathbf{z}_0 - \mathbf{z}_1 \rangle$ with the conclusion (7.16) of Claim #1 yields

$$\langle \nabla\psi(\mathbf{z}_1) - \nabla\psi(\mathbf{z}_0), \mathbf{z}_1 - \mathbf{z}_0 \rangle \geq \epsilon s_0(t,1)/t. \tag{7.18}$$

with $t = (\alpha^{-1}+1)^{-1}$. It remains to show ϵ dominates a certain power of $|\mathbf{z}_0 - \mathbf{z}_1|$.

Recall $s \leq s_0(0,\bar{t})^k$ implies $Z_{s\epsilon_0}(\mathbf{z}_0) \subset \bar{t}^k \cdot Z_{\epsilon_0}(\mathbf{z}_0)$ for all $k \in \mathbf{N}$ from Corollary 7.7. Set $s = \epsilon/\epsilon_0$, and let $k \geq 0$ be the integer satisfying $\frac{\log s}{\log s_0(0,\bar{t})} \in [k, k+1[$. Then $s \leq s_0(0,\bar{t})^k$, and since $\mathbf{z}_1 \in \partial Z_{s\epsilon_0}(\mathbf{z}_0) \subset \bar{t}^k \cdot \overline{Z_{\epsilon_0}(\mathbf{z})}$ we find

$$\begin{aligned}|\mathbf{z}_1 - \mathbf{z}_0| &\leq \bar{t}^{\frac{\log s}{\log s_0(0,\bar{t})}-1} \max_{\mathbf{z}\in\partial Z_{\epsilon_0}(\mathbf{z}_0)} |\mathbf{z} - \mathbf{z}_0| \\ &\leq s^{\frac{\log \bar{t}}{\log s_0(0,\bar{t})}} R/\bar{t}.\end{aligned}$$

Recalling $s = \epsilon/\epsilon_0$, this combines with (7.18) to complete claim (7.17) and the theorem. *End of Claim.* QED.

Hölder continuity of the map $\nabla\tilde{\psi}$ up to the free boundary $\Omega \cap \partial U_\psi$ follows from Remark 7.10. By exchange symmetry $f(\mathbf{x}) \leftrightarrow g(\mathbf{y})$, it is equivalent to show Hölder continuity of the inverse map $\nabla\psi^*$ up to $\Lambda \cap \partial U_{\psi^*}$; in fact we show Hölder continuity of $\nabla\psi^*$ globally, away from the image $\nabla\tilde{\psi}(\partial_{nc}U_\psi)$ of any nonconvexities on the first free boundary, and from any points $\partial_{nt}\Lambda$where the second free and fixed boundaries intersect tangentially.

Corollary 7.14 (Hölder continuous map up to free boundary). *Fix data $(f, g, h = k)$ of convex support $\Omega = \operatorname{int}[\operatorname{spt} f]$ and $\Lambda = \operatorname{int}[\operatorname{spt} g]$. Set $U_\psi := \{\mathbf{x} \in \mathbf{R}^n \mid \psi > h\}$, where ψ is a weak-$*$ solution to the obstacle problem (4.1). Then the Legendre transform of ψ is Hölder differentiable $\psi^* \in C^{1,\alpha}_{loc}\left(\overline{\Lambda \cap U_{\psi^*}} \setminus F\right)$ away from $F = \partial\psi\left(\partial_{nc}U_\psi\right) \cup \partial_{nt}\Lambda \subset \mathbf{R}^n \setminus \Lambda$, with $\alpha = 1/(p-1)$ universal from Theorem 7.13.*

Proof: Let $\tilde{\psi} \in C^1(\mathbf{R}^n)$ extend ψ as in Theorem 6.3, so that $\nabla\tilde{\psi} : \overline{\Omega \cap U_\psi} \longrightarrow \overline{\Lambda \cap U_{\psi^*}}$ is a homeomorphism. Any $\mathbf{y} \in \overline{\Lambda \cap U_{\psi^*}} \setminus F$ is the image $\mathbf{y} = \nabla\tilde{\psi}(\mathbf{x})$ of some $\mathbf{x} \in \overline{\Omega \cap U_\psi}$ which lies in a ball $B_{2R}(\mathbf{x})$ whose intersection with the active domain $\Omega \cap U_\psi$ is convex. Taking $R > 0$ smaller if necessary ensures $B_{2R}(\mathbf{x})$ disjoint from $X_{nt} := \overline{\Omega \cap U_\psi} \cap \nabla\tilde{\psi}^{-1}(\partial_{nt}\Lambda)$. Theorem 7.13 provides a neighbourhood $X := B_{r/2}(\mathbf{x}) \cap \overline{\Omega \cap U_\psi}$ on whose interior ψ is p-uniformly convex, with p universal. Thus the gradient of ψ^* is Hölder continuous with exponent $\alpha = 1/(p-1)$, on the interior (hence on the closure) of the neighbourhood $\nabla\tilde{\psi}(X)$ of $\mathbf{y}$ relatively open in $\overline{\Lambda \cap U_{\psi^*}}$, according to Remark 7.10. QED.

A final corollary shows that the normal to the free boundary is Hölder continuous in the interior of Ω — and up to those points of $\partial\Omega$ where the fixed and free boundaries intersect transversally, and which do not map to non-locally-convex intersection points $\partial\Lambda \cap \partial_{nc}U_{\psi^*}$ of the target's fixed and free boundaries.

