INSCRIBED RADIUS BOUNDS FOR LOWER RICCI BOUNDED METRIC MEASURE SPACES WITH MEAN CONVEX BOUNDARY

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Abstract. Consider an essentially nonbranching metric measure space with the measure contraction property of Ohta and Sturm, or with a Ricci curvature lower bound in the sense of Lott, Sturm and Villani. We prove a sharp upper bound on the inscribed radius of any subset whose boundary has a suitably signed lower bound on its generalized mean curvature. This provides a nonsmooth analog to a result of Kasue (1983) and Li (2014). We prove a stability statement concerning such bounds and — in the Riemannian curvature-dimension (RCD) setting — characterize the cases of equality.

1. Introduction

Kasue proved a sharp estimate for the inscribed radius (or inradius, denoted InRad) of a smooth, n-dimensional Riemannian manifold $M$ with nonnegative Ricci curvature and smooth boundary $\partial M$ whose mean curvature is bounded from below by $n-1$. More precisely, he concluded \( \text{InRad}_M \leq 1 \) [Kas83]. This result was also rediscovered by Li [Li14] and extended for Bakry-Emery curvature bounds by Li-Wei [LW15b, LW15a] and Sakurai [Sak19]. Their result can be seen either as a manifold-with-boundary analog of Bonnet and Myers’ diameter bound, or as a Riemannian analog of the Hawking singularity theorem from general relativity [Haw66] (for the precise statement see [Min19, Theorem 6.49]). There has been considerable interest in generalizing Hawking’s result to a nonsmooth setting [KSSV15, LMO19, Gra20]. Motivated in part by this goal, we give a generalization of Kasue’s result which is interesting in itself and can serve as a model for the Lorentzian case. Independently and simultaneously, Cavalletti and Mondino have proposed a synthetic new framework for Lorentzian geometry (also under investigation by one of us independently [McC]) in which they establish an analogue of the Hawking result [CM20].

In this note we generalize Kasue and Li’s estimate to subsets $\Omega$ of a (potentially nonsmooth) space $X$ satisfying a curvature dimension condition $CD(K,N)$ with $K \in \mathbb{R}$ and $N > 1$, provided the topological boundary $\partial \Omega$ has a lower bound on its inner mean curvature in the sense of [Ket20]. The notion of inner mean curvature in [Ket20] is defined by means of the 1D-localisation (needle decomposition) technique of Cavalletti and Mondino [CM17b] and coincides with the classical mean curvature of a hypersurface in the smooth context. We also assume that the boundary $\partial \Omega$ satisfies a measure theoretic regularity condition that is implied by an exterior ball condition. Hence, our result not only covers Kasue’s theorem but also holds for a large class of domains in Alexandrov spaces or in Finsler manifolds. Kasue (and Li) were also able to prove a rigidity result analogous to Cheng’s theorem [Che75] from the Bonnet-Myers context: namely that, among

\[ A previous draft of the manuscript circulated under the title “Diameter bounds for metric measure spaces with almost positive Ricci curvature and mean convex boundary”. The authors are grateful to Yohei Sakurai for directing us to the work of Kasue, and to two anonymous referees for very constructive comments. AB is supported by the Dutch Research Council (NWO) – Project number VI.Veni.192.208. CK is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 396662902, ”Synthetische Krümmungsschranken durch Methoden des optimalen Transports”. RM’s research is supported in part by NSERC Discovery Grants RGPIN–2015–04383 and 2020–04162. EW’s research is supported in part by NSERC Discovery Grant RGPIN-2017-04896. 2010 Mathematics Subject classification. Primary 51K10; also 53C21 30L99 83C75. Keywords: curvature-dimension condition, synthetic mean curvature, optimal transport, comparison geometry, diameter bounds, trapped surfaces, singularity theorems, inscribed radius, inradius bounds, measure contraction property. ©August 26, 2020. }
smooth manifolds, their inscribed radius bound is obtained precisely by the Euclidean unit ball. In the nonsmooth case there are also truncated cones that attain the maximal inradius; under an additional hypothesis known as RCD, we prove that these are the only nonsmooth optimizers provided $\Omega$ is compact and its interior is connected.

To state our results first we recall the following definition. For $\kappa \in \mathbb{R}$ we define $\cos_\kappa : [0, \infty) \to \mathbb{R}$ as the solution of

$$v'' + \kappa v = 0,$$

with $v(0) = 1$ and $v'(0) = 0$.

The function $\sin_\kappa : [0, \infty) \to \mathbb{R}$ is defined as solution of the same ODE with initial values $v(0) = 0$ and $v'(0) = 1$. We define

$$\pi_\kappa := \sup\{r > 0 : \sin_\kappa(r) > 0\} = \begin{cases} \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ \infty & \text{otherwise}, \end{cases}$$

and $I_\kappa = [0, \pi_\kappa)$. Let $K, H \in \mathbb{R}$ and $N > 1$. The Jacobian function is

$$r \in \mathbb{R} \mapsto J_{H,K,N}(r) := \left(\cos_{K/(N-1)}(r) + \frac{H}{N-1} \sin_{K/(N-1)}(r)\right)^{N-1}_+,$$

where $(a)_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. We denote the upper limit of its domain of positivity by

$$r_{K,H,N} := \sup\{r \in \mathbb{R} : J_{H,K,N}(r) > 0\}.$$

For instance $r_{0,\kappa(N-1),N} = \frac{1}{\kappa}$ if $\kappa > 0$, $r_{N-1,0,N} = \frac{\pi}{2}$ and $r_{0,\kappa(N-1),N} = \infty$ if $\kappa \leq 0$.

Our main theorem reads as follows:

**Theorem 1.1** (Inscribed radius bounds for metric measure spaces). Let $(X, d, m)$ be an essentially nonbranching CD$(K', N)$ space with $K' \in \mathbb{R}$, $N \in (1, \infty)$ and $\text{spt } m = X$. Let $\Omega \subset X$ be closed with $\Omega \neq X$, $m(\Omega) > 0$ and $m(\partial \Omega) = 0$ such that $\Omega$ satisfies the restricted curvature-dimension condition $CD_r(K,N)$ for $K \in \mathbb{R}$ (Definition 2.3) and $\partial \Omega = S$ has finite inner curvature (Definition 2.19). Assume the inner mean curvature $H^-_S$ satisfies $H^-_S \geq \kappa(N-1) m_S$-a.e. for $\kappa \in \mathbb{R}$ where $m_S$ denotes the surface measure (Definition 2.16). Then

$$\text{InRad } \Omega \leq r_{K,\kappa(N-1),N},$$

where $\text{InRad } \Omega = \sup_{x \in \Omega} d_{\text{vol}}(x)$ is the inscribed radius of $\Omega$.

We also show:

**Theorem 1.2** (Stability). Consider $(X, d, m)$ and $\Omega \subset X$ as in the previous theorem. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{InRad } \Omega \leq r_{K,\delta,N} + \epsilon$$

provided $K \geq K - \delta$, $H^-_S \geq \hat{H} - \delta$ $m_S$-a.e. and $N \leq \hat{N} + \delta$ for $K, \hat{H} \in \mathbb{R}$ and $\hat{N} \in (1, \infty)$.

**Remark 1.3** (Definitions and improvements). (1) The curvature-dimension conditions $CD(K, N)$ and the restricted curvature-dimension condition $CD_r(K, N)$ for an essentially nonbranching metric measure space $(X, d, m)$ are defined in Definition 2.3. If $(X, d, m)$ satisfies the condition $CD(K, N)$ then $\Omega \neq X$ trivially satisfies $CD_r(K, N)$ for the same $K$. For this we note that for essentially nonbranching $CD(K, N)$ spaces $L^2$-Wasserstein geodesics between $m$-absolutely continuous probability measures are unique [CM17a]. Appendix A extends the conclusions of Theorems 1.1 and 1.2 to the case where the $CD(K, N)$ hypothesis is replaced by the measure contraction property $MCP(K, N)$ proposed in [Stu06, Oht07], still under the essentially nonbranching hypothesis.

(2) The backward mean curvature bound introduced in Appendix B also suffices for the conclusion of the above theorems, and replaces the finiteness assumed of the inner curvature of $\partial \Omega = S$ by the requirement that the surface measure $m_S$, be Radon. These hypotheses also suffice for the rigidity result of Theorem 1.4 below. They are related to but distinct from a notion presented in [CM20].
For $S$ with finite inner curvature, the definition of generalized inner mean curvature $H_S$ is given in Definition 2.19. Let us briefly sketch the idea. Using a needle decomposition associated to the signed distance function $d_S := d - d_{\Omega}$, one can disintegrate the reference measure $m_S$ with conditional measures $m_\alpha$, $\alpha \in Q$, (for a quotient space $Q$) that are supported on curves $\gamma_\alpha$ of maximal slope with respect to $d_S$, the so-called needles. For $\alpha$-almost every curve $\gamma_\alpha$ for a quotient measure $\alpha$ on $Q$ there exists a conditional density $h_\alpha$ of $m_\alpha$ with respect to the 1-dimensional Hausdorff measure $H^1$. Then the inner mean curvature for $m_S$ almost every curve $p = \gamma_\alpha(t_0) \in S$ is defined as $\frac{2\pi}{\alpha} \log h_\alpha(t_0) = H_S(p)$. We postpone details to the Sections 2.3 and 2.4. In the case $(X,d,m) = (M,d_g,\text{vol}_g)$ for a Riemannian manifold $(M,g)$ and $\partial\Omega$ is a hypersurface the inner mean curvature coincides with the classical mean curvature.

(5) Our assumptions cover the case of a Riemannian manifold with boundary: If $(X,d,m) = (M,d_g,\text{vol}_g)$ for an $n$-dimensional Riemannian manifold $(M,g)$ with boundary and $\text{ric}_M \geq K$, then one can always construct a geodesically convex, $n$-dimensional Riemannian manifold $\tilde{M}$ with boundary such that $M$ isometrically embeds into $\tilde{M}$, and such that $\text{ric}_{\tilde{M}} \geq K'$ [Won08]. In particular, one can consider $M$ as a $CD_r(K,n)$ space that is a subset of the $CD(K',n)$ space $(\tilde{M},d_{\tilde{M}},\text{vol}_{\tilde{M}})$ (Remark 5.8 in [Ket20]).

1.1. Cones and spherical suspension. For smooth Riemannian manifolds with boundary rigidity is obtained precisely by the unit ball (see [Kas83, Li14, LW15b]). In the nonsmooth case also truncated cones attain the maximal inradius.

Let $(X,d,m)$ be a metric measure space.

(1) The Euclidean $N$-cone over $(X,d,m)$ is defined as the metric measure space
\[
\left(\{0,\infty\} \times X / \sim, d_{\text{Eucl}}, m^N_{\text{Eucl}}\right) = : [0,\infty) \times^N X,
\]
where the equivalence relation $\sim$ is defined by $(0,x) \sim (0,y)$ $\forall x,y \in X$, and $(t,x) \sim (t,y)$ for $\forall t > 0$ $\forall x,y \in X$. $o$ denotes the tip of the cone. The distance $d_{\text{Eucl}}$ is defined by
\[
d_{\text{Eucl}}((t,x),(s,y)) := t^2 + s^2 - 2ts \cos[d(x,y) \land \pi],
\]
where $a \land b := \min\{a,b\}$, and the measure $m^N_{\text{Eucl}}$ is given by $r^N dr \otimes dm_X$.

(2) The hyperbolic $N$-cone is defined similarly:
\[
\left(\{0,\infty\} \times X / \sim, d_{\text{Hyp}}, m^N_{\text{Hyp}}\right) = : [0,\infty) \times^N X,
\]
The distance $d_{\text{Hyp}}$ is defined by
\[
d^2_{\text{Hyp}}((t,x),(s,y)) := \cosh t \cosh s - \sinh t \sinh s \cos[d(x,y) \land \pi],
\]
where $a \land b := \min\{a,b\}$, and the measure $m^N_{\text{Hyp}}$ is given by $\sinh^N r dr \otimes dm_X$.

(3) Similar, the spherical $N$-suspension over $(X,d,m)$ is defined as the metric space
\[
\left(\{0,\pi\} \times X / \sim, d_{\text{Susp}}, m^N_{\text{Susp}}\right) = : [0,\pi] \times^N X,
\]
where the equivalence relation $\sim$ is defined by $(0,x) \sim (0,y)$ $\forall x,y \in X$, $(\pi,x) \sim (\pi,y)$ $\forall x,y \in X$ and $(t,x) \sim (t,y)$ for $t \in (0,\pi)$ iff $x = y$. The distance $d_{\text{Susp}}$ is defined by
\[
\cos d_{\text{Susp}}((t,x),(s,y)) := \cos t \cos s + \sin t \sin s \cos[d(x,y) \land \pi],
\]
and the measure $m^N_{\text{Susp}}$ is given by $\sin^N t dt \otimes dm$.

