MAXIMIZING POWERS OF THE ANGLE BETWEEN PAIRS OF POINTS IN PROJECTIVE SPACE

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ABSTRACT. Among probability measures on $d$-dimensional real projective space, one which maximizes the expected angle \( \arccos(\frac{x}{\|x\|} \cdot \frac{y}{\|y\|}) \) between independently drawn projective points \( x \) and \( y \) was conjectured to equidistribute its mass over the standard Euclidean basis \( \{e_0, e_1, \ldots, e_d\} \) by Fejes Tóth [12]. If true, this conjecture implies the same measure maximizes the expectation of \( \arccos^\alpha(\frac{x}{\|x\|} \cdot \frac{y}{\|y\|}) \) for any exponent \( \alpha > 1 \). For \( \alpha \) sufficiently large, we verify the conjecture and establish uniqueness of the resulting maximizer \( \hat{\mu} \) up to rotation. In the broader range \( \alpha \geq 2 \), we show \( \hat{\mu} \) and its rotations maximize this expectation uniquely on a sufficiently small ball in the \( L^\infty \)-Kantorovich-Rubinstein-Wasserstein metric \( d_\infty \) from optimal transportation; the same is true for any measure \( \mu \) which is mutually absolutely continuous with respect to \( \hat{\mu} \), but the size of the ball depends on \( \|\frac{d\mu}{d\hat{\mu}}\|_\infty \).

Keywords: potential energy minimization, spherical designs, projective space, extremal problems of distance geometry, great circle distance, attractive-repulsive potentials, mild repulsion limit, Riesz energy, \( L^\infty \)-Kantorovich-Rubinstein-Wasserstein metric, \( d_\infty \)-local

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1. Introduction

Choose \( N \) unoriented lines through the origin of \( \mathbb{R}^{d+1} \). The sum of the angles between these lines is conjectured to be maximized if the lines are distributed as evenly as possible amongst the coordinate axes of some orthonormal basis for \( \mathbb{R}^{d+1} \). When \( d = 2 \) this conjecture dates back to Fejes Tóth [12]. For \( d \geq 2 \) it has motivated a recent series of works by Bilyk, Dai, Glazyrin, Matzke, Park, Vlasiuk in different
combinations [3] [5] [6] [7], and by Fodor, Vigh and Zarnócz [13]. Other
authors have also considered versions of the problem for oriented as well
as unoriented lines, in the limit \( N = \infty \) and/or with different powers \( \alpha \)
of the angle or distance between them, e.g. [21] [8] [1] [4]. In recent work
we verified the unoriented conjecture for finite values of \( N \) in the mild
repulsion limit \( \alpha = \infty \) [17]. In the present manuscript we shall reduce
the case \( N = \infty \) to \( N < \infty \), and then show the same configuration
continues to be optimal for large finite values of \( \alpha \), uniquely optimal
apart from known symmetries. Moreover, we show this configuration
with arbitrary positive weights remains locally optimal in a suitable
sense over the broader range \( \alpha \geq 2 \) corresponding to lines whose mutual

µ ≡ ν if and only if their canonical projections onto \( \mathbb{R}P^d \) are rotations of each other. When a maximum over measures is attained uniquely up to essential equivalence, we say the maximizer is \textit{essentially unique}.

For \( N \in \mathbb{N} := \{1, 2, \ldots\} \) consider the following collections of discrete probability measures on the \( d \)-sphere:

\[
\mathcal{P}_N(S^d) := \{ \mu \in \mathcal{P}(S^d) \mid \#[\text{spt}(\mu)] \leq N \},
\]

\[
\mathcal{P}_{\text{on}}(S^d) := \{ \mu \in \mathcal{P}(S^d) \mid \text{spt}(\mu) \text{ is an orthonormal basis of } \mathbb{R}^{d+1} \},
\]

\[
\mathcal{P}^=_{\text{on}}(S^d) := \{ \mu \in \mathcal{P}(S^d) \mid \mu = \frac{1}{d+1} \sum_{i=0}^{d} \delta_{v_i} \}
\quad \text{for some orthonormal basis } \{v_i\}_{i=0}^{d} \text{ of } \mathbb{R}^{d+1},
\]

\[
\mathcal{P}_{\Delta}(S^d) := \{ \mu \in \mathcal{P}(S^d) \mid \{v_i\}_{i=0}^{d} \subseteq \text{spt}(\mu) \subseteq \{v_i, -v_i\}_{i=0}^{d} \}
\quad \text{for some orthonormal basis } \{v_i\}_{i=0}^{d} \text{ of } \mathbb{R}^{d+1},
\]

\[
\mathcal{P}^=_{\Delta}(S^d) := \{ \mu \in \mathcal{P}_{\Delta}(S^d) \mid \mu \equiv \nu \text{ for some } \nu \in \mathcal{P}^=_{\text{on}}(S^d) \}.
\]

Thus \( \mathcal{P}_{\Delta}(S^d) \) denotes the set of measures whose support coincides with an orthonormal basis \( \{e_0, \ldots, e_d\} \) — which we think of as forming the vertices of a (standard) projective simplex — and \( \mathcal{P}^=_{\Delta}(S^d) \) is the subset of measures which equidistribute their mass over these vertices.

For \( \alpha \gg 1 \) sufficiently large we claim \( \mathcal{P}^=_{\Delta}(S^d) \) coincides with the set of maximizers for (1.5), so that the maximum is essentially uniquely attained. If \( \alpha \geq 2 \), we show each measure in the broader class \( \mathcal{P}_{\Delta}(S^d) \) is an essentially unique local maximizer for the energy \( E_{\alpha}(\mu) \) on an appropriately metrized version of the landscape \( \mathcal{P}(S^d) \).