Corollary 7.15 (Hölder continuity of free boundary normal). *Fix data $(f, g, h = k)$ of convex support $\Omega = \operatorname{int}[\operatorname{spt} f]$ and $\Lambda = \operatorname{int}[\operatorname{spt} g]$. Let*

$\tilde\psi \in C^1(\mathbf{R}^n)$ from Theorem 6.3 extend a weak-$*$ solution ψ to the obstacle problem (4.1), and set $U_\psi = \{\mathbf{x} \mid \psi > h\}$. Then the normal $\mathbf{n}_{U_\psi}(\mathbf{x}) = \mathbf{x} - \nabla\tilde\psi(\mathbf{x})$ to the free boundary is Hölder continuous: $\hat{\mathbf{n}}_{U_\psi} := \mathbf{n}_{U_\psi}/|\mathbf{n}_{U_\psi}| \in C^{1,\alpha}_{loc}(\Omega \cap \partial U_\psi) \cap C^{1,\alpha}_{loc}\left(\overline{\Omega \cap \partial U_\psi} \setminus E\right)$ where $E := \nabla\tilde\psi^{-1}(\partial\Lambda \cap \partial_{nc} U_{\psi^*}) \cup \partial_{nt}\Omega$ is defined by (7.1)–(7.2).

Proof: From Corollary 7.14 and the exchange symmetry $f(\mathbf{x}) \leftrightarrow g(\mathbf{y})$, we infer the gradient of $\tilde\psi$ is Hölder continuous on compact subsets of $\overline{\Omega \cap \partial U_\psi}$ disjoint from $\partial_{nt}\Omega \cup \nabla\tilde\psi^{-1}(\partial_{nc}U_{\psi^*})$. According to Lemma 6.8, $\Omega \cap \partial U_\psi$ is disjoint from $\nabla\tilde\psi^{-1}\left(\overline{\Lambda \cap \partial U_{\psi^*}}\right)$; similarly $\overline{\Omega \cap \partial U_\psi}$ is disjoint from $\nabla\tilde\psi^{-1}(\Lambda \cap \partial U_{\psi^*})$. Since $\partial_{nc}U_{\psi^*} \subset \overline{\Lambda \cap \partial U_{\psi^*}}$ we conclude the map $\nabla\tilde\psi$ is Hölder continuous on the free boundary outside of E. The outer normal $\mathbf{n}_{U_\psi}(\mathbf{x}) = \mathbf{x} - \nabla\tilde\psi(\mathbf{x})$ to the free domain was identified in Remark 6.5; it is non-vanishing by the positive separation hypothesized for Ω and Λ. Thus the corollary is established. QED.

A. Background estimates for centered sections

This appendix is devoted to recalling and refining some central aspects of Caffarelli's $C^{1,\alpha}$ regularity theory for the Monge-Ampère equation [16]. In particular we show that the centered affine section (7.8) of height ϵ above any point $(\mathbf{x}, \psi(\mathbf{x}))$ in the graph of a convex function is uniquely defined and depends continuously on ϵ provided $\mathbf{x} \in \operatorname{int}[\operatorname{dom}\psi]$. The central estimates concerning such sections are also recalled. We use $\omega_n = |\mathbf{S}^{n-1}|$ to denote the $(n-1)$-dimensional area of the unit sphere in $\mathbf{R}^n$.

Lemma A.1 (Aleksandrov estimate and lower barrier). *If $\psi : \mathbf{R}^n \longrightarrow\ ]-\infty, \infty]$ is convex, lower semi-continuous, and $\mathbf{x} \in Z := \{\psi < 0\} \subset \overline{B_R(\mathbf{0})}$, then*

$$\psi(\mathbf{x}) \geq -\left(\mathrm{M}_\psi(Z)\frac{n(2R)^{n-1}}{\omega_{n-1}}\operatorname{dist}_{\partial Z}(\mathbf{x})\right)^{1/n}. \tag{A.1}$$

Proof: Compare ψ with a cone v with vertex at $(\mathbf{x}, \psi(\mathbf{x}))$ sharing the same zero set ∂Z. Then

$$\partial\psi(Z) \supset \partial v(\mathbf{x}) \supset B_{\frac{-v(z)}{2R}}(\mathbf{0}) \cup \left\{\frac{-\psi(\mathbf{x})}{\operatorname{dist}_{\partial Z}(\mathbf{x})}\hat{\mathbf{e}}\right\}$$

for some unit vector $\hat{\mathbf{e}} \in \mathbf{R}^n$. Since the last set contains a right circular cone of volume $(Base)(height)/n$ we estimate

$$\mathrm{M}_\psi(Z) \geq \operatorname{vol}[\partial v(\mathbf{x})] \geq \frac{\omega_{n-1}}{n}\left(\frac{-\psi(\mathbf{x})}{2R}\right)^{n-1}\left(\frac{-\psi(\mathbf{x})}{\operatorname{dist}_{\partial Z}(\mathbf{x})}\right)$$

to conclude the proof. QED.