In each case 0 denotes the equivalence class of the points $(0,x)$. The next result shows that in the following $RCD$ setting the cones and suspensions are indeed all maximizers.

The Riemannian curvature-dimension condition $RCD(K,N)$ (Definition 2.5) is a strengthening of the curvature-dimension condition that rules out Finsler manifolds and allows to prove isometric rigidity theorems for metric measure spaces. This condition is crucial in the proof of the next
Theorem 1.4 (Rigidity). Let \((X, d, m)\) be \(RCD(K, N)\) for \(K \in \mathbb{R}\) and \(N \in (1, \infty)\) and let \(\Omega \subset X\) be compact with \(\Omega \neq X\), \(m(\Omega) > 0\), connected and non-empty interior \(\Omega^0\) and \(\partial \Omega = \emptyset\). We assume that \(K \in \{N - 1, 0, -(N - 1)\}\), \(\partial \Omega = S\) has finite inner curvature, \(S \neq \{pt\}\) and the inner mean curvature satisfies \(H_S^\kappa \geq \kappa(N - 1)\) \(m_S\) a.e. Then, there exists \(x \in X\) such that
\[
d_S(x) = \text{InRad} \Omega = r_{K, \kappa(N - 1), N}^\kappa \quad \text{if and only if} \quad (K, \kappa) \in \{-N - 1, 0\} \times (0, \infty) \cup \{-(N - 1)\} \times \mathbb{R} \quad \text{and there exists an} \quad \text{RCD}(N - 2, N - 1) \quad \text{space} \quad Y \quad \text{such that} \quad (\Omega^0, d_{\Omega^0}) \quad \text{is isometric to} \quad (B_{r_{K, \kappa(N - 1), N}}(0), d) \quad \text{in} \quad L^{\sin K/(N - 1)} \times Y, \quad \text{where} \quad d_{\Omega^0} \quad \text{and} \quad d \quad \text{are the induced intrinsic distances of} \quad \Omega^0 \quad \text{and} \quad B_{r_{K, \kappa(N - 1), N}}(0), \quad \text{respectively.}
\]

Remark 1.5 (Easy Direction). In the above rigidity theorem one direction is obvious. Let us explore this just for the case \(K = 0\) and \(\kappa = 1\).

Let \(Y\) be an \(\text{RCD}(N - 2, N - 1)\) space. Then the truncated Euclidean \(N\)-cone \([0, 1] \times Y \equiv \Omega\) is geodesically convex and satisfies \(\text{RCD}(0, N)\) [Ket13]. The signed distance function \(d_S\) for \(S = \partial \Omega\) in \((X, d, m)\) restricted to \(\Omega\) is given by \(d_{\Omega^0}(t, x) = 1 - t\). In particular, \(r_{0, N - 1, N} = d_S((0, x)) = d_S(o) = 1\).

Moreover, \(S\) has (inner) mean curvature equal to \(N - 1\) in the sense of Definition 2.19 in \([0, \infty) \times Y\). Indeed, we can see that points \((s, x)\) and \((t, y)\) in \(\Omega\) lie on the same needle if \(x = y\) or either \(x\) or \(y\) is \(0\). Hence, the needles in \(\Omega\) for the corresponding \(1D\)-localization are \(t \in (0, 1) \mapsto \gamma(t) = (1 - t, x), x \in Y\). One can also easily check that \(h_{1/N - 1}(t) = t\) for all needles \(\gamma\) in the corresponding disintegration of \(m|_\Omega\). Hence \(H_{\partial \Omega}^\infty \equiv N - 1\).

2. Preliminaries

2.1. Curvature-dimension condition. Let \((X, d)\) be a complete and separable metric space and let \(m\) be a locally finite Borel measure. We call \((X, d, m)\) a metric measure space. We always assume \(\text{spt} m = X\) and \(X \neq \{pt\}\).

The length of a continuous curve \(\gamma: [a, b] \to X\) is \(L(\gamma) = \sup\{\sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i+1}))\} \in [0, \infty]\) where the supremum is w.r.t. any subdivision of \(a, b\) given by \(a = t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k = b\) and \(k \in \mathbb{N}\). A geodesic is a length minimizing curve \(\gamma: [a, b] \to X\). We denote the set of constant speed geodesics \(\gamma: [0, 1] \to X\); these are characterized by the identity
\[
d(\gamma_s, \gamma_t) = (t - s)d(\gamma_0, \gamma_1)
\]
for all \(0 \leq s \leq t \leq 1\). For \(t \in [0, 1]\) let \(e_t \in \gamma \in \mathcal{G}(X) \mapsto \gamma(t)\) be the evaluation map. A subset of geodesics \(F \subset \mathcal{G}(X)\) is said to be nonbranching if for any two geodesics \(\gamma, \tilde{\gamma} \in F\) such that there exists \(e \in (0, 1)\) with \(\gamma|_{[0, e]} = \tilde{\gamma}|_{[0, e]}\), it follows \(\gamma = \tilde{\gamma}\).

Example 2.1 (Euclidean geodesics). When \(X \subset \mathbb{R}^n\) is convex and \(d(x, y) = |x - y|\) then \(\mathcal{G}(X)\) consists of the affine maps \(\gamma: [0, 1] \to X\).

The set of (Borel) probability measures on \((X, d, m)\) is denoted with \(\mathcal{P}(X)\), the subset of probability measures with finite second moment is \(\mathcal{P}^2(X)\), the set of probability measures in \(\mathcal{P}^2(X)\) that are \(m\)-absolutely continuous is denoted with \(\mathcal{P}^a(X, m)\) and the subset of measures in \(\mathcal{P}^2(X, m)\) with bounded support is denoted with \(\mathcal{P}^b(X, m)\).

The space \(\mathcal{P}^2(X)\) is equipped with the \(L^2\)-Wasserstein distance \(W_2\), e.g. [Vil09]. A dynamical optimal coupling is a probability measure \(\Pi \in \mathcal{P}(\mathcal{G}(X))\) such that \(t \in [0, 1] \mapsto (e_t)_\# \Pi\) is a \(W_2\)-geodesic in \(\mathcal{P}^2(X)\) where \((e_t)_\# \Pi\) denotes the pushforward under the map \(\gamma \mapsto e_t(\gamma) := \gamma(t)\). The set of dynamical optimal couplings \(\Pi \in \mathcal{P}(\mathcal{G}(X))\) between \(\mu_0, \mu_1 \in \mathcal{P}^2(X)\) is denoted with \(\text{OptGeo}(\mu_0, \mu_1)\).
A metric measure space \((X, d, m)\) is called essentially nonbranching if for any pair \(\mu_0, \mu_1 \in \mathcal{P}^2(X, m)\) any \(\Pi \in \text{OptGeo}(\mu_0, \mu_1)\) is concentrated on a set of nonbranching geodesics.

**Definition 2.2** (Distortion coefficients). For \(K \in \mathbb{R}, N \in (0, \infty)\) and \(\theta \geq 0\) we define the distortion coefficient as

\[
t \in [0, 1] \mapsto \sigma^{(t)}_{K,N}(\theta) := \begin{cases} \frac{\sin_{K/N}(t\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\ \infty & \text{otherwise}, \end{cases}
\]

where \(\pi_{\kappa} := \infty\) if \(\kappa \leq 0\) and \(\pi_{\kappa} := \frac{\pi}{\sqrt{\kappa}}\) if \(\kappa > 0\). Here \(\sin_{K/N}\) was defined after (1), and \(\sigma^{(t)}_{K,N}(0) = t\). Moreover, for \(K \in \mathbb{R}, N \in [1, \infty)\) and \(\theta \geq 0\) the modified distortion coefficient is defined as

\[
t \in [0, 1] \mapsto \tau^{(t)}_{K,N}(\theta) := t^{\frac{2}{N}} \left[\sigma^{(t)}_{K,N}^{-1}(\theta)\right]^{1 - \frac{2}{N}}
\]

where our conventions are \(0 \cdot \infty := 0\) and \(\infty^0 := 1\).

**Definition 2.3** (Curvature-dimension conditions [Stu06, LV09]). An essentially nonbranching metric measure space \((X, d, m)\) satisfies the curvature-dimension condition \(CD(K, N)\) for \(K \in \mathbb{R}\) and \(N \in [1, \infty)\) if for every \(\mu_0, \mu_1 \in \mathcal{P}^2(X, m)\) there exists a dynamical optimal coupling \(\Pi\) between \(\mu_0\) and \(\mu_1\) such that for all \(t \in (0, 1)\)

\[
(4) \quad \rho_{t}(\gamma_t)^{-\frac{1}{N}} \geq \tau^{(1-t)}_{K,N}(d(\gamma_0, \gamma_1))\rho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{K,N}(d(\gamma_0, \gamma_1))\rho_1(\gamma_1)^{-\frac{1}{N}} \quad \text{for } \Pi\text{-a.e. } \gamma \in \mathcal{G}(X),
\]

where \((e_t)_{\#}\Pi = \rho_t\).

We say that \(\Omega \subset X\) with \(m(\Omega) > 0\) for an essentially nonbranching metric measure space \((X, d, m)\) satisfies the restricted curvature-dimension condition \(CD_r(K, N)\) if for every dynamical optimal coupling \(\Pi\) between \(\mu_0, \mu_1 \in \mathcal{P}^2_b(X, m)\) with \((e_t)_{\#}\Pi(\Omega) = 1\) for all \(t \in [0, 1]\), (4) holds for all \(t \in [0, 1]\).

**Remark 2.4** (Consequences). A \(CD(K, N)\) space \((X, d, m)\) for \(N \in [1, \infty)\) is geodesic and locally compact. Hence, by the metric Hopf-Rinow theorem the space is proper [BBI01, Theorem 2.5.28].

We recall briefly the Riemannian curvature-dimension condition that is a strengthening of the \(CD(K, N)\) condition and the result of the combined efforts by several authors [AGS14b, Gig15, EKS15, AGMR15, AMS19, CM16].

The **Cheeger energy** \(\text{Ch} : L^2(m) \to [0, \infty]\) of metric measure space \((X, d, m)\) is defined as

\[
(5) \quad 2 \text{Ch}(f) := \liminf_{\text{Lip}(X) \ni u \to \infty} \int \frac{1}{L^4} d m
\]

where \(\text{Lip}(X)\) is the space of Lipschitz functions on \((X, d, m)\) and \(\text{Lip}(u(x)) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}\) is the local slope of \(u \in \text{Lip}(X)\). The \(L^2\)-Sobolev space is defined as \(W^{1,2}(X) = \{f \in L^2(m) : \text{Ch}(f) < \infty\}\) and equipped with the norm \(\|f\|^2 := \|f\|_{L^2(\mu)}^2 + 2 \text{Ch}(f) [\text{AGS13}, \text{AGS14a}].

**Definition 2.5** (RCD(K,N)). A metric measure space \((X, d, m)\) satisfies the **Riemannian curvature-dimension condition** \(\text{RCD}(K, N)\) if \((X, d, m)\) satisfies the condition \(CD(K, N)\) and \(W^{1,2}(X)\) is a Hilbert space, meaning the 2-homogeneous Cheeger energy (5) satisfies the parallelogram law:

\[
\text{Ch}(f + g) + \text{Ch}(f - g) = 2 \text{Ch}(f) + 2 \text{Ch}(g).
\]

**2.2. Disintegration of measures.** For further details about the content of this section we refer to [Fre06, Section 452].

Let \((R, \mathcal{R})\) be a measurable space, and let \(\mathcal{Q} : R \to Q\) be a map for a set \(Q\). One can equip \(Q\) with the \(\sigma\)-algebra \(\mathcal{Q}\) that is induced by \(\mathcal{Q}\) where \(B \in \mathcal{Q}\) if \(\mathcal{Q}^{-1}(B) \in \mathcal{R}\). Given a probability measure \(m\) on \((R, \mathcal{R})\), one can define a probability measure \(\mathfrak{q}\) on \(Q\) via the pushforward \(\mathcal{Q}_\# m = \mathfrak{q}\).