These theorems echo our results concerning particles interacting through strongly attractive - mildly repulsive potentials on Euclidean space \([16]\). In the present context, the interaction kernel \( \Lambda^\alpha \) acts purely repulsively on \( \mathbb{R}P^d \) (or attractive-repulsively on \( S^d \)), with compactness of the space substituting for strong attraction at large distances. The restriction \( \alpha \geq 2 \) imposed in our theorems corresponds to the mildly repulsive range of potentials from \([2, 16]\). Using an estimate from \([18]\) we shall show the maximum (1.5) is attained only by discrete measures, whose support has cardinality bounded by \( N = N(d) \) independently of \( \alpha \geq 4 \); we identified the maximizers in the limiting case \( \alpha = \infty \) of mildest repulsion in \([17]\). In the final section below, we combine \( \Gamma \)-convergence techniques with geometrical energy estimates to extend this characterization to large finite values of \( \alpha \). To do so, we must first discuss appropriate metrics both for our discussion of local maximizers in the next section, and for the asserted \( \Gamma \)-convergence.
2. Measures supported on an orthonormal basis are local energy maximizers

For $1 \leq p < +\infty$ define the $L^p$-Kantorovich-Rubinstein-Wasserstein (optimal transport) distance between $\mu, \nu \in \mathcal{P}(S^d)$ by

$$d_p(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \left( \int_{S^d \times S^d} \rho(x, y)^p d\gamma(x, y) \right)^{1/p},$$

where the infimum is taken over the set $\Gamma(\mu, \nu)$ of joint probability measures on $S^d \times S^d$ having $\mu$ and $\nu$ as their left and right marginals.

For $p \neq \infty$, the distance $d_p$ is well-known to metrize narrow convergence (against continuous bounded test functions), e.g. Theorem 7.12 of [22], so $\mathcal{P}(S^d)$ becomes a compact metric space under $d_p$. The limit

$$d_{\infty}(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \sup_{(x, y) \in \text{supp} \gamma} \rho(x, y)$$

is also a distance, but metrizes a much finer and non-compact topology on $\mathcal{P}(S^d)$. For $\alpha \geq 2$, this finer topology allows us to establish local energy maximality of even the wildly unbalanced measures in $P_{\Delta}(S^d)$, echoing its uses in other settings [20] [2] [16].

Let $e_0 = (0, 0, ..., 0, 1), e_1 = (1, 0, ..., 0), \ldots, e_d = (0, ..., 0, 1, 0)$ be the standard basis of $R^{d+1}$. Let $D(x, r)$ and $B(x, r)$ denote the open balls of center $x$ and radius $r$ in $S^d$ and $R^d$ respectively, and denote the Euclidean norm by $|\cdot|$.

**Lemma 2.1** (Spherical distance near orthogonal points). Let $g$ denote the standard round metric, $\exp$ the Riemannian exponential map, and $\rho(x, y)$ its induced distance function on the unit sphere $S^d \subseteq R^{d+1}$. For each $i = 1, \ldots, d$,

$$\rho(e_i, \exp_{e_0} v) = \frac{\pi}{2} - g(e_i, v) + O(|v|^3),$$

and hence the Taylor expansion of

$$\varphi(y) = \sum_{i=1}^d |\rho(e_i, y) - \frac{\pi}{2}|^2$$

around its minimum $y = e_0$ is given by $\varphi(\exp_{e_0} v) = |v|^2 + O(|v|^4)$.  

**Proof.** Let $\exp$ denote the exponential map on the round sphere, i.e.

$$\exp_{e_0} v := \cos(|v|_g)e_0 + \sin(|v|_g)\frac{v}{|v|_g},$$

and note that $e_1, \ldots e_d$ form a basis for the tangent space to $S^d$ at $e_0$. The first order term in the Taylor expansion (2.3) is easy to compute and well-known (since the 1-Lipschitz function $y \mapsto \rho(e_i, y)$ increases
at constant rate one along the geodesic joining \( e_i \) to \( e_0 \), the unit tangent \(-e_i\) to this geodesic at \( y = e_0 \) gives the gradient of \( \rho(e_i, y) \) there; see e.g. Proposition 6 of [19]. This gradient \(-e_i\) is also an eigenvector with eigenvalue 0 of the Hessian of \( y \mapsto \rho(e_i, y) \) at \( y = e_0 \). Since \( \rho(e_i, \cdot) \) has a totally geodesic (great circle) level set through \( e_0 \) for \( 1 \leq i \leq d \), we see the orthogonal directions to \( e_i \) at \( e_0 \) are also eigenvectors with eigenvalue zero. Thus there is no quadratic term in the Taylor expansion (2.3). Summing up yields \( \varphi(\exp_{e_0} v) = \sum_{i=1}^{d} g(e_i, v)^2 + O(|v|^4) \) in (2.4) as desired.

Set \( e = \frac{1}{\sqrt{d+1}}(1, 1, \ldots, 1) \in S^d \) and let \( p : S^d \setminus \{e\} \to \{e\}^\perp \cong \mathbb{R}^d \) be the stereographic projection from the pole \(-e\) opposite the principal diagonal, i.e. \( p(z) \) is the unique intersection point between the subspace \( \{e\}^\perp \) and the line connecting \(-e\) and \( z \). In the coordinates given by \( p \) on \( S^d \setminus \{-e\} \), the standard (round) metric takes the (conformally flat) form

\[
g_{S^d} = \frac{4}{(1 + |p|^2)^2} g_{\mathbb{R}^d}.
\]

Let \( u_i = p(e_i), i = 0, \ldots, d \), and note that \( \{u_i\}_i \) forms the set of vertices of a \( d \)-dimensional regular simplex in \( \mathbb{R}^d \) centered at the origin \( p(e) = 0 \).