Corollary A.2 (Bounded sections have positive MA mass). *Let $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ be proper and convex, with $S := \{\mathbf{x} \mid \psi(\mathbf{x}) < \langle \mathbf{v}, \mathbf{x}\rangle + \epsilon\}$ bounded and non-empty for some $\mathbf{v} \in \mathbf{R}^n$ and $\epsilon \in \mathbf{R}$. Then $\mathrm{M}_\psi(S) > 0$.*

Proof: Let $\mathbf{z} \in S$ and apply the lemma to the convex function $\tilde\psi(\mathbf{x}) := \psi(\mathbf{x}) - \langle \mathbf{v}, \mathbf{x}\rangle - \epsilon$, so $\mathrm{M}_\psi = \mathrm{M}_{\tilde\psi}$ and $Z = S$. Since $\tilde\psi(\mathbf{z}) < 0$ we conclude $\mathrm{M}_\psi(Z) > 0$ in (A.1). QED.

The next lemma is a version of a theorem by Fritz John [47], adapted to ellipsoids with fixed center of mass. The sharp constant $\alpha = n^{3/2}$ for this version may be found in Gutiérrez [45], where much of this theory is described.

Lemma A.3 (Normalization of convex bodies [16, Lemma 2]). *There is a universal constant $\alpha \geq n$, such that each bounded convex domain $\Omega \subset \mathbf{R}^n$ with barycenter at the origin contains an ellipsoid $E \subset \Omega$, also centered at the origin, whose dilation by $\alpha = n^{3/2}$ encloses $\Omega \subset \alpha \cdot E$.*

Lemma A.4 (Doubling property implies upper barrier). *Suppose $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ convex, lower semi-continuous, attains its minimum value at $\mathbf{y}$. If $Z := \{\psi < 0\}$ is a bounded set with positive volume and $\mathbf{0}$ as its barycenter, then $1 \leq \psi(\mathbf{y})/\psi(\mathbf{0}) \leq \alpha + 1$ where $\alpha = \alpha(n)$ is the balancing constant of Lemma A.3. Moreover, the doubling condition (A.3) on the Monge-Ampère measure M_ψ provides $\delta > 0$ for which*

$$c \leq \mathrm{vol}\,(Z)\mathrm{M}_\psi(Z)/|\psi(\mathbf{y})|^n \leq C/\delta^2 \tag{A.2}$$

where the constants $c = \omega_{n-1}\omega_n/(n2^{n-1}\alpha)$ and $C = \alpha^{n+2}c$ depend on dimension only.

Proof: Since both conclusions are invariant under $\psi \mapsto \psi \circ T$, when $T(\mathbf{x})$ is an affine unit-determinant transformation, Lemma A.3 allows us to assume

$$B_{\alpha r}(\mathbf{0}) \subset Z \subset \overline{B_{\alpha^2 r}(\mathbf{0})}$$

for some $r > 0$ without loss of generality. The estimate $\mathrm{dist}\,(Z/\alpha^2, \mathbf{R}^n \setminus Z) \geq (\alpha - 1)r$ then yields $|\nabla\psi(\mathbf{x})| \leq |\psi(\mathbf{y})|/(\alpha-1)r$ for all $\mathbf{x} \in Z/\alpha^2$. Thus

$$\mathrm{M}_\psi\left(Z/\alpha^2\right) \leq \frac{\omega_n}{(\alpha-1)^n}\frac{|\psi(\mathbf{y})|^n}{r^n}.$$

On the other hand, the preceding lemma yields

$$\mathrm{M}_\psi(Z) \geq \frac{\omega_{n-1}}{n2^{n-1}}\frac{|\psi(\mathbf{y})|^n}{R^{n-1}\mathrm{dist}_{\,\partial Z}(\mathbf{y})}$$

with $R = \alpha^2 r$. If for some $\epsilon > 0$ we have the doubling property

$$\frac{\omega_n \delta^2}{(\alpha-1)^n} \mathrm{M}_\psi(Z) \leq \frac{\omega_{n-1}}{n(2\alpha^2)^{n-1}} \mathrm{M}_\psi(Z/\alpha^2) \tag{A.3}$$

then $\epsilon r \leq \mathrm{dist}_{\partial Z}(\mathbf{y}) \leq \alpha^2 r$ and hence $|\psi(\mathbf{y})| \sim r\mathrm{M}_\psi(Z)^{1/n}$ as desired. In fact, (A.2) holds with $c = \omega_{n-1}\omega_n/(n2^{n-1}\alpha)$ and $C = \alpha^{n+2}c$. Since Z is star-shaped around $\mathbf{y}$, expressing $\mathbf{0} = (1-t)\mathbf{x} + t\mathbf{y}$ as a convex combination of $\mathbf{y}$ and a boundary point $\mathbf{x} \in \partial Z$ yields $\psi(\mathbf{0}) \leq 0 + t\psi(\mathbf{y})$. Now $|\mathbf{x}| \geq \alpha r$ but $|\mathbf{y}| \leq \alpha^2 r$ yields $(1-t)/t \leq \alpha$, hence $t^{-1} \leq \alpha + 1$. Since $\psi(\mathbf{y})$ is a minimum, we have $1 \leq \psi(\mathbf{y})/\psi(\mathbf{0}) \leq t^{-1}$ as desired. QED.

The preceding lemma asserts not only that the maximum height of a convex function over any affine section is comparable to its height at the barycenter, but that, as for a parabola, the section volume corresponds to the $\frac{n}{2}$th power of this height, provided only that the Monge-Ampère measure of the section is comparable to its volume, and doubles affinely around its barycenter. Since the two latter properties can be deduced when ψ satisfies a Monge-Ampère equation with appropriate right hand side, this already hints at a regularity theory. The next lemma refines this observation to show that a sequence of sections at any given height cannot become arbitrarily thin, unless they simultaneously become arbitrarily long. At a given height, bounded sections therefore have bounded eccentricity.