**Definition 2.6** (Disintegration of measures). A disintegration of \(m\) that is consistent with \(\mathcal{Q}\) is a map \((B, \alpha) \in \mathcal{R} \times Q \mapsto \mathfrak{m}_\alpha(B) \in [0, 1]\) such that it follows
• $m_\alpha$ is a probability measure on $(R, \mathcal{R})$ for every $\alpha \in Q$,
• $\alpha \mapsto m_\alpha(B)$ is $q$-measurable for every $B \in \mathcal{R}$,

and for all $B \in \mathcal{R}$ and $C \in Q$ the consistency condition

$$m(B \cap \Omega^{-1}(C)) = \int_C m_\alpha(B)q(\,d\alpha)$$

holds. We use the notation $\{m_\alpha\}_{\alpha \in Q}$ for such a disintegration. We call the measures $m_\alpha$ conditional probability measures.

A disintegration $\{m_\alpha\}_{\alpha \in Q}$ consistent with $\Omega$ is called strongly consistent if for $q$-a.e. $\alpha$ we have the normalization $m_\alpha(\Omega^{-1}(\alpha)) = 1$.

**Theorem 2.7** (Existence of unique disintegrations). Assume that $(R, \mathcal{R}, m)$ is a countably generated probability space and $R = \bigcup_{\alpha \in Q} R_\alpha$ is a partition of $R$. Let $\Omega : R \to Q$ be the quotient map associated to this partition, that is $\alpha = \Omega(x)$ if and only if $x \in R_\alpha$ and assume the corresponding quotient space $(Q, Q)$ is a Polish space.

Then, there exists a strongly consistent disintegration $\{m_\alpha\}_{\alpha \in Q}$ of $m$ with respect to $\Omega : R \to Q$ that is unique in the following sense: if $\{m'_\alpha\}_{\alpha \in Q}$ is another consistent disintegration of $m$ with respect to $\Omega$ then $m_\alpha = m'_\alpha$ for $q$-a.e. $\alpha \in Q$.

### 2.3. 1D-localization.

In this section we will recall the basics of the localization technique introduced by Cavalletti and Mondino for 1-Lipschitz functions as a nonsmooth analogue of Klartag’s needle decomposition; here needle refers to any geodesic along which the Lipschitz function attains its maximum slope that Klartag and others also call transport rays [EG99, FM02, Kla17]. The presentation follows Section 3 and 4 in [CM17b]. We assume familiarity with basic concepts in optimal transport (for instance [Vil09]).

Let $(X, d, m)$ be a proper metric measure space. We assume that $\text{spt} \, m = X$.

Let $u : X \to \mathbb{R}$ be a 1-Lipschitz function. Then

$$\Gamma_u := \{(x, y) \in X \times X : u(y) - u(x) = d(x, y)\}$$

is a $d$-cyclically monotone set, and one defines $\Gamma^{-1}_u = \{(x, y) \in X \times X : (y, x) \in \Gamma_u\}$.

Note that we switch orientation in comparison to [CM17b] where Cavalletti and Mondino define $\Gamma_u$ as $\Gamma^{-1}_u$.

The union $\Gamma_u \cup \Gamma^{-1}_u$ defines a relation $R_u$ on $X \times X$, and $R_u$ induces the transport set with endpoint and branching points

$$\mathcal{T}_{u,e} := P_1(R_u \setminus \{(x, y) : x = y \in X\}) \subset X$$

where $P_1((x, y) = x$. For $x \in T_{u,e}$ one defines $\Gamma_u(x) := \{y \in X : (x, y) \in \Gamma_u\}$, and similarly $\Gamma^{-1}_u(x)$ and $R_u(x)$. Since $u$ is 1-Lipschitz, $\Gamma_u, \Gamma^{-1}_u$ and $R_u$ are closed as well as $\Gamma_u(x), \Gamma^{-1}_u(x)$ and $R_u(x)$.

The forward and backward branching points are defined respectively as

$$A_+ := \{x \in \mathcal{T}_{u,e} : \exists z, w \in \Gamma_u(x) \& (z, w) \notin R_u\}, \quad A_- := \{x \in \mathcal{T}_{u,e} : \exists z, w \in \Gamma^{-1}_u(x) \& (z, w) \notin R_u\},$$

Then one considers the (nonbranched) transport set as $\mathcal{T}_u := \mathcal{T}_{u,e} \setminus (A_+ \cup A_-)$ and the (nonbranched) transport relation as the restriction of $R_u$ to $\mathcal{T}_u \times \mathcal{T}_u$.

The sets $\mathcal{T}_{u,e}, A_+$ and $A_-$ are $\sigma$-compact ([CM17b, Remark 3.3] and [Cav14, Lemma 4.3] respectively), and $\mathcal{T}_u$ is a Borel set. In [Cav14, Theorem 4.6] Cavalletti shows that the restriction of $R_u$ to $\mathcal{T}_u \times \mathcal{T}_u$ is an equivalence relation. Hence, from $R_u$ one obtains a partition of $\mathcal{T}_u$ into a disjoint family of equivalence classes $\{X_\alpha\}_{\alpha \in Q}$. There exists a measurable section $s : \mathcal{T}_u \to \mathcal{T}_u$ [Cav14, Proposition 5.2], that is if $(x, s(x)) \in R_u$ and $(y, x) \in R_u$ then $s(x) = s(y)$, and $Q$ can be identified with the image of $\mathcal{T}_u$ under $s$. Every $X_\alpha$ is isometric to an interval $I_\alpha \subset \mathbb{R}$ (c.f. [CM17b, Lemma 3.1] and the comment after Proposition 3.7 in [CM17b]) via a distance preserving map $\gamma_\alpha : I_\alpha \to X_\alpha$ where $\gamma_\alpha$ is parametrized such that $d(\gamma_\alpha(t), s(\gamma_\alpha(t))) = \text{sign}(\gamma_\alpha(t))t$, $t \in I_\alpha$, and where sign $x$ is the sign of $u(x) - u(s(x))$. The map $\gamma_\alpha : I_\alpha \to X$ extends to a geodesic also denoted $\gamma_\alpha$ and defined on the closure $\mathcal{T}_\alpha$ of $I_\alpha$. We set $\mathcal{T}_\alpha = [a(X_\alpha), b(X_\alpha)]$. 
Then, the quotient map \( \Omega : \mathcal{T}_u \to Q \) given by the section \( s \) above is measurable, and we set \( q := \Omega_x \circ m |_{\mathcal{T}_u} \). Hence, we can and will consider \( Q \) as a subset of \( X \), namely the image of \( s \), equipped with the induced measurable structure, and \( q \) as a Borel measure on \( X \). By inner regularity we replace \( Q \) with a Borel set \( Q' \) such that \( q(Q \setminus Q') = 0 \) and in the following we denote \( Q' \) by \( Q \).

The following theorem is \cite[Theorem 3.3]{CM18}.

**Theorem 2.8 (Disintegration into needles/transport rays).** Let \( (X, d, m) \) be a geodesic metric measure space with \( \text{spt } m = X \) and \( m \) \( \sigma \)-finite. Let \( u : X \to \mathbb{R} \) be a \( 1 \)-Lipschitz function, let \( \{X_\alpha\}_{\alpha \in Q} \) be the induced partition of \( \mathcal{T}_u \) via \( R_u \), and let \( \Omega : \mathcal{T}_u \to Q \) be the induced quotient map as above. Then, there exists a uniquely strongly consistent disintegration \( \{m_\alpha\}_{\alpha \in Q} \) of \( m |_{\mathcal{T}_u} \) with respect to \( \Omega \).

Now, we assume that \( (X, d, m) \) is an essentially nonbranching \( CD(K, N) \) space for \( K \in \mathbb{R} \) and \( N > 1 \). The following is \cite[Lemma 3.4]{CM18}.

**Lemma 2.9 (Negligibility of branching points).** Let \( (X, d, m) \) be an essentially nonbranching \( CD(K, N) \) space for \( K \in \mathbb{R} \) and \( N \in (1, \infty) \) with \( \text{spt } m = X \). Then, for any \( 1 \)-Lipschitz function \( u : X \to \mathbb{R} \), it follows \( m(\mathcal{T}_u \setminus \mathcal{T}_u) = 0 \).

The initial and final points are defined by

\[
a := \{x \in \mathcal{T}_u : \Gamma_u^{-1}(x) = \{x\}\}, \quad b := \{x \in \mathcal{T}_u : \Gamma_u(x) = \{x\}\}.
\]

In \cite[Theorem 7.10]{CM16} it was proved that under the assumption of the previous lemma there exists \( \tilde{Q} \subset Q \) with \( q(Q \setminus \tilde{Q}) = 0 \) such that for \( \alpha \in \tilde{Q} \) one has \( \mathcal{X}_\alpha \setminus \mathcal{T}_u \subset a \cup b \). In particular, for \( \alpha \in \tilde{Q} \) we have

\[
R_u(x) = \mathcal{X}_\alpha \supset X_\alpha \supset (R_u(x))^\circ \quad \forall x \in \Omega^{-1}(\alpha) \subset \mathcal{T}_u.
\]

where \( (R_u(x))^\circ \) denotes the relative interior of the closed set \( R_u(x) \).

The following is \cite[Theorem 3.5]{CM18}.

**Theorem 2.10 (Factor measures inherit curvature-dimension bounds).** Let \( (X, d, m) \) be an essentially nonbranching \( CD(K, N) \) space with \( \text{spt } m = X \), \( K \in \mathbb{R} \) and \( N \in (1, \infty) \).

Then, for any \( 1 \)-Lipschitz function \( u : X \to \mathbb{R} \) there exists a disintegration \( \{m_\alpha\}_{\alpha \in Q} \) of \( m |_{\mathcal{T}_u} \) that is strongly consistent with \( R_u \).

Moreover, there exists \( \tilde{Q} \) such that \( q(Q \setminus \tilde{Q}) = 0 \) and \( \forall \alpha \in \tilde{Q}, m_\alpha \) is a Radon measure with \( m_\alpha = h_\alpha \mathcal{H}^{1}|_{X_\alpha} \) and \( (X_\alpha, d, m_\alpha) \) verifies the condition \( CD(K, N) \).

More precisely, for all \( \alpha \in \tilde{Q} \) it follows that

\[
h_\alpha(\gamma_t)^{\frac{1}{N-1}} \geq \sigma^{(1-t)}_{K/N-1}(|\gamma|)h_\alpha(\gamma_0)^{\frac{1}{N-1}} + \sigma^{(t)}_{K/N-1}(|\gamma|)h_\alpha(\gamma_1)^{\frac{1}{N-1}}
\]

for every affine map \( \gamma : [0, 1] \to (a(X_\alpha), b(X_\alpha)) \).

**Remark 2.11 (Consequences).** The property (7) yields that \( h_\alpha \) is locally Lipschitz continuous on \( (a(X_\alpha), b(X_\alpha)) \) \cite[Section 4]{CM17b}, and that \( h_\alpha : (a(X_\alpha), b(X_\alpha)) \to (0, \infty) \) satisfies

\[
\frac{d^2}{dt^2} h_\alpha^{\frac{1}{N-1}} + \frac{K}{N-1} h_\alpha^{\frac{1}{N-1}} \leq 0 \quad \text{on} \quad (a(X_\alpha), b(X_\alpha))
\]

in distributional sense.

In particular, \( h_\alpha^{\frac{1}{N-1}} \) is semiconcave on \( (a(X_\alpha), b(X_\alpha)) \), hence admits left and right derivatives at each point.

It is also well-known that (8) is equivalent to (7) provided \( h_\alpha \) is continuous.

**Remark 2.12.** We observe the following from the proof of \cite[Theorem 4.2]{CM17b}: when \( \Omega \subset X \) satisfies the restricted condition \( CD_{\alpha}(K', N) \) and the \( 1 \)-Lipschitz function \( u \) is the distance function to \( \Omega^c \), that is \( d_{\chi_\Omega} \), then \( h_{\alpha} \) satisfies (7) with \( K' \) replacing \( K \).

Let us be a little bit more precise here. For the proof of Theorem 4.2 in \cite{CM17b} the authors construct \( L^2 \)-Wasserstein geodesics between \( m \)-absolutely continuous probability measures such
that the corresponding optimal dynamical plans are supported on transport geodesics of the 1-Lipschitz function $\phi$ that appears in the statement of [CM17b, Theorem 4.2].

In our situation, when $\phi$ is actually $d_Q$, all transport geodesics of positive length are inside of $\Omega$. Hence, the $L^2$-Wasserstein geodesics constructed by Cavalletti and Mondino are concentrated in $\Omega$ and the restricted condition $CD_p(K', N)$ applies. Then we can follow verbatim the proof of Theorem 4.2 in [CM17b].