Recall the \( \mathcal{P}_\Delta(S^d) \) consists of probability measures \( \mu \) whose support covers an orthonormal basis \( V \subseteq \mathbb{R}^{d+1} \) and is contained in the double \( V \cup -V \) of that basis. For \( \alpha \geq 2 \), the following theorem provides a \( d_\infty \)-ball around each such \( \mu \) on which it maximizes the energy \( E_\alpha(\mu) \) essentially uniquely (i.e. uniquely among probability measures on the projective sphere, apart from rotations). It is inspired by Corollary 4.3 of [16], which gives the analogous result in a different context.

**Theorem 2.2** (\( d_\infty \)-local energy maximizers). Given \( m > 0 \) and \( d \in \mathbb{N} \), there exists \( r = r(d, m) > 0 \) such that for every \( \alpha \geq 2 \) and \( \xi, \hat{\xi} \in \mathcal{P}(S^d) \) with \( d_\infty(\xi, \hat{\xi}) < r \): if \( \hat{\xi} \in \mathcal{P}_\Delta(S^d) \) and \( \xi(\{z, -z\}) \geq m \) for each \( z \in \mathrm{spt} \hat{\xi} \), then \( E_\alpha(\xi) \leq E_\alpha(\hat{\xi}) \) and the inequality is strict unless \( \xi \) is a rotation of \( \hat{\xi} \).

**Proof.** Observe it is sufficient to prove for \( \alpha = 2 \) since \( E_2(\xi) \geq E_\alpha(\xi) \) for all \( \alpha \geq 2 \) and \( \xi \in \mathcal{P}(S^d) \), while \( E_2(\hat{\xi}) = E_\alpha(\hat{\xi}) \). Thus we set \( \alpha = 2 \). Fix \( \hat{\xi} \in \mathcal{P}_\Delta(S^d) \) and assume \( \xi \in \mathcal{P}(S^d) \) satisfies \( d_\infty(\hat{\xi}, \xi) < r \) for some \( 0 < r \ll \pi/4 \) which will be specified later.

By rotation, we may assume \( \hat{\xi} = \sum_{i=0}^{d} (p_i \delta_{e_i} + q_i \delta_{-e_i}) \). By transferring the mass at \(-e_i\) to \( e_i \), define \( \hat{\mu} = \sum_{i=0}^{d} m_i \delta_{e_i} \) with \( m_i = p_i + q_i > 0 \), and set \( m := \min_i m_i \). We similarly transform \( \xi \) to \( \mu \), retaining \( d_\infty(\mu, \hat{\mu}) < r \)
and $E_2(\hat{\xi}) = E_2(\hat{\mu})$, $E_2(\xi) = E_2(\mu)$. (We can alternately consider that we convert $\hat{\xi}$ and $\xi$ to measures $\hat{\omega}$ and $\omega$ on the projective space $\mathbb{RP}^d$, by pushing them forward through the map $z \in S^d \to \tilde{z} = \{z, -z\} \in \mathbb{RP}^d$. This neither increases their separation nor changes their energy, when we convert $\hat{\omega}$ and $E_\alpha$ are adopted for measures on projective space. We shall derive conditions under which $\omega$ must be a rotation of $\hat{\omega}$, hence supported at $d + 1$ well-separated points. When $d(\hat{\xi}, \xi) < \pi/4$ this in turn implies $\xi$ is a rotation of $\hat{\xi}$.

We need to show that, for sufficiently small $r$, $E_\alpha (\hat{\mu}) \geq E_\alpha (\mu)$ and the inequality is strict unless $\mu$ is a rotation of $\hat{\mu}$, meaning in particular that $\mu \in \mathcal{P}_\text{on}(S^d)$ as well. Note that, since $d(\mu, \hat{\mu}) < r \ll \pi/4$, we have $\mu(D(e_i, r)) = m_i$ for all $i$. Let $\mu_i = \mu|_{D(e_i, r)}$ be the restriction and $\nu_i = m_i^{-1} \mu_i$ its normalization.

We will transform the study of $E_2$ on $S^d$ to $\mathbb{R}^d$ via the projection $p$ to stereographic coordinates. Define the interaction kernel $W = W_2$ by

$$W(x, x') = \Lambda^2 (p^{-1}(x), p^{-1}(x')) \quad \text{for} \quad x, x' \in \mathbb{R}^d.$$ 

Now push-forward $\mu, \hat{\mu}$ via $p$ but keep the notation. Since (2.6) shows $p$ to act as a biLipschitz contraction on the hemisphere bounded by $\{e\}^1$ which contains the $e_i$, (and as an expansion on the complementary hemisphere), it follows from our assumptions that $\mu, \hat{\mu} \in \mathcal{P}(\mathbb{R}^d)$ also satisfy $\hat{\mu}(u_i) = \mu(B(u_i, r)) = m_i$ for $i = 0, 1, ..., d$ where $u_i = p(e_i)$. Set $F(\mu) = 2 (E_2(\mu) - E_2(\hat{\mu}))$, and observe

$$F(\mu) = \sum_{j=0}^{d} \left[ \int (\mu_j \ast W) d\mu_j + \sum_{i \neq j} \iint (W(x_i, x_j) - 1) d\mu_i(x_i) d\mu_j(x_j) \right].$$