Lemma A.5 (Sections degenerate doubly or not at all). *Fix $\epsilon_0 > 0$ and a proper convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$. Suppose $\mathbf{v} \in \mathbf{R}^n$ yields a section $Z = Z_{\epsilon_0}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{R}^n \mid \psi(\mathbf{x}) < \psi(\mathbf{z}) + \langle \mathbf{v}, \mathbf{x} - \mathbf{z}\rangle + \epsilon_0\}$ with barycenter at $\mathbf{z} \in \mathrm{int}\,[\mathrm{dom}\,\psi]$. If $\mathrm{vol}\,[Z] \geq \delta_0 M_\psi[Z]$ for some $\delta_0 > 0$, then there is a (universal) constant $\beta := (c\delta_0)^{1/2}/(\omega_n \alpha^n)$ such that $Z_{\epsilon_0}(\mathbf{z}) \subset B_R(\mathbf{z})$ implies $B_{\beta {\epsilon_0}^{n/2} R^{1-n}}(\mathbf{z}) \subset Z_{\epsilon_0}(\mathbf{z})$.*

Proof: Without loss of generality set $\mathbf{z} = \mathbf{0}$ so that $Z = Z_{\epsilon_0}(\mathbf{0})$ is a section of the convex function ψ with height $\epsilon_0 > 0$ over its barycenter $\mathbf{0} \in \mathrm{int}\,[\mathrm{dom}\,\psi]$. Since the height of the section is at least ϵ_0, from $\mathrm{vol}\,[Z] \geq \delta_0 M_\psi[Z]$ and (A.2) we recover

$$c{\epsilon_0}^n < \mathrm{vol}\,[Z]^2/\delta_0; \tag{A.4}$$

for this lower bound we do not need the doubling condition (A.3), as evidenced by the absence of δ. According to (John's) Lemma A.3, there is an ellipsoid centered at the origin with $E \subset Z \subset \alpha E$. The principle axes of this ellipsoid have lengths $a_1 \leq a_2 \leq \cdots \leq a_n \leq R$ if $Z \subset B_R(\mathbf{0})$, so the volume of $Z \subset \alpha E$ can be estimated by

$$\mathrm{vol}\,[Z] \leq \omega_n a_1 R^{n-1} \alpha^n.$$

Combined with (A.4) this yields the desired bound $a_1 \geq \sqrt{(c\delta_0{\epsilon_0}^n)}/(\omega_n \alpha^n R^{n-1}) = \beta{\epsilon_0}^{n/2}/R^{n-1}$. QED.

It is not hard to rule out the possibility of a sequence of centered sections becoming doubly infinite using the following remark. Some of the work concerning Hölder estimates will be devoted to ruling out limiting sections which contain only a half-line.

REMARK A.6. *The subgradient image $\partial\psi(\mathbf{R}^n)$ of a convex function ψ has an empty interior if and only if the graph of ψ contains a straight line, or equivalently, if and only $\psi(x_1, \ldots, x_n)$ is independent of x_1 in some orthogonal basis for $\mathbf{R}^n$.*

Let us finally recall the basic existence result concerning centered affine sections [16, Lemma 1] [17, Theorem 2.2]. We prove uniqueness of these sections and their continuous dependence on height afterwards.

THEOREM A.7 (CENTERED SECTIONS OF A CONVEX FUNCTION). *Let $\psi : \mathbf{R}^n \longrightarrow [0, \infty]$ be a non-negative convex function, continuous at $\psi(\mathbf{0}) < \frac{1}{2}$ and with $\partial\psi(\mathbf{R}^n)$ having non-empty interior. For some affine function $L(\mathbf{x}) = 1 + \langle \mathbf{v}, \mathbf{x}\rangle$ the section $Z_L := \{\mathbf{x} \in \mathbf{R}^n \mid \psi(\mathbf{x}) < L(\mathbf{x})\}$ is bounded, convex, and has zero as its barycenter.*

Proof: First assume ψ is smooth and strictly convex with quadratic growth (A.5) as $|\mathbf{x}| \to +\infty$. Then Z_L is a bounded non-empty convex domain with a first moment vector

$$\mathbf{z}_L := \int_{Z_L} \mathbf{x}\, d\mathrm{vol}\,(\mathbf{x})$$

equal to its volume times its barycenter. We claim that for a suitable choice of $\mathbf{v} \in \mathbf{R}^n$, the first moment $|\mathbf{z}_L|$ achieves its minimum value among all affine functions $L(\mathbf{x}) = 1 + \langle \mathbf{v}, \mathbf{x}\rangle$. This follows by continuity and compactness once the following claim is established.

Claim #1: The moment $|\mathbf{z}_L|$ grows without bound as the slope $|\mathbf{v}| \to \infty$ grows.