**Remark 2.13 (Extended densities).** The Bishop-Gromov volume monotonicity implies that $h_\alpha$ can always be extended to continuous function on $[a(X_\alpha), b(X_\alpha)]$ [CM18, Remark 8.4]. Then (7) holds for every affine map $\gamma : [0, 1] \to [a(X_\alpha), b(X_\alpha)]$. We set $(h_\alpha \circ \gamma_\alpha(r)) \cdot 1_{[a(X_\alpha), b(X_\alpha)]} = h_\alpha(r)$ and consider $h_\alpha$ as function that is defined everywhere on $\mathbb{R}$. We also consider $h'_\alpha : X_\alpha \to \mathbb{R}$ defined a.e. via $h'_\alpha(\gamma_\alpha(r)) = h_\alpha(r)$.

It is standard knowledge that the derivatives from the left and from the right

$$
\frac{d^+}{dr} h_\alpha(r) = \lim_{t \downarrow 0} \frac{h_\alpha(r + t) - h_\alpha(r)}{t}, \quad \frac{d^-}{dr} h_\alpha(r) = \lim_{t \uparrow 0} \frac{h_\alpha(r + t) - h_\alpha(r)}{t}
$$

exist for $r \in [a(X_\alpha), b(X_\alpha)]$ and $r \in (a(X_\alpha), b(X_\alpha)]$ respectively. Moreover, we set $\frac{d^+}{dr} h_\alpha = -\infty$ in $b(X_\alpha)$ and $\frac{d^-}{dr} h_\alpha = \infty$ in $a(X_\alpha)$.

**Remark 2.14 (Generic geodesics).** In the following we set $Q^\dagger := \hat{Q} \cap \bar{Q}$, where $\hat{Q}$ and $\bar{Q}$ index the transport rays identified between Lemma 2.9 and Theorem 2.10. Then, $q(Q) = 0$ and for every $\alpha \in Q^\dagger$ the inequality (7) and (6) hold. We also set $\Omega^{-1}(Q^\dagger) := \mathcal{T}^\dagger_u \subset \mathcal{T}_u$ and $\bigcup_{x \in \mathcal{T}^\dagger_u} R_u(x) =: \mathcal{T}^\dagger_{u,c} \subset \mathcal{T}_{u,c}$.

**2.4. Generalized mean curvature.** Let $(X, d, m)$ be a metric measure space as in Theorem 2.10. Let $\Omega \subset X$ be a closed subset, and let $S = \partial\Omega$ such that $m(S) = 0$. The function $d_\Omega : X \to \mathbb{R}$ is given by

$$
\inf_{y \in \Omega} d(x, y) =: d_\Omega(x).
$$

Let us also define $d_\Omega^* := d_{\Omega^c}$. The signed distance function $d_S$ for $S$ is given by

$$
d_S := d_\Omega - d_\Omega^* : X \to \mathbb{R}.
$$

It follows that $d_S(x) = 0$ if and only if $x \in S$, and $d_S \leq 0$ if $x \in \Omega$ and $d_S \geq 0$ if $x \in \Omega^c$. It is clear that $d_S|_{\Omega} = -d_\Omega$ and $d_S|_{\Omega^c} = d_{\Omega^c}$. Setting $v = d_S$ we can also write

$$
d_S(x) = \text{sign}(v(x))d(\{v = 0\}, x), \forall x \in X.
$$

Since $X$ is proper, $d_S$ is 1-Lipschitz [CM18, Remark 8.4, Remark 8.5]. Let $\Omega^c$ denote the topological interior of $\Omega$.

Let $\mathcal{T}_{d_S, e}$ be the transport set of $d_S$ with end- and branching points. We have $\mathcal{T}_{d_S, e} \supset X \setminus S$. In particular, we have $m(X \setminus \mathcal{T}_{d_S}) = 0$ by Lemma 2.9 and $m(S) = 0$.

Therefore, by Theorem 2.10 the 1-Lipschitz function $d_S$ induces a partition $\{X_\alpha\}_{\alpha \in Q}$ of $(X, d, m)$ up to a set of measure zero for a measurable quotient space $Q$, and a disintegration $\{m_\alpha\}_{\alpha \in Q}$ that is strongly consistent with the partition. The subset $X_\alpha$, $\alpha \in Q$, is the image of a geodesic $\gamma_\alpha : I_\alpha \to X$.

We consider $Q^\dagger \subset Q$ as in Remark 2.14. One has the representation

$$
m(B) = \int_Q m_\alpha(B) d\mu(\alpha) = \int_{Q^\dagger} \int_{\gamma^{-1}_\alpha(B)} h_\alpha(r) dr d\mu(\alpha)
$$

for all Borel $B \subset X$.

For any transport ray $X_\alpha$, $\alpha \in Q^\dagger$, it follows that $d_S(\gamma_\alpha(b(X_\alpha))) \geq 0$ and $d_S(\gamma_\alpha(a(X_\alpha))) \leq 0$ (for instance compare with [CM18, Remark 4.12]).
Remark 2.15 (Measurability and zero-level selection). It is easy to see that \( A := \Omega^{-1}(\Omega(S \cap T_{ds})) \subset T_{ds} \) is a measurable subset. The set \( A \subset T_{ds} \) is defined such that \( \forall \alpha \in \Omega(A) \) we have \( X_\alpha \cap S = \{ \gamma(t_\alpha) \} \neq \emptyset \) for a unique \( t_\alpha \in I_\alpha \). Then, the map \( s : \gamma(t) \in A \mapsto \gamma(t_\alpha) \in S \cap T_{ds} \) is a measurable section (i.e., selection) on \( A \subset T_{ds} \), one can identify the measurable set \( \Omega(A) \subset Q \) with \( A \cap S \) and one can parameterize \( \gamma_\alpha \) such that \( t_\alpha = 0 \).

This measurable section \( s \) on \( A \) is fixed for the rest of the paper. The set \( A \) is the union of all disjoint needles that intersect \( \partial \Omega \). We shall also define \( B_{in} \) as the union of all needles disjoint from \( \Omega^c \) and \( B_{out} \) as the union of all needles disjoint from \( \Omega \). The superscript \( \dagger \) will be used to indicate intersection with \( T_{ds}^\dagger \).

Thus \[ A \cap T_{ds}^\dagger := A^\dagger \quad \text{and} \quad \bigcup_{x \in A^\dagger} R_{ds}(x) =: A^\dagger_c. \]

The sets \( A^\dagger \) and \( A^\dagger_c \) are measurable, and also

\[
B_{in}^\dagger := \Omega^c \cap T_{ds}^\dagger \quad \text{and} \quad B_{out}^\dagger := \Omega^c \cap T_{ds}^\dagger \setminus A^\dagger \subset T_{ds}
\]

as well as \( \bigcup_{x \in B_{in}^\dagger} R_{ds}(x) =: B_{in}^{\dagger,c} \) and \( \bigcup_{x \in B_{out}^\dagger} R_{ds}(x) =: B_{out}^{\dagger,c} \) are measurable.

The map \( \alpha \in \Omega(A^\dagger) \mapsto h_\alpha(0) \in \mathbb{R} \) is measurable (see [CM16, Proposition 10.4]).

Definition 2.16 (Surface measures). Taking \( S = \partial \Omega \) as above, we use the disintegration of Remark 2.15 to define the surface measure \( m_S \) via

\[
\int \phi(x) \, dm_S(x) := \int_{\Omega(A^\dagger)} \phi(\gamma_\alpha(0)) h_\alpha(0) \, dq(\alpha)
\]

for any bounded and continuous function \( \phi : X \to \mathbb{R} \).

Remark 2.17. That is, \( m_S \) is the pushforward of the measure \( h_\alpha(0)q(d\alpha)|_{\Omega(A^\dagger)} \) under the map \( S : \gamma \in \Omega(A^\dagger) \mapsto \gamma(0) \).

Remark 2.18 (Recast using ray maps). Let us briefly explain the previous definition from the viewpoint of the ray map [CM17b, Definition 3.6] or its precursor from the smooth setting [FM02]. For the definition we fix measurable section \( s_0 : T_{ds} \to T_{ds} \) such that \( s_0|_{A^\dagger} = s \) as in Remark 2.15. As was explained in Subsection 2.3 such a section allows us to identify the quotient space \( Q \) with a Borel subset in \( X \) up to a q-set of measure 0. Following [CM17b, Definition 3.6] we define the ray map inside \( \Omega \) defined as

\[ g : \mathcal{V} \subset \Omega(A \cup B_{in}) \times \mathbb{R} \to X \]

via its graph

\[
\text{graph}(g) = \{(\alpha, x, t) \in \Omega(A) \times \mathbb{R} \times \Omega : x \in X_\alpha, -d(x, \alpha) = t\}
\]

\[ \cup \{(\alpha, x, t) \in \Omega(B_{in}) \times \mathbb{R} \times \Omega : x \in X_\alpha, -d(x, \gamma_\alpha(b(X_\alpha))) = t\}. \]

This is exactly the ray map as in [CM17b] up to a reparametrisation for \( \alpha \in \Omega(B_{in}) \). Note that \( g(\alpha, 0) = \gamma_\alpha(0) = \alpha \) and \( g(\alpha, t) = \gamma_\alpha(t) \) if \( \alpha \in \Omega(A) \) but \( \gamma_\alpha(t + d(b(X_\alpha)), \alpha)) = g(\alpha, t) \) for \( \alpha \in \Omega(B_{in}) \). Then the disintegration for a non-negative \( \phi \in C_b(X) \) takes the form

\[
\int_X \phi \, dm = \int_Q \int_{\mathcal{V}_\alpha} \phi \circ g(\alpha, t) h_\alpha \circ g(\alpha, t) \, d\mathcal{L}^1(t) \, dq(\alpha)
\]

where \( \mathcal{V}_\alpha = P_2(\mathcal{V} \cap \{ \alpha \} \times \mathbb{R}) \subset \mathbb{R} \) and \( P_2(\alpha, t) = t \). With Fubini’s theorem the right hand side is

\[
\int_{\mathcal{V}} \phi \circ g(\alpha, t) h_\alpha \circ g(\alpha, t) \, dq \otimes L^1(\alpha) |_{\mathcal{V}_\alpha} = \int_{\mathcal{V}_\alpha} \phi \circ g(\alpha, t) h_\alpha \circ g(\alpha, t) \, dq(\alpha) \, dt
\]

where \( \mathcal{V}_\alpha = P_1(\mathcal{V} \cap Q \times \{ t \}) \subset Q \) and \( P_1(\alpha, t) = \alpha \). In particular, for \mathcal{L}^1\text{-a.e.} \( t \in \mathbb{R} \) the map \( \alpha \mapsto h_\alpha \circ g(\alpha, t) \) is measurable. Hence, for \mathcal{L}^1\text{-a.e.} \( t \in \mathbb{R} \) we define \( p_t = h_\alpha \circ g(\alpha, t) \) on \( Q \). Then disintegration takes the form

\[
m|_\Omega = m|_{\Omega \cap T_{ds}} = \int g(\cdot, t) \# p_t \, dt.
\]
Remark 2.21 (Smooth case). We point out two differences in comparison to [Ket20]: For the definition of $A^1$ we do not remove points that lie in $a$ and $b$, and we switched signs in the definition of inner mean curvature. The latter allows us to work with mean curvature bounded below instead of bounded above.

Remark 2.22 (Exterior ball condition). Let $\Omega \subset X$ and $\partial \Omega = S$. Then $S$ satisfies the exterior ball condition if for all $x \in S$ there exists $r > 0$ and $p_x \in \Omega^c$ such that $d(x, p_x) = r$ and $B_r(p_x) \subset \Omega^c$.

Lemma 2.23 (Exterior ball criterion for finite inner curvature). Let $\Omega \subset X$. If $S = \partial \Omega$ satisfies the exterior ball condition, then $S$ has finite inner curvature.

Proof. Let $S$ satisfy the exterior ball condition. Then for every $x \in S$ there exists a point $p_x \in \Omega^c$ and a geodesic $\gamma_x : [0, r_x] \to \Omega^c$ from $x$ to $p_x$ such that $L(\gamma_x) = d(x, p_x) = r_x$ and $d(p_x, y) > r_x$ for any $y \in S \setminus \{x\}$. Hence, $d_S(p_x) = r_x$ and the image of $\gamma_x$ is contained in $R_{d_S}(x)$.