We write $x_i = p(z_i)$ and $\rho(x_i, x_j) = \rho(p^{-1}(x_i), p^{-1}(x_j))$ by abusing notation. Now observe that for any $M > 0$, there exists a neighbourhood of $\pi/2 \in \mathbb{R}$ on which $\Lambda_0^2(t) - 1 \leq -M|t - \pi/2|^2$. Thus for $r > 0$ sufficiently small, any $x_i \in B(u_i, r)$ and $i \neq j$ satisfy

$$W(x_i, x_j) - 1 = \Lambda_0^2 (\rho(z_i, z_j)) - 1 \leq -M|\rho(x_i, x_j) - \pi/2|^2.$$ 

Note $\psi(x, x') := |\rho(x, x') - \pi/2|^2$ vanishes if and only if $x$ and $x'$ correspond to perpendicular points on the sphere. Now define $\bar{x}_i \in \mathbb{R}^d$ to be the barycenter of $\nu_i$ in stereographic coordinates, i.e. $\bar{x}_i = \int_{\mathbb{R}^d} x d\nu_i(x)$. Let $y_i = x_i - \bar{x}_i$. Then the Taylor expansion of $\psi$ at $(\bar{x}_i, \bar{x}_j)$ is

$$\psi(x_i, x_j) = \psi(\bar{x}_i, \bar{x}_j) + \nabla \psi(\bar{x}_i, \bar{x}_j)(y_i, y_j) + \frac{1}{2} H \psi(\bar{x}_i, \bar{x}_j)((y_i, y_j), (y_i, y_j)) + O(|y_i|^3 + |y_j|^3)$$
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where $H\psi$ is the Hessian of $\psi$. By the definition of barycenter, observe

$$\int\int \nabla \psi(\bar{x}_i, \bar{x}_j)(y_i, y_j) d\nu_i(x_i) d\nu_j(x_j) = 0,$$

and

$$\int\int H\psi(\bar{x}_i, \bar{x}_j)((y_i, y_j), (y_i, y_j)) d\nu_i(x_i) d\nu_j(x_j)$$

$$= \int\int H_1\psi(\bar{x}_i, \bar{x}_j)(y_i, y_i) d\nu_i(x_i) + \int\int H_2\psi(\bar{x}_i, \bar{x}_j)(y_j, y_j) d\nu_j(x_j),$$

where $H_1$ and $H_2$ are the Hessians in the first and second variable, respectively. From Lemma 2.1 it follows that $\sum_{i,i \neq j} H_2 \psi(w_i, w_j)$ is positive definite for all $w_k \in B(u_k, r)$ and $r > 0$ sufficiently small: more explicitly, (2.6) implies

$$\frac{1}{2} \sum_{i,i \neq j} H_2 \psi(w_i, w_j) \geq \frac{4 + O(r)}{(1 + |u_j|^2)^2} g_{R^d}.$$

Taking $\lambda < 4/(1 + |u_j|^2)^2$ and $r$ sufficiently small therefore yields

$$\frac{1}{2} \sum_{i,i \neq j} \int\int H_2 \psi(\bar{x}_i, \bar{x}_j)(y_j, y_j) d\nu_j(x_j) \geq \lambda \text{Var}(\nu_j),$$

where

$$\text{Var}(\nu_j) := \int_{\mathbb{R}^d} |x - \bar{x}_j|^2 d\nu_j(x)$$

denotes the variance of $\nu_j$. In addition,

$$\int\int O(|y_i|^2 + |y_j|^2) d\nu_i(x_i) d\nu_j(x_j) \geq -Cr(\text{Var}(\nu_i) + \text{Var}(\nu_j))$$

for some $C = C(d) > 0$. Together we obtain

$$\sum_{j=0}^d \sum_{i \neq j} \int\int (W(x_i, x_j) - 1) d\mu_i(x_i) d\mu_j(x_j)$$

$$= \sum_{j=0}^d \sum_{i \neq j} m_i m_j \int\int (W(x_i, x_j) - 1) d\nu_i(x_i) d\nu_j(x_j)$$

$$\leq -M m^2 \sum_{j=0}^d \sum_{i \neq j} \int\int \psi(x_i, x_j) d\nu_i(x_i) d\nu_j(x_j)$$

$$\leq -M m^2 \left[2\lambda - 2dCr \right] \sum_{j=0}^d \text{Var}(\nu_j) + \sum_{j=0}^d \sum_{i \neq j} \psi(\bar{x}_i, \bar{x}_j),$$
recalling \( m := \min_i m_i \). Next, let us address the localized self-interaction terms \( \int (\mu_j * W) d\mu_j \). For \( x, x' \in B(u_j, r) \),

\[
\int \int W(x, x') d\mu_j(x) d\mu_j(x') \leq \int \int W(x, x') d\nu_j(x) d\nu_j(x') = \frac{4}{\pi^2} \int \rho(x, x')^2 d\nu_j(x) d\nu_j(x').
\]

Note that \( \rho(x, x')^2 \) is a smooth function for \( x, x' \in B(u_j, r) \). Obviously \( \rho \geq 0 \) and \( \rho(\bar{x}_j, \bar{x}_j) = 0 \). Hence by Taylor expansion at \( (\bar{x}_j, \bar{x}_j) \),

\[
\rho(x, x')^2 = O(|x - \bar{x}_j|^2 + |x' - \bar{x}_j|^2).
\]

Thus there exists \( C' = C'(d) > 0 \) such that

\[
\int \int W(x, x') d\mu_j(x) d\mu_j(x') \leq C' \text{Var}(\nu_j).
\]