Proof of claim: It costs no generality to suppose $\mathbf{v} = (\lambda, 0, \ldots, 0)$ parallels the postive x_1-axis. Decompose $Z_L = Z_L^+ \cup Z_L^-$ into $Z_L^{\pm} = \{(x_1, \ldots, x_n) \in Z_L \mid \pm x_1 > 0\}$. The quadratic growth assumptions assert

$$\frac{2|\mathbf{x}|^2}{R^2} - 1 \leq \psi(\mathbf{x}) \leq \frac{1}{2}(\frac{|\mathbf{x}|^2}{r^2} + 1) \tag{A.5}$$

for some $0 < r < R < \infty$, whence $Z_L^- \subset B_R(\mathbf{0}) \cap \{\mathbf{x} \mid x_1 \geq -1/\lambda\}$ since ψ is also non-negative. Thus

$$\int_{Z_L^-} x_1\, d\mathrm{vol}\,(\mathbf{z}) \geq -\frac{R^{n-1}}{\lambda^2}. \tag{A.6}$$

On the other hand, the convex set $\overline{Z_L^+}$ includes the $n-1$ dimensional ball $B_r^{n-1}(\mathbf{0})$ in the hyperplane $x_1 = 0$, as well as the point $(2\lambda r^2, 0, \ldots, 0)$ on the x_1-axis. Thus Z_L^+ contains a right circular cone along the x_1-axis with a cylinder of diameter r and height λr^2 inside it. It is easy to estimate

$$\int_{Z_L^+} x_1 \, d\mathrm{vol}\,(\mathbf{z}) \geq \frac{\lambda^2 r^4}{2} \omega_{n-1} \left(\frac{r}{2}\right)^{n-1}. \tag{A.7}$$

Summing (A.6)–(A.7) shows that $|z_L|$ diverges with $\lambda \to \infty$. *End of claim.*

Claim #2: The lemma is true for ψ smooth and strictly convex satisfying (A.5).

Proof of claim: A sequence $\mathbf{v}_k \in \mathbf{R}^n$ minimizing $|\mathbf{z}_{L_k}|$ is bounded according to Claim #1, so a subsequence converges to a limit $\mathbf{v}_{k(i)} \to \mathbf{v}_\infty$ for which $|\mathbf{z}_{L_\infty}|$ is a minimum. We need only show $\mathbf{z}_{L_\infty} = \mathbf{0}$. It costs no generality to assume $\mathbf{z}_{L_\infty} = (-\lambda, 0, \ldots, 0)$ with $\lambda \geq 0$. Consider the dependence of the first moment $\mathbf{z}_L = (z_1(\epsilon), \ldots, z_n(\epsilon))$ on $\mathbf{v} = \mathbf{v}_\infty + (\epsilon, 0, \ldots, 0)$. If the derivatives existed and $z_1'(0) > 0$, minimality of $|\mathbf{z}_{L_\infty}|$ would imply

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{|\mathbf{z}_L|^2}{2} = \sum_{j=1}^n z_j(0) z_j'(0) = -\lambda z_1'(0) \tag{A.8}$$

forcing $\lambda = 0$ as desired. Below we shall argue the same conclusion without addressing existence of the derivatives.

Set $Z^\pm(\epsilon) = \{(x_1, \ldots, x_n) \in Z_L \mid \pm x_1 > 0\}$. Now $Z^\pm(\epsilon)$ is a monotone sequence of bounded convex sets, whose union $Z_L = \{(x_1, \ldots, x_n) \mid \tilde{\psi}(\mathbf{x}) < \epsilon x_1\}$ is conveniently expressed in terms of $\tilde{\psi}(\mathbf{x}) := \psi(\mathbf{x}) - L_\infty(\mathbf{x})$. By smoothness and strict convexity, $\nabla\tilde{\psi}(\mathbf{x})$ is bounded away from zero and infinity in a neigbourhood of the compact set ∂Z_{L_∞}. Thus $\mathrm{vol}\,[Z_L \Delta Z_{L_\infty}] \leq K|\epsilon|$ for some large constant K and ϵ small enough. This shows $|z_j(\epsilon)|^2 = O(\epsilon^2)$ for $j \neq 1$. On the other hand, it is clear that $z_1(\epsilon) \geq -\lambda$ for $\epsilon > 0$, since $Z^+(0) \subset Z^+(\epsilon)$ and $Z^-(\epsilon) \subset Z^-(0)$. To quantify this inequality, observe that any maximal interval $\{(x_1, \mathbf{X}) \mid a(\mathbf{X}) < x_1 < b(\mathbf{X})\}$ in $Z^+(0)$ for fixed $\mathbf{X} \in \mathbf{R}^{n-1}$ lies in an interval $\{(x_1, \mathbf{X}) \mid a(\mathbf{X}) < x_1 < b(\mathbf{X}) + \epsilon k(\mathbf{X})\} \subset Z^+(\epsilon)$ which is strictly longer since $\nabla\tilde{\psi}(\mathbf{x})$ is non-vanishing on $\partial Z^+(0)$. Thus $z_1(\epsilon) \geq -\lambda + \epsilon k$ for some $k > 0$, whence $|\mathbf{z}(\epsilon)|^2 = \lambda^2 - 2\lambda k\epsilon + O(\epsilon^2)$. Minimality of $|\mathbf{z}(0)|^2$ establishes the claim $\lambda = 0$. *End of claim.*