Recall the definition of $Q^1 \subset Q$ (Remark 2.14). Since $Q^1$ has full $\varrho$-measure, it is enough to show that for all $\alpha \in Q^1$ the endpoint $b(X_\alpha) > 0$. Then also $B_{r_x}^\varrho = \emptyset$. Assume the contrary. Let $\alpha' \in Q^1$ and let $\gamma' := \gamma_{\alpha'}$ be the corresponding geodesic such that $b(X_{\alpha'}) = 0$, that is $\Im(\gamma'|_{(a(X_{\alpha'}), 0)}) \subset \Omega$. The concatenation $\gamma'' : (a(X_{\alpha'}), r_x) \to X$ of $\gamma'$ with $\gamma_x$ for $\gamma'(0) = x$ satisfies $\gamma''(0) = x$ and

$$d(\gamma''(s), \gamma''(t)) \leq d(\gamma''(s), x) + d(x, \gamma''(t)) = d_S(\gamma''(t)) - d_S(\gamma''(s)).$$

for $s \in (a(X_{\alpha'}), 0]$ and $t \in [0, r_x)$.

The first inequality in (10) is actually an equality. To see this let $\tilde{\gamma}$ be the geodesic between $\gamma''(s)$ and $\gamma''(t)$. There exists $t_0 \in [0, L(\tilde{\gamma})]$ such that $\tilde{\gamma}(t_0) \in S$. Since $-d(\gamma''(s), x) = d_S(\gamma''(s))$ and
In radius bounds for metric measure spaces

Recall the definitions (1)–(3) of the Jacobian $J_{H,K,N}(r)$ and its domain $0 < r < r_{K,H,N}$ of positivity. Observe $J_{H,K,N}$ is pointwise monotone non-decreasing in $H$ and $K$, and monotone non-increasing in $N$. To prove our main theorems requires one more fact from [Ket20]:

**Lemma 3.1 (Comparison inequality).** Let $h : [a,b] \rightarrow [0,\infty)$ be continuous such that $a \leq 0 < b$ and every affine map $\gamma : [0,1] \rightarrow [a,b]$ satisfies

\[
h(\gamma_t) \frac{1}{1-t} \geq \sigma_{K/\langle N-1 \rangle}^t(|\gamma|)h(\gamma_0) \frac{1}{1-t} + \sigma_{K/\langle N-1 \rangle}^t(|\gamma|)h(\gamma_1) \frac{1}{1-t}. \tag{11}\]

Then

\[
\frac{h(r)}{r^{1-t}} \leq \frac{h(0)}{r^{1-t}} \cos \frac{\theta}{r} + \frac{d^+}{dr} \bigg|_{r=0} (h(r)) \frac{1}{1-t} \sin \frac{\theta}{r}(r). \tag{12}\]

In particular, $b \leq r_{K,H,N}$, and if $h(0) > 0$, then $h(r)h(0)^{1-r} \leq J_{K,H,N}(r)$ for $r \in (0,b)$ where $H = \frac{d^+}{dr} \log h(0)$.

**Proof.** If $a < 0$, the lemma is exactly the statement of Corollary 4.3 in [Ket20].

For $a = 0$ we pick $r_a \downarrow 0$. Then, the statement follows since $\frac{d^+}{dr} h(r)$ is continuous from the left for a semiconcave function $h$. \hfill $\square$

**Proof of the Theorems 1.1 and 1.2.** Let $(X,d,m)$ be $CD(K',N)$ and consider $\Omega \subset X$ satisfying $CD_\kappa(K,N)$ as assumed in Theorem 1.1. Let $u = d_{S}$ be corresponding signed distance function. Let $\{X_\alpha\}_{\alpha \in Q}$ be the decomposition of $T_{d_{S}}$ and $\int m_{\alpha} dq(\alpha)$ be the disintegration of $m$ given by Theorem 2.10 and Remark 2.15. In Remark 2.14 we define $Q^1 \subset Q$. Recall that $Q^1$ is a subset of $Q$ with full $q$-measure and for all $\alpha \in Q^1$ one has $d_{m_{\alpha}} = h_{\alpha}dH^1$, $X_{\alpha,e} = X_{\alpha}$ and $h_{\alpha}$ satisfies

\[
(h_{\alpha})^{\frac{1}{1-r}} + \frac{K}{N-1} (h_{\alpha})^{\frac{1}{1-r}} \leq 0 \text{ on } (a(X_{\alpha}),0) \tag{12}\]

in the distributional sense. For $\hat{h}_{\alpha}(r) := h_{\alpha}(-r)$, (12) still holds. 1. Assume $K \in \mathbb{R}$ and $H_{S} \geq \kappa(N-1)$ $m_{S}$-a.e. \hfill $\square$

Recall that

\[
H_{S}(\gamma_{\alpha}(0)) = \frac{d^-}{dr} \log h_{\alpha}(0) = -\frac{d^+}{dr} \log \hat{h}_{\alpha}(0). \tag{13}\]

In particular, $h_{\alpha}(0) > 0$ for $q$-a.e. $\alpha \in Q^1$, since $h_{\alpha}(0) = 0$ yields $\frac{d^-}{dr} \log h_{\alpha}(r) = -\infty < \kappa(N-1)$ by Definition 2.19.
By Lemma 3.1 it follows that \( r \in [0, -a(X_\alpha)] \mapsto \tilde{h}_\alpha(r)\tilde{h}_\alpha(0)^{-1} \) is bounded from above by \( J_{K,\kappa(N-1),N}^{-1} \). Hence \( -a(X_\alpha) \leq \kappa r_{K,\kappa(N-1),N} \) for any \( \alpha \in Q^1 \).

We show \( d_{\Omega^c}|\Omega = -d_{\Omega^c}|\Omega \leq r_{K,\kappa(N-1),N} \). Assume there exists \( x \in \Omega \) such that \( d_{\Omega^c}(x) > r_{K,\kappa(N-1),N} \). If \( r_{K,\kappa(N-1),N} = \infty \), this is already impossible.

Otherwise there exists \( \epsilon > 0 \) such that \( B_\epsilon(x) \cap B_{r_{K,\kappa(N-1),N}}(\Omega^c) = \emptyset \) and \( m(B_\epsilon(x)) > 0 \). Since \( 0 < m(B_\epsilon(x)) = \int_{\Omega^c} \int_{\gamma^{-1}(B_\epsilon(x))} h_\alpha(r)drd\nu(\alpha) \), there exists at least one \( \alpha \in Q^1 \) such that

\[
\int_{\gamma^{-1}(B_\epsilon(x))} h_\alpha(r)dr > 0
\]

and every \( r \in \gamma^{-1}(B_\epsilon(x)) \) satisfies \( r > r_{K,\kappa(N-1),N} \) because \( B_\epsilon(x) \subset (B_{r_{K,\kappa(N-1),N}}(\Omega^c))^c \). Hence \( r = -d_S(\gamma_\alpha(r)) > r_{K,\kappa(N-1),N} \) for such \( r \), and therefore \( d_{\Omega^c}|B_\epsilon(x) > r_{K,\kappa(N-1),N} \). This is a contradiction. In particular, the inscribed radius satisfies \( \text{InRad} \Omega \leq r_{K,\kappa(N-1),N} \).

2. Assume \( K \geq K - \delta, H_\delta \geq H - \delta \text{ m}_S\text{-a.e. and } N \leq \bar{N} + \delta \). Arguing as before yields \( r \in [0, -a(X_\alpha)] \mapsto \tilde{h}_\alpha(r)\tilde{h}_\alpha(0)^{-1} \) is bounded from above by \( J_{K-\delta,\bar{H}-\delta,\bar{N}+\delta} \) for every \( \alpha \in Q^1 \) and consequently

\[
\text{InRad} \Omega \leq r_{K-\delta,\bar{H}-\delta,\bar{N}+\delta}.
\]

It follows that for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( r_{K-\delta,\bar{H}-\delta,\bar{N}+\delta} \leq r_{K,H,N} + \epsilon \) since \( f(r) = \cos \frac{\pi}{N} + \frac{\bar{H}}{N-1} \sin \frac{\pi}{N} \) that appears in the definition of \( J_{K,H,N} \) depends continuously on \( K, H \) and \( N \).

\[ \square \]

4. Rigidity

4.1. Volume cone implies metric cone. The following theorem by Gigli and De Philippis will be crucial in the proof the main result.

**Theorem 4.1** (Volume cone implies metric cone [DPG16]). Let \( K \in \{0, N-1\}, N \in (1, \infty) \) and \( X \) an \( RCD(K,N) \) space with \( \text{spt m}_X = X \). Assume there exists \( o \in X \) and \( R > r > 0 \) such that

\[
m_X(B_R(o)) = \frac{\int_0^R \left( \sin \frac{\pi}{N} u \right)^{N-1} du}{\int_0^r \left( \sin \frac{\pi}{N} v \right)^{N-1} dv} m_X(B_r(o)).
\]

Then the following holds:

1. If \( \partial B_{R/2}(o) \) contains only one point, then \( X \) is isometric to \([0, \text{diam}_X] \) or \([0, \infty) \) in the case \( K \leq 0 \) with an isometry that sends \( o \) to \( 0 \), or \( X \) is isometric to \([0, \pi_{K/(N-1)}] \) in the case \( K > 0 \) with an isometry that sends \( o \) to \( 0 \). The measure \( m_X|B_R(o) \) is given by \( c\sin \frac{\pi}{N} dx \) for some positive constant \( c \).
2. If \( B_{R/2}(o) \) contains more than one point then \( N \geq 2 \) and there exists an \( RCD(N-2,N-1) \) space \( Z \) with \( \text{diam}_Z \leq \pi \) and a local isometry \( U : B_R(o) \to [0,R) \times \frac{N}{\sin \frac{\pi}{N}} Z \) that is also a measure preserving bijection.

**Remark 4.2.** In the proof of Theorem 4.1 Gigli and De Philippis show that the map \( U \) has an inverse \( V : B_R(0) \to B_R(o) \) that is also a local isometry.

There is also a third case in the conclusion of the theorem when \( B_R(o) \) contains exactly two points. However, in this case, one has necessarily \( N = 1 \) which is excluded by our assumptions.

4.2. Distributional Laplacian and strong maximum principle. We recall the notion of the distributional Laplacian for \( RCD \) spaces (cf. [Gig15, CM18]).

Let \( X \) be an \( RCD \) space, and \( \text{Lip}_c(\Omega) \) denote the set of Lipschitz function compactly supported in an open subset \( \Omega \subset X \). A Radon functional over \( \Omega \) is a linear functional \( T : \text{Lip}_c(\Omega) \to \mathbb{R} \) such
that for every compact subset $W$ in $\Omega$ there exists a constant $C_W \geq 0$ such that
\begin{equation}
|T(f)| \leq C_W \max_W |f| \quad \forall f \in \Lip_c(\Omega) \text{ with spt } f \subset W.
\end{equation}

One says $T$ is non-negative if for all $f \in \Lip_c(\Omega)$ with $f \geq 0$ it follows that $T(f) \geq 0$.

The classical Riesz-Markov-Kakutani representation theorem says that for every non-negative Radon functional $T$ from (14) there exists a non-negative Radon measure $\mu_T$ such that $T(f) = \int f \, d\mu_T$ for all $f \in \Lip_c(\Omega)$.

When $(X,d,m)$ is an RCD space, its Cheeger energy (5) is a quadratic form and one can introduce a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the Sobolev space $W^{1,2}(X)$ with values in $L^1(m)$ via
\begin{equation}
(f,g) \in W^{1,2}(X) \times W^{1,2}(X) \mapsto \langle \nabla f, \nabla g \rangle := \frac{1}{4} |\nabla(f+g)|^2 - \frac{1}{4} |\nabla(f-g)|^2 \in L^1(m).
\end{equation}

**Definition 4.3** (Nonsmooth Laplacian). Let $\Omega \subset X$ be an open subset and let $u \in \Lip(X)$. One says $u$ is in the domain of the distributional Laplacian on $\Omega$ provided there exists a Radon functional $T$ over $\Omega$ such that
\begin{equation}
T(f) = \int \langle \nabla u, \nabla f \rangle \, d\, m_X \quad \forall f \in \Lip_c(\Omega).
\end{equation}

In this case we write $u \in D(\Delta, \Omega)$. If $T$ is represented as a measure $\mu_T$, one writes $\mu_T \in \Delta u|_\Omega$, and if there is only one such measure $\mu_T$ by abuse of notation we will identify $\mu_T$ with $T$ and write $\mu_T = \Delta u|_\Omega$.