All together,

\[
F(\mu) \leq (-Mm^2(2\lambda - 2dCr) + C') \sum_{j=0}^d \text{Var}(\nu_j) - Mm^2 \sum_{j=0}^d \sum_{i \neq j} \psi(\bar{x}_i, \bar{x}_j).
\]

Recalling \( M \) can be chosen arbitrarily large, for sufficiently small \( r = r(d, m) > 0 \) we have \( F(\mu) \leq 0 \), and moreover, \( F(\mu) = 0 \) if and only if \( \text{Var}(\nu_j) = 0 \) for all \( j \) and \( \rho(\bar{x}_i, \bar{x}_j) = \pi/2 \) for all \( i \neq j \), that is, if and only if \( \mu \) is a rotation of \( \hat{\mu} \).

\[\text{QED}\]

3. Uniform Cardinality Estimate of Support of Maximizers

For \( A \subseteq \mathbb{R}^d \) let us denote by \( \angle A \) the smallest value of \( \theta \in [0, \pi] \) such that no distinct \( x, y, z \in A \) form an angle \( \angle xyz > \theta \). Thus \( \angle A \leq \theta \). Also let \( |A| \in \{0, 1, ..., \infty\} \) denote the cardinality of \( A \). Paul Erdős conjectured around 1950 that \( A \subseteq \mathbb{R}^d \) satisfying \( \angle A \leq \pi/2 \) yields \( |A| \leq 2^d \), which was then verified by Danzer-Grünbaum [11] in 1962. But it seems no corresponding results have been given for obtuse angle bounds \( \theta > \pi/2 \). In a separate work we showed for any \( \theta < \arccos(-1/d) \in (\pi, \pi) \) that \( |A| \) is uniformly bounded among all \( A \subseteq \mathbb{R}^d \) satisfying \( \angle A \leq \theta \); the explicit value of the bound is contained in [18]. We deduce from this result that competitors in the maximization (1.5) can be assumed to have finite support (uniformly over \( \alpha \geq 4 \)).

**Theorem 3.1** (Maximizers have uniformly finite support). For \( d \in \mathbb{N} \), there exists \( N = N(d) \in \mathbb{N} \) such that any \( d_2 \)-local maximizer \( \mu \in \mathcal{P}(\mathbb{S}^d) \) of \( E_\alpha \) for \( \alpha \geq 4 \) satisfies \( |\text{spt}(\mu)| \leq N \). Here \( d_2 \) is from [2,1].
Proof. For any $\alpha > 0$ and any $d_2$-local maximizer $\mu^*$ of $E_\alpha$, we claim an enhanced version of the projective triangle inequality holds:

\begin{equation}
\Lambda^{\alpha/2}(x, y) + \Lambda^{\alpha/2}(y, z) \geq \Lambda^{\alpha/2}(z, x)
\end{equation}

for any $x, y, z \in \text{spt } \mu^*$.

This was shown by Carrillo et al [10] and Vlasiuk [23] in more general contexts, but we recall their proof for clarity and completeness. We can assume $\rho(z, x)$ is closest to $\pi/2$ among $\rho(x, y), \rho(y, z), \rho(z, x)$, hence measures the longest side of the corresponding projective triangle $\Delta \tilde{x}\tilde{y}\tilde{z}$. Let $t \in \mathbb{R}$, $\nu = t\delta_x + (1 - t)\delta_z - \delta_y$, and

\[
\nu_r = \frac{t}{\mu(D_{x,r})}\mu\left(D_{x,r}\right) + \frac{1 - t}{\mu(D_{z,r})}\mu\left(D_{z,r}\right) - \frac{1}{\mu(D_{y,r})}\mu\left(D_{y,r}\right) \text{ for } r > 0
\]

where $D_{x,r} \subseteq S^d$ is the ball with center $x$ and radius $r$. Then $\mu^* \pm \epsilon\nu_r \in \mathcal{P}(S^d)$ for small enough $\epsilon$, hence $E_\alpha(\mu^*) \geq E_\alpha(\mu^* \pm \epsilon\nu_r)$, or equivalently $\pm 2\epsilon B_\alpha(\mu^*, \nu_r) \geq \epsilon^2 B_\alpha(\nu_r, \nu_r)$. Notice this can hold for all small $\epsilon$ only if $B_\alpha(\mu^*, \nu_r) = 0$, yielding $B_\alpha(\nu_r, \nu_r) \leq 0$. Taking $r \searrow 0$ now yields $B_\alpha(\nu, \nu) \leq 0$, i.e., $t(1 - t)\Lambda^\alpha(x, z) \leq t\Lambda^\alpha(x, y) + (1 - t)\Lambda^\alpha(y, z)$ for all $t \in \mathbb{R}$, or $(1 - t)b_a + (1 - t)b$ with $a = \frac{\Lambda^\alpha(x, y)}{\Lambda^\alpha(x, z)}$, $b = \frac{\Lambda^\alpha(y, z)}{\Lambda^\alpha(x, z)}$. Then the inequality holding at the minimum of $t \mapsto ta + (1 - t)b - t(1 - t)$ gives $b^2 - 2b(a + 1) + (a - 1)^2 \leq 0$, yielding $b \geq (1 - \sqrt{a})^2$, which is (3.1).

Next we adapt the reasoning of Kang et al [14] [15] to our context.