A series of approximations will show a general convex function $\psi : \mathbf{R}^n \longrightarrow [0, \infty]$ can be approximated by one which is smooth, strictly convex, and has quadratic growth (A.5). Indeed, let Ω_0 denote the interior of $\mathrm{dom}\,\psi := \{\mathbf{x} \in \mid u(\mathbf{x}) < \infty\}$ and define

$$\psi_\epsilon^1(\mathbf{x}) := \max\{\psi(\mathbf{x}), \frac{\epsilon^{1/2}}{\mathrm{dist}_{\,\mathbf{R}^n \setminus \Omega_0}(\mathbf{x})}\}$$

as a maximum of two convex functions [58, Lemma 4.2], so that its rate of divergence is known at $\partial\Omega_0$. Fix a subdomain $\Omega_\epsilon := \{\mathbf{x} \in B_{1/\epsilon}(\mathbf{0}) \mid \mathrm{dist}_{\,\mathbf{R}^n \setminus \Omega_0}(\mathbf{x}) > \epsilon\}$

bounded and convex, and define $\psi^2_\epsilon(\mathbf{x})$ as a supremum of supporting hyperplanes

$$\psi^2_\epsilon(\mathbf{x}) = \sup\{\langle\, \mathbf{v},\, \mathbf{x}-\mathbf{z}\rangle + \psi^1_\epsilon(\mathbf{z}) \mid \mathbf{z}\in\Omega_\epsilon, \mathbf{v}\in\partial\psi^1_\epsilon(\mathbf{z})\}$$

to be the smallest convex extension of $\psi^1_\epsilon(\mathbf{x})$ from Ω_ϵ to $\mathbf{R}^n$. Then ψ^2_ϵ is globally Lipschitz and converges uniformly to ψ on compact subsets of Ω_0. Strict convexity, smoothness, and quadratic growth are ensured by adding a parabola $\psi^3_\epsilon = \psi^2_\epsilon(\mathbf{x}) + \epsilon|\mathbf{x}|^2$, and convolving with a standard mollifier $\psi_\epsilon = \psi^3_\epsilon * \eta_\epsilon$, neither of which spoils the uniform convergence to ψ on compact subsets of Ω_0. Divergence outside Ω_0 is addressed by the next statement.

Claim #3: For $\epsilon > 0$ small, if $|\mathbf{x}| < \frac{3}{4}\epsilon^{-1}$ but $\mathbf{x}\notin\Omega_0$ then $\psi_\epsilon(\mathbf{x}) \geq \frac{1}{2}\epsilon^{-1/2}$.

Proof of claim: We shall rather prove $\psi^2_\epsilon(\mathbf{x}_0) \geq \epsilon^{-1/2}$ on the part of $B_{1/\epsilon}(\mathbf{0})$ lying outside the convex set Ω_0. The claim then follows since ψ^2_ϵ is non-negative in Ω_0 and our mollifier η_ϵ is spherically symmetric.

Take $0 < \epsilon < 1/2$ small enough that $\psi^2_\epsilon(\mathbf{0}) < 1/2$ and Ω_ϵ contains the origin. Then some $\mathbf{z}_\epsilon \in \partial\Omega_\epsilon$ lies on the segment joining $\mathbf{0}$ to $\mathbf{x}_0 \in B_{1/\epsilon}(\mathbf{0})\setminus\Omega_0$. Since $\operatorname{dist}_{\mathbf{R}^n\setminus\Omega_0}(\mathbf{z}_\epsilon) = \epsilon$ one finds $\mathbf{y}_\epsilon \in \partial\psi^2_\epsilon(\mathbf{z}_\epsilon)$ satisfies

$$\begin{aligned}\psi^2_\epsilon(\mathbf{x}) &\geq \langle\, \mathbf{y}_\epsilon,\, \mathbf{x}-\mathbf{z}_\epsilon\rangle + \psi^2_\epsilon(\mathbf{z}_\epsilon)\\ &\geq \langle\, \mathbf{y}_\epsilon,\, \mathbf{x}-\mathbf{z}_\epsilon\rangle + \frac{1}{\epsilon^{1/2}}\end{aligned} \tag{A.9}$$

for all $\mathbf{x}\in\mathbf{R}^n$. Taking $\mathbf{x}=\mathbf{0}$ in (A.9) shows $\langle\, \mathbf{y}_\epsilon,\, \mathbf{z}_\epsilon\rangle > 0$. Since $\mathbf{z}_\epsilon$ is a positive fraction of $\mathbf{x}_0$, the desired inequality $\psi^2_\epsilon(\mathbf{x}_0)\geq\epsilon^{-1/2}$ follows by setting $\mathbf{x}=\mathbf{x}_0$ in (A.9). *End of claim.*

Now $\psi_\epsilon \to \psi$ uniformly on compact subsets of $\mathbf{R}^n\setminus\partial\Omega_0$. Claim #2 provides a convex section $Z_\epsilon := \{\mathbf{x} \mid \psi_\epsilon \leq L_\epsilon\}$ with barycenter at the origin. F. John's Lemma, A.3, yields a centered ellipsoid $E_\epsilon \subset Z_\epsilon \subset \alpha{\cdot}E_\epsilon$.

Claim #4: ∂E_ϵ remains bounded away from the origin and ∞ as $\epsilon\to 0$, and the slope of L_ϵ remains bounded.

Proof of claim: Since $\psi(\mathbf{x})$ is continuous at $\psi(\mathbf{0}) < 1/2$, for some small fixed $r>0$, taking ϵ small enough ensures $\psi_\epsilon(\mathbf{x}) < 1/2$ for all $|\mathbf{x}| < \alpha r$. Now $L_\epsilon(\mathbf{x}) = 1 + \langle\, \mathbf{v}_\epsilon,\, \mathbf{x}\rangle$, so at least half of the ball $B_{\alpha r}(\mathbf{0})$ is contained in $Z_\epsilon \subset \alpha E_\epsilon$. This shows ∂E_ϵ remains outside $B_r(\mathbf{0})$ to prove the first half of the claim. Now $B_r(\mathbf{0})\subset Z_\epsilon$, which incidentally implies $|\mathbf{v}_\epsilon| \leq 1/r$ since $\psi_\epsilon\geq 0$.