We also recall that $u \in W^{1,2}_0(\Omega)$ for an open set $\Omega \subset X$ if and only if for any Lipschitz function $\phi$ with compact support in $\Omega$ we have $\phi \cdot u \in W^{1,2}(\Omega)$. In particular, if $u \in \Lip(X)$ then $u \in W^{1,2}_0(\Omega)$.

**Remark 4.4** (Locality and linearity).

(i) If $u \in D(\Delta, \Omega)$ and $\Omega'$ is open in $X$ with $\Omega' \subset \Omega$, then $u \in D(\Delta, \Omega')$ and for $\mu \in \Delta u|_\Omega$ it follows that $\mu|_{\Omega'} \in \Delta u|_{\Omega'}$.

(ii) If $u, v \in D(\Delta, \Omega)$, then $u + v \in D(\Delta, \Omega)$ and for $\mu_u \in \Delta u|_\Omega$ and $\mu_v \in \Delta v|_\Omega$ it follows that $\mu_u + \mu_v \in \Delta (u + v)|_\Omega$.

Recall that $u \in W^{1,2}(\Omega)$ is sub-harmonic if
\begin{equation}
\int_\Omega |\nabla u|^2 \, d\, m_X \leq \int_\Omega |\nabla(u + g)|^2 \, d\, m_X \quad \forall g \in W^{1,2}(\Omega) \text{ with } g \leq 0.
\end{equation}

One says $u$ is super-harmonic if $-u$ is sub-harmonic, and $u$ is harmonic if it is both sub- and super-harmonic. The following is well known.

**Theorem 4.5** (Characterizing sub-harmonicity). Let $X$ be RCD, let $\Omega \subset X$ be open and $u \in W^{1,2}_0(\Omega)$. Then $u$ is sub-harmonic if and only if $u \in D(\Delta, \Omega)$ and there exists $\mu \in \Delta u|_\Omega$ such that $\mu \geq 0$.

The following is [BB11, Theorem 9.13]:

**Theorem 4.6** (Strong Maximum Principle). Let $X$ be an RCD($K,N$) space with $K \in \mathbb{R}$ and $N \in [1,\infty)$, let $\Omega \subset X$ be open with compact closure and connected and let $u \in W^{1,2}_0(\Omega) \cap C(\Omega)$ be sub-harmonic. If there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{\Omega} u$ then $u$ is constant.

Let us recall another result of Cavalletti-Mondino:

**Theorem 4.7** (Laplacian of a signed distance [CM18]). Let $(X,d,m)$ be a CD($K,N$) space, and $\Omega$ and $S = \partial \Omega$ as above. Then $d_S \in D(\Delta, X \backslash S)$, and one element of $\Delta d_S|_{X \backslash S}$ that we also denote with $\Delta d_S|_{X \backslash S}$ is the Radon functional on $X \backslash S$ given by the representation formula
\begin{equation}
\Delta d_S|_{X \backslash S} = (\log h_\alpha)' m|_{X \backslash S} + \int_Q (h_\alpha \delta_{\alpha}(x_\alpha \cap \{d_S > 0\} - h_\alpha \delta_{\beta}(x_\beta \cap \{d_S < 0\})) \, dq(\alpha).
\end{equation}

We note that the Radon functional $\Delta d_S|_{X \backslash S}$ can be represented as the difference of two measures $[\Delta d_S]^+$ and $[\Delta d_S]^-$ such that
\begin{equation}
[\Delta d_S]^+_{reg} - [\Delta d_S]^-_reg = (\log h_\alpha)' \quad m\text{-a.e.}
\end{equation}
where $[\Delta d_S |_{X \setminus S}]^{\pm}_{reg}$ denotes the m- absolutely continuous part in the Lebesgue decomposition of $[\Delta d_S |_{X \setminus S}]^{\pm}$. In particular, $-(\log h_\alpha)'$ coincides with a measurable function m-a.e.

To prove the rigidity asserted in Theorem 1.4, we need one more lemma:

**Lemma 4.8** (Riccati comparison). Let $u : [0, b] \to \mathbb{R}$ be continuous such that $u'' + \kappa u \leq 0$ in the distributional sense, $u(0) = 1$ and $u'(0) \leq d$. Let $v : [0, \tilde{b}] \to \mathbb{R}$ be the maximal positive solution of $v'' + \kappa u = 0$ with $v(0) = 1$ and $v'(0) = d$. Then $\tilde{b} \geq b$ and $\frac{d^+}{dt} \log u \leq (\log v)'$ on $[0, b)$.

**Proof.** Note that $v(r) = \cos\alpha(r) + d \sin\alpha(r)$ and $\tilde{b} = r_{\kappa(N-1),d(N-1),N}$. Then Lemma 3.1 already yields that $b \leq \tilde{b}$ and $v \leq v$ on $[0, b)$. Therefore, without loss of generality we restrict $v$ to $[0, b]$.

We pick $\varphi \in C^2_c((-1, 1))$ with $\int \varphi \, d\mathcal{L}^1 = 1$ and define $\varphi_\epsilon(x) = \epsilon \varphi(x)$. Let $\epsilon > 0$ and $\epsilon \in (0, \epsilon)$, and let $u_\epsilon = \int \varphi_\epsilon(t) u(s-t) \, dt$ be the mollification of $u$ by $\varphi_\epsilon$. One can check that $u_\epsilon$ is well-defined on $[\epsilon, b-\epsilon]$ and $u_\epsilon \in C^2([\epsilon, b-\epsilon])$ satisfies

$$u''_\epsilon + (\kappa + \delta) u_\epsilon \leq 0$$

in the classical sense with $\delta = \delta(\epsilon) \to 0$ for $\epsilon \to 0$. Since $u$ is continuous, $u_\epsilon(t) \to u(t)$ for all $t \in [\epsilon, b-\epsilon]$. Moreover, $u'_\epsilon(t) \to u'(t)$ for every $t \in [\epsilon, b-\epsilon]$ where $u$ is differentiable.

Let $v_\epsilon : [0, \tilde{b}_\epsilon] \to [0, \infty)$ be the maximal positive solution of $v''_\epsilon + (\kappa + \delta(\epsilon)) v_\epsilon = 0$ with $v(0) = 1$ and $v'(0) = d$. By the previous comparison principle of Lemma 3.1 it follows that $\tilde{b} \leq \tilde{b}_\epsilon$, and since $\delta(\epsilon) \to 0$ for $\epsilon \to 0$ we have $\tilde{b}_\epsilon \to \tilde{b}$, $v_\epsilon \to v$ and $v'_\epsilon \to v$ pointwise on $[0, \tilde{b}]$ if $\epsilon \to 0$.

We pick $\epsilon \in (0, b)$ and $t \in [\epsilon, b-\epsilon]$, where $u$ is differentiable. Then

$$0 \geq \int_\epsilon^t \left[ v_\epsilon(u''_\epsilon + (\kappa + \delta) u_\epsilon) - u_\epsilon(v''_\epsilon + (\kappa + \delta) v_\epsilon) \right] \, d\mathcal{L}^1 = \int_\epsilon^t \left\{ u_\epsilon v''_\epsilon - u_\epsilon v''_\epsilon \right\} \, d\mathcal{L}^1 = v_\epsilon(t) u'_\epsilon(t) - u_\epsilon(t) v'_\epsilon(t) + u_\epsilon(t) v'_\epsilon(t) - v_\epsilon(t) u'_\epsilon(t) + u_\epsilon(t) v'(t) + u(t) v'(t)$$

Since $u$ is semiconcave and continuous on $[0, b]$, the right derivative $\frac{d^+}{dt} u : [0, b] \to \mathbb{R} \cup \{\infty\}$ is continuous from the right. Hence, for $\epsilon \downarrow 0$ and any $t \in (0, b)$ it follows

$$0 \geq v(t) \frac{d^+}{dt} u(t) - v(t) u'(t) + u(t) v'(t) - v(t) u'(t) + u(t) v'(t)$$

Hence $\frac{d^+}{dt} \log u = \frac{d^+}{dt} \log u \leq \frac{d^+}{dt} \log v$ as desired. \qed

**4.3. Proof of rigidity (Theorem 1.4).** Let $(X, d, m)$ be $RCD(K, N)$ and consider $\Omega \subset X$ as in the main theorem. Let $u = d_S$ be the corresponding signed distance function. Let $(X_\alpha)_{\alpha \in Q}$ be the decomposition of $\mathcal{T}_\alpha$ and $\int m_\alpha \, d\alpha$ be the disintegration of $m$ given by Theorem 2.8 and Remark 2.15. We consider again the set $Q^\dagger$ of full $q$-measure in $Q$. For every $\alpha \in Q^\dagger$ we have that $m_\alpha = h_\alpha H^t_{reg}$, $X_{\alpha, e} = \overline{X}_\alpha$ and $h_\alpha$ satisfies

$$(h_\alpha^{\gamma_{\alpha}})' + \frac{K}{N-1} h_\alpha^{\frac{N-1}{N}} \leq 0$$

in the distributional sense. As usual we write $h_\alpha = h_\alpha \circ \gamma_{\alpha}$. Note the properties of $h_\alpha$ discussed in Remark 2.11. For $h_\alpha(r) := h_\alpha(-r)$, (15) is still true.

Also recall that by the lower mean curvature bound the set of $\alpha$ in $Q$ with $h_\alpha(0) = 0$ has $q$-measure 0. Hence $h_\alpha(r) > 0$ for $q$-a.e. $\alpha$.

We only consider the cases $K \geq 0$. A straightforward modification of the $K > 0$ proof also yields the analogous result for $K < 0$.

1. Assume $K = 0$ and $H^-_S \geq \kappa(N-1) m_S$-a.e. with $\kappa \in \mathbb{R}$. Recall that

$$H^-_S(\gamma_\alpha(0)) = \frac{d^-}{dr} \log h_\alpha(0) = - \frac{d^-}{dr} \log \bar{h}_\alpha(0).$$
By Theorem 4.7, \( d_S|_{\Omega^o} \in D(\Delta, \Omega^o) \) and 
\[
\Delta d_S|_{\Omega^o} = (\log h_\alpha)' m|_{\Omega^o} + \int_Q (h_\alpha \delta_{\alpha}(x_a) \cap \{d_S > 0\} - h_\alpha \delta_{b(X_a)} \cap \{d_S < 0\}) dq(\alpha).
\]
Any \( \gamma_\alpha \) for \( \alpha \in Q \) that starts inside of \( \Omega^o \) satisfies \( \gamma_\alpha(b(X_a)) \in \partial \Omega^o \). Hence \( \int_Q h_\alpha \delta_{b(X_a)} \cap \Omega^o dq(\alpha) = 0 \). It follows that 
\[
\Delta d_{\Omega^o}|_{\Omega^o} = \Delta (-d_S|_{\Omega^o})|_{\Omega^o} \leq (\log h_\alpha)' m|_{\Omega^o}.
\]
Recall that \(-d_S|_{\Omega^o} = d_{\Omega^o}|_{\Omega^o}\) and by locality of the distributional Laplacian \( \Delta d_S|_{\Omega^o \setminus \Omega^s} = \Delta d_S|_{\Omega^o} \).

Recall that \(- (\log h_\alpha)'(r) = (\log h_\alpha)'(-r) \). By the mean curvature bound and Riccati comparison lemma (Lemma 4.8) it follows that 
\[
\Delta d_{\Omega^o}|_{\Omega^o} \leq (N - 1) \frac{-\kappa}{\kappa - 1} d_{\Omega^o} m|_{\Omega^o}.
\]
We can add the previous two inequalities on \( \Omega^o \) and obtain 
\[
\Delta (d_{\Omega^o} + d_p)|_{\Omega^o} \leq (N - 1) \frac{-\kappa d_{\Omega^o} - \kappa d_p}{(\kappa - 1) d_p} m|_{\Omega^o} \leq 0
\]
Since \( d_{\Omega^o} + d_p \geq \frac{1}{\kappa}, \) \( d_{\Omega^o} + d_p \) is a super-harmonic function on \( \Omega^o \). Moreover, \( (d_p + d_{\Omega^o})|_{\Omega^o} \geq \frac{1}{\kappa} \) and attains its minimum on the geodesic \( \gamma^* \) in \( \Omega^o \).