Note that for $\alpha \geq 4$, the function $t \mapsto \frac{\Lambda^\alpha(t)}{t^2}$ is nondecreasing on $[0, \frac{\pi}{2}]$. Since the problem is effectively set on projective space, we can assume $0 < \rho(x, y) \leq \rho(y, z) \leq \rho(z, x) \leq \frac{\pi}{2}$. Then

\[
\Lambda^{\alpha/2}(x, y) \leq \frac{\rho(x, y)^2}{\rho(x, z)^2}\Lambda^{\alpha/2}(x, z), \quad \Lambda^{\alpha/2}(y, z) \leq \frac{\rho(y, z)^2}{\rho(x, z)^2}\Lambda^{\alpha/2}(x, z).
\]

With (3.1), this implies an interpolation

\begin{equation}
\rho(x, y)^2 + \rho(y, z)^2 \geq \rho(z, x)^2 \text{ for any } x, y, z \in \text{spt } \mu^*,
\end{equation}

between (3.1) and the triangle inequality. Now let $p \in S^d$ and $\epsilon, r > 0$, and consider the ball $D_{p,r} \subseteq S^d$. If $r = r(\epsilon)$ is sufficiently small, we claim (3.2) implies $\angle(D_{p,r} \cap \text{spt } \mu^*) \leq \frac{\pi + \epsilon}{2}$ (here the angles are of the geodesic triangles determined by the set $D_{p,r} \cap \text{spt } \mu^*$). To see this, Set $a = \rho(x, y), b = \rho(y, z), c = \rho(z, x) \text{ so that } a^2 + b^2 \geq c^2$ and the largest angle $\theta$ in the geodesic triangle $\Delta xyz$ is given by $\angle xyz$. Striving to achieve maximum angle $\theta$, we may assume $a^2 + b^2 = c^2$. Then the spherical law of cosines asserts

\[
\cos \theta = \frac{\cos \sqrt{a^2 + b^2} - \cos a \cos b}{\sin a \sin b}.
\]
For the claim that \( r \) small implies \( \angle(D_{p,r} \cap \text{spt } \mu^*) \leq \frac{\pi + \epsilon}{2} \), it is enough to show

\[
f(a, b) := \frac{\cos \sqrt{a^2 + b^2} - \cos a \cos b}{\sin a \sin b} \to 0 \quad \text{as} \quad (a, b) \to (0, 0).
\]

To see this, note that the Taylor expansion of cosine at zero yields

\[
\cos \sqrt{a^2 + b^2} = \sum_{k=0}^{\infty} \frac{(-1)^k (a^2 + b^2)^k}{(2^k)!} = -1 + \sum_{k=0}^{\infty} \frac{(-1)^k (a^2 + b^2)^k}{(2^k)!} + o(ab),
\]

hence \( \cos \sqrt{a^2 + b^2} = \cos a + \cos b - 1 + o(ab) \), and similarly \( \cos a \cos b = \cos a + \cos b - 1 + o(ab) \). This yields (3.3), thus the claim.

Conformality (2.6) shows that the stereographic image of \( D_{p,r} \cap \text{spt } \mu^* \) from the pole \(-p\) onto \( \{p\}^\perp \) satisfies the angle upper bound \( \pi/2 + \epsilon \) (with smaller \( r \) if necessary). Now taking \( \epsilon \) small enough that \( \frac{\pi}{2} + 3\epsilon < \arccos(-1/d) \), and \( r \) sufficiently small that the Euclidean geodesics differ from the projected spherical geodesics by at most angle \( \epsilon \), shows that there exists \( N = N(d) \in \mathbb{N} \) such that \( |D_{p,r} \cap \text{spt } \mu^*| \leq N \) independently of the \( d_2 \)-local maximizer \( \mu^* \), and of \( p \). Since \( S^d \) can be covered by finitely many such balls of radius \( r \), the theorem follows.

QED

4. Global maximizers near the mildest repulsion limit

To verify convergence of maximizers to maximizers in the mildest repulsion limit \( \alpha \to +\infty \), we use DeGiorgi’s notion of \( \Gamma \)-convergence [9]. Since the sign conventions in this theory are normally set up so that \( \Gamma \)-convergence guarantees accumulation points of minimizers are minimizers, we must in fact show \( -E_\infty = \Gamma \text{-}\lim_{\alpha \to \infty} (-E_\alpha) \).

Definition 4.1 (\( \Gamma \)-convergence). A sequence \( F_i : M \to \mathbb{R} \) on a metric space \((M, d)\) is said to \( \Gamma \)-converge to \( F_\infty : M \to \mathbb{R} \), denoted \( F_\infty = \Gamma \text{-}\lim_{i \to \infty} F_i \), if (a)

\[
F_\infty(\mu) \leq \liminf_{i \to \infty} F_i(\mu_i) \quad \text{whenever} \quad d(\mu_i, \mu) \to 0,
\]

and (b) each \( \mu \in M \) is the limit of a sequence \((\mu_i)_i \subseteq M \) along which

\[
F_\infty(\mu) \geq \limsup_{i \to \infty} F_i(\mu_i).
\]

Lemma 4.2 (\( \Gamma \)-convergence to the mildest repulsion limit). Fix \( N \in \mathbb{N} \). The functionals \( (-E_\alpha) \) \( \Gamma \)-converge to \( (-E_\infty) \) on \( (\mathcal{P}_N(S^d), d_2) \) as \( \alpha \to \infty \). Here \( d_2 \) is from (2.1).
Proof. Let \( \{\alpha_n\}_n \) be an increasing sequence with \( \lim_{n \to \infty} \alpha_n = \infty \). To show the \( \Gamma \)-convergence, we need to show:

\[
\tag{4.3}
E_\infty(\mu) \geq \limsup_{n \to \infty} E_{\alpha_n}(\mu_n) \quad \text{whenever} \quad d_2(\mu_n, \mu) \to 0,
\]

and each \( \mu \in \mathcal{P}_N(S^d) \) is the limit of a sequence \( (\mu_n)_n \subseteq \mathcal{P}_N(S^d) \) with

\[
\tag{4.4}
E_\infty(\mu) \leq \liminf_{n \to \infty} E_{\alpha_n}(\mu_n).
\]