On the other hand, if the longest axis of E_ϵ grows without bound we derive a contradiction as follows. For some subsequence $\epsilon(k)\to 0$ we have a convergent sequence of unit vectors $\hat{\mathbf{u}}_k \to \hat{\mathbf{e}}_1$ without loss of generality, such that $\pm 2k\hat{\mathbf{u}}_k \in E_{\epsilon(k)}$ while $\langle\, \hat{\mathbf{u}}_k,\, \hat{\mathbf{e}}_1\rangle \geq 1-(r/k)^2$. Since $B_r(\mathbf{0})\subset E_{\epsilon(k)}$ it follows that $\pm k\hat{\mathbf{e}}_1 \in E_{\epsilon(k)} \subset Z_{\epsilon(k)}$. This means

$$0 \leq \psi_{\epsilon(k)}(\pm k\hat{\mathbf{e}}_1) \leq 1 \pm k\langle\, \mathbf{v}_{\epsilon(k)},\, \hat{\mathbf{e}}_1\rangle.$$

Thus $|\langle\, \mathbf{v}_{\epsilon(k)},\, \hat{\mathbf{e}}_1\rangle| \leq 1/k$ which in turn implies $0\leq \psi_{\epsilon(k)}(\pm k\hat{\mathbf{e}}_1) \leq 2$. Since $\psi_\epsilon\to\psi$ in Ω_0 and grows large outside Ω_0, convexity implies $0\leq\psi(\mathbf{x})\leq 2$

along the entire x_1-axis. But the only bounded convex function is a constant: the graph of ψ would contain a line, contradicting Remark A.6 to establish the claim. *End of claim.*

Using the preceding claim, the Blaschke selection theorem yields a subsequence $\epsilon(k) \to 0$ and a bounded convex domain $Z_0 \subset \mathbf{R}^n$ such that $Z_{\epsilon(k)} \to Z_0$ in Hausdorff distance while $L_{\epsilon(k)} \to L_0$ converges to an affine function $L_0(\mathbf{x}) = 1 + \langle \mathbf{v}, \mathbf{x} \rangle$. Clearly Z_0, like $Z_{\epsilon(k)}$, has barycenter at the origin since convergence takes place in a bounded set and $\mathrm{vol}\,[Z_{\epsilon(k)} \Delta Z_0] \to 0$. Setting $Z = \{\mathbf{x} \in \Omega_0 \mid \psi < L_0\}$ and $Z' = \{\mathbf{x} \in \mathbf{R}^n \mid \psi \leq L_0\}$, one verifies $Z \subset Z_0 \subset Z'$ from the convergence $\psi_\epsilon \to \psi$ and $Z' \subset \overline{Z}$ because $\psi(\mathbf{0}) < L_0(\mathbf{0})$. Since the distinction $Z_0 \setminus Z \subset \partial Z$ is negligible, the proof is complete. QED.

Lemma A.8 (Centered sections are unique). *Fix a convex function $\psi : \mathbf{R}^n \longrightarrow]-\infty, \infty]$ for which $\partial\psi(\mathbf{R}^n)$ has non-empty interior. If $\mathbf{0} \in \mathrm{int}\,[\mathrm{dom}\,\psi]$, then the section $Z_\epsilon(\mathbf{0}) := \{\mathbf{x} \in \mathbf{R}^n \mid \psi(\mathbf{x}) < \psi(\mathbf{0}) + \langle \mathbf{v}^\epsilon, \mathbf{x} \rangle + \epsilon\}$ with center of mass at the origin has an interior uniquely determined by $\epsilon > 0$. Metrized by Hausdorff distance, the section $Z_\epsilon(\mathbf{0})$ varies continuously with $\epsilon > 0$. The slope $\mathbf{v}^\epsilon$ is also continuous and uniquely determined by $\epsilon > 0$, except when $\mathrm{vol}\,[\mathrm{dom}\,\psi \setminus Z_\epsilon(\mathbf{0})] = 0$.*

Proof: Let $\mathbf{p} \in \partial\psi(\mathbf{0})$. Replacing ψ by $\tilde{\psi}(\mathbf{x}) = \psi(\mathbf{x}) - \psi(\mathbf{0}) - \langle \mathbf{p}, \mathbf{x} \rangle$ shows it costs no generality to assume $\psi(\mathbf{x}) \geq \psi(\mathbf{0}) = 0$ is non-negative. Theorem A.7 then implies the existence of at least one $\mathbf{v}^\epsilon$ for which $Z_\epsilon = Z_\epsilon(\mathbf{0})$ is bounded and has zero as its barycenter. To derive a contradiction, suppose for some $\epsilon > 0$ there are two distinct solutions, $\mathbf{v}^\epsilon \neq \tilde{\mathbf{v}}$, corresponding to bounded sections Z_ϵ and $\tilde{Z}$ both having height ϵ over their common center of mass. In some coordinate system $\tilde{\mathbf{v}} - \mathbf{v}^\epsilon = \lambda \mathbf{e}_n$ with $\lambda > 0$, so that $\tilde{L}(\mathbf{x}) = \epsilon + \langle \tilde{\mathbf{v}}, \mathbf{x} \rangle$ satisfies $x_n(\tilde{L}(\mathbf{x}) - L_\epsilon(\mathbf{x})) > 0$. Thus the parts $\tilde{Z}^\pm := \tilde{Z} \cap H_\pm$ of $\tilde{Z} := \{\mathbf{x} \mid \psi < \tilde{L}\}$ which lie in the upper and lower halfspaces $H_\pm := \{\mathbf{x} \mid \pm x_n > 0\}$ satisfy reverse inclusions $\tilde{Z}^- \subset Z_\epsilon^-$ and $Z_\epsilon^+ \subset \tilde{Z}^+$ with respect to the corresponding parts of $Z_\epsilon^\pm := Z_\epsilon \cap H_\pm$. Moreover, the inclusions are generally strict: both Z_ϵ and $\tilde{Z}$ contain a neighbourhood of the origin, and any boundary point $\mathbf{x} \in H_+ \cap \partial Z_\epsilon \cap \mathrm{int}\,[\mathrm{dom}\,\psi]$ will have a neighbourhood contained in $\tilde{Z}$. Then it follows that at least one of the inequalities