Hence, by the strong maximum principle (Theorem 4.6) it follows that \( d_{\Omega^o} + d_p = \frac{1}{\kappa} \) and 
\[
(\Delta d_p) \circ \gamma_\alpha(r) = - (\Delta d_{\Omega^o}) \circ \gamma_\alpha(r) = (\log h_\alpha)' \circ \gamma_\alpha(r) \quad \text{for } r \in \left[-\frac{1}{\kappa}, 0\right], \text{q.a.e. } \alpha \in Q.
\]
Recall at this point that \((\log h_\alpha)' \circ \gamma_\alpha(r)\) is in fact given by \((\log h_\alpha)'(r)\) for the density \( h_\alpha \) of \( m_\alpha \) with respect to \( \mathcal{L}^1 \) on \([a(X_a), b(X_a)]\).

It follows for q.a.e. \( \alpha \in Q \) and \( r \in \left[-\frac{1}{\kappa}, 0\right] \) that 
\[
(N - 1) \frac{-\kappa}{\kappa - 1} + d_{\Omega^o} \circ \gamma_\alpha(r) = \frac{1}{\kappa - 1} - d_{\Omega^o} \circ \gamma_\alpha(r).
\]
Solving this ODE we obtain that 
\[
h_\alpha(r) = h_\alpha(0)(1 + \kappa r)^{N-1} = h_\alpha(0)J_{\partial,\kappa(N-1),N}(-r) \quad \text{for } r \in \left[-\frac{1}{\kappa}, 0\right], \text{q.a.e. } \alpha \in Q.
\]
Then it follows that 
\[
m(B_R(p)) = \int_{-\frac{1}{\kappa}}^{R - \frac{1}{\kappa}} h_\alpha(r) dr dq(\alpha) = \int_{-\frac{1}{\kappa}}^{R} h_\alpha(0)K^{N-1}r^{N-1} dr dq = \lambda \int_{0}^{R} r^{N-1} dr \quad \forall R \in \left[0, \frac{1}{\kappa}\right],
\]
where \( \lambda = \kappa^{N-1} \int h_\alpha(0) dq(\alpha) \). Hence \( B_{\frac{1}{\kappa}}(p) \) is a volume cone in the sense of (13) in Theorem 4.1.

2. Assume that \( K > 0 \) and \( H_{N,\kappa} \geq \kappa(N - 1) m_N \)-a.e. for \( \kappa \in \mathbb{R} \). By rescaling, we may then assume without loss of generality that \( K = N - 1 \).
The proof is similar to the case $K = 0$. We deduce from Theorem 4.7, the curvature bounds and the Riccati comparison lemma (Lemma 4.8) that $d_S|_{\Omega^o} = -d_{\Omega^o}|_{\Omega^o} \in D(\Delta, \Omega^o)$ and

$$\Delta d_{\Omega^o}|_{\Omega^o} \leq (N - 1) \frac{-\sin d_{\Omega^o} + \kappa \cos d_{\Omega^o}}{\cos d_{\Omega^o} + \kappa \sin d_{\Omega^o}} m|_{\Omega^o}. \tag{18}$$

Assume that there is a $p \in \Omega^o$ such that $d_{\Omega^o}(p) = r_{K, \kappa(N-1), N}$. Then $B_{r_{K, \kappa(N-1), N}(p)} \subset \Omega^o$ and

$$\Delta d_p|_{\Omega^o} \leq (N - 1) \frac{\cos d_p}{\sin d_p} m|_{\Omega^o}. \tag{19}$$

Adding the previous two inequalities yields

$$\Delta(d_p + d_{\Omega^o})|_{\Omega^o} \leq (N - 1) \frac{\cos d_p \cos d_{\Omega^o} + \kappa \cos d_p \sin d_{\Omega^o} - \sin d_p \sin d_{\Omega^o} + \kappa \sin d_p \cos d_{\Omega^o}}{\sin d_p (\cos d_{\Omega^o} + \kappa \sin d_{\Omega^o})} m|_{\Omega^o}$$

$$= (N - 1) \frac{\cos(d_p + d_{\Omega^o}) + \kappa \sin(d_p + d_{\Omega^o})}{\sin d_p (\cos d_{\Omega^o} + \kappa \sin d_{\Omega^o})} m|_{\Omega^o} \leq 0.$$

The last inequality follows since $r_{K, \kappa(N-1), N} \leq d_p + d_{\Omega^o}$ by the triangle inequality and since $\cos(r) + \kappa \sin(r) \leq 0$ for $r \geq r_{K, \kappa(N-1), N}$ by definition of $r_{K, \kappa(N-1), N}$. For this we also note that

$$\sin d_p (\cos d_{\Omega^o} + \kappa \sin d_{\Omega^o}) = \sin d_p (J_{K, \kappa(N-1), N} \circ d_{\Omega^o}) \frac{1}{\pi} \geq 0$$

on $\Omega^o$ because $d_p \leq \pi$ for any $p \in X$ where $X$ is $CD(N - 1, N)$ by the Bonnet-Myers diameter estimate (e.g. [Stu06]) and because $J_{K, \kappa(N-1), N} (d_{\Omega^o}) > 0$ on $\Omega^o$ by Theorem 1.1.

Hence $d_p + d_{\Omega^o}$ on $\Omega^o$ is a super-harmonic function that attains its minimum inside $\Omega^o$ and is therefore constant equal to $d_p + d_{\Omega^o} = r_{K, \kappa(N-1), N}$ on $\Omega^o$. In particular, it follows $\Omega^o = B_{r_{K, \kappa(N-1), N}(p)}$. Moreover, the inequality in (18) must be an equality on $\Omega^o$:

$$\Delta d_{\Omega^o}|_{\Omega^o} = (N - 1) \frac{-\sin d_{\Omega^o} + \kappa \cos d_{\Omega^o}}{\cos d_{\Omega^o} + \kappa \sin d_{\Omega^o}} m|_{\Omega^o}.$$

Solving the ODE analogous to (17) yields

$$h_\alpha(r) = h_\alpha(0)J_{N-1, \kappa(N-1), N}(-r) \quad \text{for} \quad r \in [-r_{N-1, \kappa(N-1), N}, 0) \quad \text{and} \quad \text{q.a.e.} \; \alpha \in Q$$

and therefore

$$m(B_R(p)) = \lambda \int_0^R \sin^{N-1} r \, dr \; \forall R \in [0, r_{K, \kappa(N-1), N}]$$

for some constant $\lambda > 0$. Hence $B_{r_{K, \kappa(N-1), N}(p)}$ is a volume cone in the sense of (13).

3. Now we can finish the proof of the main theorem for both cases by application of Theorem 4.1. First, we note that by assumption $S = \partial B_{r_{K, \kappa(N-1), N}(p)}$ is not just a point. Hence, one can conclude that $\partial B_{r_{K, \kappa(N-1), N}(p)}$ is also not just a point. Otherwise, one can easily construct an $L^2$-Wasserstein geodesic between $\delta_p$ and an $m$-absolutely continuous measure $\mu$ in $\mathcal{P}^2(X)$ that is supported on set of branching geodesics. But this contradicts the fact that the space is essentially nonbranching because of the RCD condition. For instance

$$\mu := \frac{c}{m(X_{\alpha} \cap [-\eta + r_{K, \kappa(N-1), N}, r_{K, \kappa(N-1), N}])} dq(\alpha)$$

with $\eta < R/2$ is such a measure ($c$ is a normalisation constant).

Second, $S$ must contain more than 2 points. Otherwise, we can apply Remark 4.2 and it follows that $N = 1$, which is also excluded.

Hence, only the last case in Theorem 4.1 remains relevant to us. By Remark 4.2 there exist local isometries $U$ and $V$ between $\Omega^o = B_{r_{K, \kappa(N-1), N}(p)}$ and $B_{r_{K, \kappa(N-1), N}(0)}$ in the corresponding cone that are also measure preserving bijections.

Now, it is standard knowledge that $U$ is an isometry with respect to the induced intrinsic distances.
Let us be more precise. Set $r_{K,\kappa(N-1),N} = R$ and let $\tilde{d}_\epsilon$ be the induced intrinsic distance on $\tilde{B}_{R-\epsilon}(p)$. We denote by $d^*$ the cone (or suspension distance) and by $\tilde{d}^*$ and $\tilde{d}^*_\epsilon$ the induced intrinsic distances of $B_R(0)$ and $\tilde{B}_{R-\epsilon}(0)$, respectively. Then $U$ is an isometry between $\tilde{B}_{R-\epsilon}(p)$ and $B_{R-\epsilon}(0)$ with respect to the induced intrinsic distances.

To prove this let $\gamma : [0, 1] \to \tilde{B}_{R-\epsilon}(p)$ be a geodesic with respect to $\tilde{d}_\epsilon$ between $x, y \in \tilde{B}_{R-\epsilon}(p)$. We can divide $\gamma$ into $k \in \mathbb{N}$ small pieces $\gamma|_{[t_{i-1}, t_i]}$ with $i = 1, \ldots, k$ and $t_0 = 0, t_k = 1$ such that each piece stays inside a small ball that is mapped isometrically with respect to $d^*$ via $U$ to a small ball in $B_R(0)$. We obtain

$$\sum_{i=1}^{k} d^*(U(\gamma(t_{i-1})), U(\gamma(t_i))) = \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)) \leq \sum_{i=1}^{k} \tilde{d}_\epsilon(\gamma(t_{i-1}), \gamma(t_i)) = \tilde{d}_\epsilon(x, y).$$

The first equality holds because $U$ is an isometry with respect to $d$ and $d^*$ on the small balls that contain $\gamma|_{[t_{i-1}, t_i]}$. The last equality holds because $\gamma$ is geodesic with respect to $\tilde{d}_\epsilon$, and the inequality holds because the intrinsic distance is always equal or larger than $d^*$ itself.

On the left hand side we take the supremum with respect to all such subdivisions $(t_i)_{i=0, \ldots, k-1}$. This yields $\tilde{d}_\epsilon(U(x), U(y)) \leq L(U \circ \gamma) \leq \tilde{d}_\epsilon(x, y)$, where $L(U \circ \gamma)$ is the length of the continuous curve $U \circ \gamma$. In particular $U \circ \gamma$ is a rectifiable curve (that means has finite length) in $\tilde{B}_{R-\epsilon}(0)$.

We can argue in the same way for the inverse map $V$ and obtain that $U : \tilde{B}_{R-\epsilon}(p) \to B_{R-\epsilon}(0)$ is an isometry with respect to the induced intrinsic distances $\tilde{d}_\epsilon$ and $\tilde{d}^*_\epsilon$.

Finally, we let $\epsilon \to 0$ and observe that $\tilde{d}_\epsilon \to \tilde{d}$ on $\tilde{B}_{R-\epsilon}(p)$ and the same for $\tilde{d}^*_\epsilon$ and $\tilde{d}^*$. This finishes the proof. \hfill $\square$

### Appendix A. Substituting measure contraction for lower Ricci bounds

In this appendix we will sketch why the results of Theorem 1.1 also hold when one replaces the condition $CD(K, N)$ with the weaker measure contraction property $MC\overline{P}(K, N)$ that was introduced in [Stu06, Oht07]. We will not repeat the technical details but focus on necessary modifications for this setup.

For a proper metric measure space $(X, d, m)$ that is essentially nonbranching there are several equivalent ways to define the $MC\overline{P}(K, N)$. The following one can be found in [CM16, Section 9].

**Definition A.1** (Measure contraction property). Let $(X, d, m)$ be proper and essentially nonbranching. The measure contraction property $MC\overline{P}(K, N)$, $K \in \mathbb{R}$ and $N \in (1, \infty)$ holds if for every pair $\mu_0, \mu_1 \in P^2(X)$ such that $\mu_0$ is $m$-absolutely continuous there exists a dynamical optimal plan $\Pi$ such that $(\epsilon_t)_{t \geq 0} \Pi = \mu_t \in P^2(m)$ and

$$\rho_t(\gamma_t)^{-\frac{1}{t}} \geq \sigma_{K,N}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_0(\gamma_0)^{-\frac{1}{t}}$$

for $\Pi$-a.e. geodesic $\gamma$.

where $\mu_t = \rho_t m$.

For $\Omega \subset X$ with $m(\Omega) > 0$ the restricted measure contraction property $MC\overline{P}_r$ is defined similarly as the condition $CD_r$ (compare with Definition 2.3).

All the technical results in Section 2.3 still hold when we replace the condition $CD(K, N)$ with $MC\overline{P}(K, N)$ (c.f. [CM18]). Only in Theorem 2.10 the density $h_{\alpha}$ of the conditional measure $m_{\alpha}$ will not satisfy (7) and therefore need not be semiconcave, although it does remain Lipschitz. Instead one has only

$$h_{\alpha}(\gamma_t)^{\frac{1}{1-t}} \geq \sigma_{K,N}^{(1-t)}(d(\gamma_0, \gamma_1)) h_{\alpha}(\gamma_0)^{\frac{1}{1-t}}$$

for every affine function $\gamma : [0, 1] \to [a(X_\alpha), b(X_\alpha)]$.