Firstly, (4.4) is clear by taking \( \mu_n = \mu \) for all \( i \), since \( E_{\alpha_n} \geq E_\infty \). Next, due to the monotone decreasing property of the kernel \( \Lambda^\alpha \) as \( \alpha \to \infty \), we have \( \limsup E_{\alpha_n}(\mu_n) \leq \limsup E_{\alpha_m}(\mu_n) = E_{\alpha_m}(\mu) \) for any \( m \in \mathbb{N} \). Now \( m \to \infty \) yields (4.3) by the Lebesgue dominated convergence theorem.

It is well-known (and easy to see) that Lemma 4.2 implies any \( d_2 \)-accumulation point \( \mu \) of \( \mu_{\alpha_n} \in \text{argmax}_{\mathcal{P}_N(S^d)} E_{\alpha_n} \) as \( \alpha \to \infty \) belongs to

\[
\text{argmax}_{\mathcal{P}_N(S^d)} E_\infty.
\]

For each small \( r > 0 \), we use an intersection of strips

\[
T^r(y) := \{ x \in S^d \mid \frac{\pi}{2} - r < \rho(y, x) < \frac{\pi}{2} + r \},
\]

of width \( 2r \) around different waists of the sphere to define neighbourhoods \( A^+_i := \bigcap_{j \neq i} T^r_j \) of each pair \( \pm e_i \) of antipodes, and their union and complement:

\[
\tag{4.5}
A^+ := \bigcup_{i=0}^d A^+_i \quad \text{and} \quad A^- := S^d \setminus A^+.
\]

Lemma 4.3 (Mildly repelling maximimizers equidistribute most mass near the vertices of a projective simplex). Given small \( r, \epsilon > 0 \) and \( N \geq d + 1 \), there exists \( \alpha^* \in \mathbb{R} \) such that for all \( \alpha > \alpha^* \), \( \mu_{\alpha} \in \text{argmax}_{\mathcal{P}_N(S^d)} E_{\alpha} \) satisfies, after a rotation,

\[
\tag{4.6}
\frac{1}{d+1} - \epsilon < \mu_{\alpha}(A^+_i) < \frac{1}{d+1} + \epsilon, \quad \text{and} \quad \mu_{\alpha}(A^-) < \epsilon.
\]

Proof. To derive a contradiction, suppose the lemma fails to be true. Then there exist \( r, \epsilon > 0 \) and \( N \geq d + 1 \) with \( \mu_{\alpha(k)} \in \text{argmax}_{\mathcal{P}_N(S^d)} E_{\alpha(k)} \) such that every rotation of \( \mu_{\alpha(k)} \) violates (4.6) for each \( k \in \mathbb{N} \). The compactness that \( \mathcal{P}_N(S^d) \) inherits from \( \mathcal{P}(S^d) \) under \( d_2 \) implies convergence of a subsequence of \( \mu_{\alpha(k)} \) to a limit \( \mu \in \mathcal{P}_N(S^d) \), each rotation of which must also fail to satisfy one of the \( d + 2 \) estimates in (4.6). The \( \Gamma \)-convergence established in Lemma 4.2 implies \( \mu \in \text{argmax}_{\mathcal{P}_N(S^d)} E_\infty \). On the other hand, our companion
paper [17] shows $E_\infty$ to be maximized essentially uniquely on $P_N(S^d)$ by $\hat{\mu} = \frac{1}{d+1} \sum_{i=0}^d \delta_{e_i}$. Thus $\mu \equiv \hat{\mu}$ and one of its rotations satisfies (4.6), the desired contradiction. QED

**Lemma 4.4** (The only spherical waist passing through $d$ elements of a standard basis is orthogonal to the $(d+1)^{st}$ element). For each small $r > 0$, there exists $\delta = \delta(r) > 0$ such that

$$\# \{ i \mid T^\delta(y) \cap A^\delta_i \neq \emptyset \} \leq d - 1 \text{ for every } y \in A^{-r}. \tag{4.7}$$

**Proof.** Suppose the contrary, so that there is $r > 0$ such that for any small $\delta > 0$ one can find $y \in A^{-r}$ and $y_i \in A^\delta_i$, $i = 1, \ldots, d$, such that $\frac{\pi}{2} - \delta < \rho(y, y_i) < \frac{\pi}{2} + \delta$. However, notice

$$\bigcup_{\{z_i\}_{i=1}^d \in \otimes_{i=1}^d A^\delta_i} \cap \bigcap_{i=1}^d T^\delta(z_i) \setminus \{e_0, -e_0\} \text{ as } \delta \to 0,$$

a contradiction. QED

Finally, for $\alpha$ large we are able to characterize the global maximizers of (1.5) as the measures which equidistribute their mass over the vertices of the regular, diameter $\pi/2$ (i.e. standard) projective simplex:

**Theorem 4.5** (Characterizing mildly repelling global maximizers). There exists a least $\alpha_\Delta = \alpha_\Delta(d) \in [0, \infty)$ such that for all $\alpha > \alpha_\Delta$, the set of maximizers of $E_\alpha$ over $P(S^d)$ coincides with $P_\Delta(S^d)$.