$$\int_{Z_\epsilon^\pm} x_n d\mathrm{vol}\,(\mathbf{x}) \leq \int_{\tilde{Z}^\pm} x_n d\mathrm{vol}\,(\mathbf{x})$$

is strict, violating the hypothesis that both Z_ϵ and $\tilde{Z}$ share the same center of mass. This contradiction implies $\tilde{\mathbf{v}} = \mathbf{v}^\epsilon$. The only other possibility is that both $(H_+ \cap \mathrm{dom}\,\psi) \setminus Z_\epsilon^+$ and $(H_- \cap \mathrm{dom}\,\psi) \setminus \tilde{Z}^-$ have zero volume, in which case the inclusions above imply the interiors of the convex sets $Z_\epsilon \subset \mathrm{dom}\,\psi$ and $\tilde{Z} \subset \mathrm{dom}\,\psi$ coincide.

Let us now show continuous dependence of the section Z^ϵ on $\epsilon > 0$; again by replacing ψ by $\epsilon^{-1}\psi$ it suffices to show continuity near $\epsilon = 1$. Choose any sequence $\lim_{k\to\infty} \epsilon_k = 1$. We claim the corresponding sections $Z_k = Z_{\epsilon_k}(\mathbf{0})$ are bounded independently of k. Otherwise there is an unbounded sequence of points $\alpha\mathbf{x}_k \in Z_k$ whose reflections $-\mathbf{x}_k$ also lie in Z_k, since centered sections are roughly balanced. This implies

$$0 \leq \psi(\pm\mathbf{x}_k) \leq \epsilon_k \pm \langle \mathbf{v}_k, \mathbf{x}_k\rangle \leq 2\epsilon_k$$

where $\mathbf{v}_k := \mathbf{v}^{\epsilon_k}$. A subsequential limit forces $\psi(\lambda\hat{\mathbf{x}}_\infty) \leq 2$ for all $\lambda \in \mathbf{R}$ along the line $\hat{\mathbf{x}}_\infty = \lim_{k\to\infty} \mathbf{x}_k/|\mathbf{x}_k|$. In this case the graph of ψ must contain a full line, contradicting the hypothesis that $\partial\psi(\mathbf{R}^n)$ has non-empty interior.

Having shown $Z_k \subset B_R(\mathbf{0})$ for some $R < \infty$, let us also observe that the section boundaries also remain bounded away from the origin. Indeed, taking $r > 0$ small enough implies $\psi(\mathbf{x}) < 1/2$ on the ball of radius αr around zero. Extracting a further subsequence ensures $1/2 < \epsilon_k < 2$ for all k, so the half of $B_{\alpha r}(\mathbf{0})$ on the positive side of the hyperplane $\langle \mathbf{v}_k, \mathbf{x}\rangle = 0$ will be contained in Z_k. Since Z_k is roughly balanced, this implies $B_r(\mathbf{0}) \subset Z_k$ for all k. From this fact we conclude that $|\mathbf{v}_k| \leq 2/r$, so a subsequential limit $\mathbf{v}_k \to \mathbf{v}_\infty$ exists. We claim the corresponding sections Z_k converge to $Z_\infty := \{\mathbf{x} \mid \psi(\mathbf{x}) \leq 1 + \langle \mathbf{v}_\infty, \mathbf{x}\rangle\}$ in Hausdorff distance or — what is equivalent for convex bodies — the sense that the volume of the symmetric difference $\mathrm{vol}\,[Z_\infty \Delta Z_k] \to 0$. This is a consequence of Lesbegue's dominated convegence theorem applied on the ball $B_R(\mathbf{0})$: since the equality defining Z_∞ is strict both inside Z_∞ and outside the closure of its complement, it is not hard to show $\chi_{Z_\infty} = \lim_{k\to\infty} \chi_{Z_k}(\mathbf{x})$ pointwise for all $\mathbf{x} \notin \partial Z_\infty$. Lebesgue's dominated convergence theorem also shows the center of mass of Z_∞ must vanish, so the uniqueness established above implies $\partial Z_\infty = \partial Z_1$ as desired. This forces $\mathbf{v}_\infty = \mathbf{v}^1$ unless $\mathrm{vol}\,[\mathrm{dom}\,\psi \setminus Z_\epsilon(\mathbf{0})] = 0$. QED.

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