**Proof.** For all $\alpha \geq 4$, Theorem 3.1 ensures all maximizers of (1.5) enjoy a uniform bound $N \in \mathbb{N}$ on the cardinality of their support, hence lie in $P_N(S^d)$. Fix this $N$, and any $\epsilon, r > 0$ sufficiently small, so that $0 < \epsilon < \frac{1}{2d}$ and $0 < r \ll r(d, \frac{1}{2d})$ where the latter bound is from Theorem 2.2. For $\alpha > \alpha(N, r, \epsilon)$ sufficiently large, up to a rotation we can assume $\mu_\alpha \in \arg\max_{P_N(S^d)} E_\alpha$ concentrates most of its mass near the vertices of a standard projective simplex, as in Lemma 4.3. We are going to use energy estimates to improve this statement, and establish that for $r$ sufficiently small and $\alpha$ correspondingly large, all of the mass of $\mu_\alpha$ lies on the vertices of a standard projective simplex. Abbreviate $\mu = \mu_\alpha$ and $E = E_\alpha$. Denote $\mu = \mu' + \sum_{i=0}^d \mu_i$ with $\mu_i = \mu|_{A'_i}$ in the notation of (4.5). Then define $\nu = \nu' + \sum_{i=0}^d \nu_i$ by $\nu_i = ||\mu_i||\delta_{e_i}$, $\nu' = ||\mu'||\delta_{e_0}$. From

$$E(\mu) = E(\mu - \mu') - E(\mu') + B(\mu, \mu').$$
we observe \( E(\mu) - E(\nu) = G(\mu) + H(\mu) \), where

\[
G(\mu) = E(\mu - \mu') - E(\nu - \nu'), \quad \text{and}
\]

\[
H(\mu) = B(\mu', \mu) - E(\mu') - B(\nu', \nu) + E(\nu')
\]

\[
\leq B(\mu', \mu) - B(\nu', \nu)
\]

\[
= \sum_{y \in \text{spt} \mu'} \mu'(y) \left[ \int \Lambda^\alpha(y, z) d\mu(z) - \nu(\{e_0\}^\perp) \right].
\]

Recall for sufficiently small \( r > 0 \) that Theorem 2.2 asserts \( G(\mu) \leq 0 \), and moreover \( G(\mu) = 0 \) if and only if \( \mu - \mu' \equiv \nu - \nu' \).

Next we shall argue that \( H \leq 0 \) for all sufficiently large \( \alpha \), and the inequality is strict unless \( \mu' = 0 \). Let \( y \in \text{spt} \mu' \) so that \( y \in A^{-r} \). Note

\[
\int \Lambda^\alpha(y, z) d\mu(z) = \int_{A^{-\delta}} \Lambda^\alpha(y, z) d\mu(z) + \int_{\bigcup_i A_i^\delta} \Lambda^\alpha(y, z) d\mu(z)
\]

where \( \delta = \delta(r) \) is given in Lemma 4.4. Let \( \epsilon > 0 \) be arbitrarily small.

The first integral is less than \( \epsilon \) for large \( \alpha \) by Lemma 4.3. Then notice by Lemmas 4.3 and 4.4 we have, for sufficiently large \( \alpha \),

\[
\int_{\bigcup_i A_i^\delta} \Lambda^\alpha(y, z) d\mu(z)
\]

\[
= \int_{\bigcup_i (A_i^\delta \cap T^\delta(y))} \Lambda^\alpha(y, z) d\mu(z) + \int_{\bigcup_i (A_i^\delta \setminus T^\delta(y))} \Lambda^\alpha(y, z) d\mu(z)
\]

\[
\leq (d - 1) \left( \frac{1}{d + 1} + \epsilon \right) + \epsilon = \frac{d - 1}{d + 1} + d\epsilon.
\]

On the other hand \( \nu(\{e_0\}^\perp) \geq d \left( \frac{1}{d + 1} - \epsilon \right) \) by (4.6). Hence

\[
\int \Lambda^\alpha(y, z) d\mu(z) - \nu(\{e_0\}^\perp)
\]

\[
= \int_{\bigcup_i A_i^\delta} \Lambda^\alpha(y, z) d\mu(z) - \nu(\{e_0\}^\perp) + \int_{A^{-\delta}} \Lambda^\alpha(y, z) d\mu(z)
\]

\[
\leq -\frac{1}{d + 1} + 2d\epsilon + \epsilon
\]

which is negative for small enough \( \epsilon \). We conclude \( H \leq 0 \), and moreover \( H = 0 \) if and only if \( \mu'(y) = 0 \) for every \( y \in \text{spt} \mu' \), that is, \( \mu' = 0 \).

We have shown that for all sufficiently large \( \alpha \), every maximizer \( \mu \) of \( E_\alpha \) lies in \( P_\Delta(S^d) \). On the other hand, continuity of \( E_\alpha \) and compactness of \( P(S^d) \) with respect to \( d_2 \) imply the maximum (1.5) is attained. But \( E_\alpha \) is independent of \( \alpha \) on \( P_\Delta(S^d) \). By either the Perron-Frobenius theorem or our previous results concerning \( \alpha = \infty \) [17], we conclude the set of maximizers must coincide with \( P_\Delta(S^d) \) as desired. QED
Remark 4.6. As observed in [17], the Fejes Tóth conjecture [12] is equivalent to the assertion $\alpha_\Delta(d) = 1$. It is known $\alpha_\Delta(1) = 1$, and the monotonicity of $\alpha_\Delta(d)$ observed in [7] then implies $\alpha_\Delta(d) \geq 1$. Theorem 4.5 complements this by showing $\alpha_\Delta(d) < \infty$.

References


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