Considering $\Omega \subset X$ with $\partial \Omega = S$ and $m(S) = 0$ then the same definition for $m_{\alpha}$ as in Definition 2.16 makes sense. We also can define the notion of finite inner curvature of $\Omega$. However, since $h_{\alpha}$ is not semiconcave in general, the right and the left derivative might not exist for every $t \in$
where we set
\[ t \text{ for } \int_{\gamma_0} \]

Hence
\[ H \]

Proof.
\[ \text{From } \]

Thus
\[ \text{inequality is negative by the definition of } r \]

\[ \text{bounded from below which yields Theorems 1.1 and Theorem 1.4 without requiring finite inner curvature of } \Omega \]

Now, one can finish the argument exactly as for \( CD(K,N) \) spaces.

APPENDIX B. A WEAKER FORM OF MEAN CURVATURE BOUND ALSO SUFFICES

Inspired by [CM20], in this appendix we introduce another new notion of mean curvature bounded from below which yields Theorems 1.1 and Theorem 1.4 without requiring finite inner curvature of \( \Omega \) but assuming that the measure \( p_0 \) in Remark 2.18 is a Radon measure on \( Q \).
Definition B.1 (Backward mean curvature bounded below). Let \((X, d, m)\) be an essentially nonbranching metric measure space that satisfies MCP or CD. Recall the family of measures \(p_t = h_α \circ g(α, t)|_{V_α}, \ t \in [0, ∞)\) on \(Q\), that we introduced in Remark 2.18. Recall that \(Q\) is constructed as a Borel subset of \(X\) and \((α, t) \to g(α, t)\) is the ray map constructed in Remark 2.18.

Then \(S = \partial Ω\) has backward mean curvature bounded from below by \(H \in \mathbb{R}\) if the measure \(p_0\) is a Radon measure and

\[
\limsup_{t↑0} \frac{1}{t} \left( ∫ \phi dp_t - ∫ \phi dp_0 \right) \geq H ∫ \phi dp_0
\]

for all bounded and continuous functions \(\phi : X → [0, ∞)\).

Remark B.2. Since it is not assumed that \(p_t\) for \(t < 0\) is a Radon measure, \(∫ \phi dp_t\) can be infinite.

Lemma B.3 (Backward versus inner mean curvature). Let \((X, d, m)\) be an essentially nonbranching MCP space. Assume that \(S = \partial Ω\) for some Borel set \(Ω\) that has finite inner curvature in the sense of Definition A.2 and that \(p_0\) is a Radon measure. Then \(m_Σ\)-almost everywhere, the pointwise inner mean curvature of \(S\) is bounded from below by \(H\) if and only if \(S\) has backward mean curvature bounded from below by \(H\).

Proof. 1. Assume backward mean curvature bounded below by \(K \in \mathbb{R}\). We can compute for \(t < 0\):

\[
∫ \phi dp_t - ∫ \phi dp_0 = ∫ φ(α) (1_{V_α} h_α \circ g(α, t) - 1_{V_0} h_α \circ g(α, 0)) dq(α).
\]

Fatou’s lemma yields

\[
H ∫_{V_0} φ(α) h_α \circ g(α, 0) dq(α) = H ∫_{V_0} φ(α) dp_0(α)
\]

\[
\leq ∫ φ(α) \limsup_{t↑0} \frac{1}{t} (1_{V_α} h_α \circ g(α, t) - 1_{V_0} h_α \circ g(α, 0)) dq(α)
\]

\[
\leq ∫ φ(α) \limsup_{t↑0} \frac{1}{t} (1_{V_α \cap V_0} h_α \circ g(α, t) - 1_{V_0} h_α \circ g(α, 0)) dq(α)
\]

\[
= ∫_{V_0} φ(α) \frac{d}{dt} |_{t=0} h_α \circ g(α, t) dq(α)
\]

for any bounded and continuous function \(φ : X → [0, ∞)\). It follows that

\[
(21) \quad H h_α \circ g(α, 0) ≤ \frac{d}{dt} |_{t=0} h_α \circ g(α, t) \text{ for } q\text{-a.e. } α ∈ V_0.
\]

In particular, for \(q\text{-a.e. } α ∈ V_0\) with \(h_α \circ g(α, 0) = 0\), it follows that \(\frac{d}{dt} |_{t=0} h_α \circ g(α, t) ≥ 0\). On the other hand for \(α ∈ V_0\) with \(h_α \circ g(α, 0) = 0\) one has

\[
\frac{d}{dt} |_{r=0} h_α \circ g(α, r) = \lim_{t↑0} \frac{1}{t} h_α \circ g(α, t) \leq 0.
\]

Hence for \(q\text{-almost every } α ∈ V_0\) with \(h_α \circ g(α, 0) = 0\) one has \(\frac{d}{dt} |_{r=0} h_α \circ g(α, r) = 0\). Then, it follows that

\[
\frac{d}{dt} |_{r=0} h_α \circ g(α, r) \frac{1}{t^{α-1}} = \lim_{t↑0} h_α \circ g(α, t) \frac{1}{t^{α-1}} = 0 \text{ for } q\text{-a.e. } α \text{ s.t. } h_α \circ g(α, 0) = 0
\]

In the case of a CD space directly by Lemma 3.1 and in the case of an MCP space by (20) it follows that \(h_α \circ g(α, t) ≡ 0\) for \(t < 0\) for such \(α\), and by the disintegration formula the set of such \(α\) has \(q\)-measure 0.
In particular, we can assume $h_\alpha \circ g(\alpha, 0) > 0$ for $q$-almost every $\alpha \in \mathcal{V}_0$. Therefore, it follows that

$$H \int_{\mathcal{V}_0} \phi(\alpha)h_\alpha \circ g(\alpha, 0)d\alpha(q(\alpha)) \leq \int_{\mathcal{V}_0} \phi(\alpha) \frac{d^-}{dt}_{t=0} h_\alpha \circ g(\alpha, t)d\alpha(q(\alpha)) = \int_{\mathcal{V}_0} \phi(\alpha) \frac{d^-}{dt}_{t=0} \log(h_\alpha \circ g(\alpha, t))h_\alpha \circ g(\alpha, 0)d\alpha(q(\alpha))$$

Now, we recall that $\mathcal{V}_0 \subset \Omega(A^1) \cup B_{l_0}^1$ with $q(\Omega(A^1) \cup B_{l_0}^1) \setminus \mathcal{V}_0 = 0$ and $m(B_{l_0}^1) = q(\Omega(B_{l_0}^1)) = 0$ (because we assume finite inner curvature). Moreover $g(\alpha, t) = \gamma_\alpha(t)$, $h_\alpha \circ g(\alpha, t) = h_\alpha \circ \gamma_\alpha(t)$ and $\alpha = \gamma_\alpha(0)$ if $\alpha \in \Omega(A)$. Hence

$$H \int_{\mathcal{V}_0} \phi d\mathcal{M}_S = H \int_{\Omega(A^1)} \phi(\gamma_\alpha(0))h_\alpha(0)d\alpha(q(\alpha)) \leq \int_{\Omega(A^1)} \phi(\gamma_\alpha(0))H_{S^{-}} \circ \gamma_\alpha(0)h_\alpha(0)d\alpha(q(\alpha)) = \int \phi H_{S^{-}} d\mathcal{M}_S.$$

and consequently $H \leq H_{S^{-}} d\mathcal{M}_S$ almost everywhere.

2. Assuming inner mean curvature bounded from below by $H$ we have already observed in the prove of Theorem 1.1 that for $q$-a.e. $\alpha \in Q^1$ one has $h_\alpha(0) > 0$. Using Fatou’s lemma again, it follows that

$$H \int \phi dp_0 = \int_{\mathcal{V}_0} \phi(\alpha)Hh_\alpha \circ g(\alpha, 0)d\alpha(q(\alpha)) = \int_{\Omega(A^1)} \phi(\alpha)Hh_\alpha(0)d\alpha(q(\alpha)) \\
\leq \int_{\mathcal{V}_0} \phi(\alpha) \frac{d^-}{dt}_{t=0} \log h_\alpha(0)h_\alpha(0)d\alpha(q(\alpha)) \\
\leq \liminf_{t \to 0} \int_{\mathcal{V}_0} \phi(\alpha) \frac{1}{t} (1_{\mathcal{V}_0}(\alpha)h_\alpha(t) - 1_{\mathcal{V}_0}(\alpha)h_\alpha(0))d\alpha(q(\alpha)) \\
\leq \limsup_{t \to 0} \int_{\mathcal{V}_0} \phi(\alpha) \frac{1}{t} (1_{\mathcal{V}_0}(\alpha)h_\alpha(t) - 1_{\mathcal{V}_0}(\alpha)h_\alpha(0))d\alpha(q(\alpha)) \\
= \limsup_{t \to 0} \frac{1}{t} \left( \int \phi dp_t - \int \phi dp_0 \right).$$

Hence, the backward mean curvature is bounded from below $H$. \hfill \Box

We state a theorem under MCP. The corresponding statement for CD then follows since CD implies MCP for essentially nonbranching proper metric measure spaces.

**Theorem B.4** (Inradius bounds under backward mean curvature bounded below). Let $(X,d,m)$ be an essentially nonbranching MCP$(K,N)$ space with $K \in \mathbb{R}$, $N \in (1,\infty)$ and $\text{spt } m = X$. Let $\Omega \subset X$ be closed with $\Omega \neq X$, $m(\Omega) > 0$ and $m(\partial \Omega) = 0$. Assume $\partial \Omega = S$ has backward mean curvature bounded from below by $H \in \mathbb{R}$. Then

$$\text{InRad } \Omega \leq r_{K,H,N}.$$

**Proof.** As in the previous appendix we have

$$h_\alpha(t)^{\frac{1}{N-1}} \geq \sigma_{K/N-1}^{\frac{1}{N-1}}(a(X_\alpha))h_\alpha(0)^{\frac{1}{N-1}},$$

for an $t \in (a(X_\alpha), 0)$ and any $\alpha \in Q^1$. Therefore, it follows that

$$\frac{d^-}{dt}_{t=0} h_\alpha \circ g(\alpha, t) \leq \limsup_{t \to 0} \frac{1}{t} (h_\alpha(g(\alpha, t)) - h_\alpha(g(\alpha, 0))) \leq \frac{d^-}{dt}_{t=0} \sigma_{K/N-1}^{\frac{1}{N-1}}(a(X_\alpha))^{-1} h_\alpha(g(\alpha, 0))$$

Since the backward mean curvature is bounded below by $H$, it follows that

$$H \int_{\mathcal{V}_0} \phi(\alpha)h_\alpha \circ g(\alpha, 0)d\alpha(q(\alpha)) \leq \int_{\mathcal{V}_0} \phi(\alpha) \frac{d^-}{dt}_{t=0} h_\alpha \circ g(\alpha, t)d\alpha(q(\alpha)).$$
We obtain again the inequality (21) exactly as in the beginning of step 1 of the proof of Lemma B.3 (where we did not use finite inner curvature), and this yields again \( h_\alpha(0) > 0 \) for \( \alpha \)-almost every \( \alpha \). Hence, it follows that

\[
H \int_{V_0} \phi(\alpha) h_\alpha \circ g(\alpha, t) d\alpha(\alpha) \leq \int_{V_0} \phi(\alpha) \frac{d}{dt} \bigg|_{t=0} \log(h_\alpha \circ g(\alpha, t)) h_\alpha \circ g(\alpha, t) d\alpha(\alpha)
\]

Hence \( \frac{H}{N-1} \leq \frac{d}{dt} \bigg|_{t=0} \log h_\alpha \circ g(\alpha, t) \leq \frac{\cos K/(N-1)(\sigma(X_\alpha))}{\sin K/(N-1)(\pi(X_\alpha))} \) for \( \alpha \)-a.e. \( \alpha \in V_0 = \Omega(\cup B_{in}). \)

At this point it is clear that we can finish the proof as before. \( \square \)

**Theorem B.5** (Rigidity under backward mean curvature bounded from below). Let \((X, d, m)\) be \( RCD(K, N) \) for \( K \in \mathbb{R} \) and \( N \in (1, \infty) \) and let \( \Omega \subset X \) be compact with \( \Omega \neq X \), \( m(\Omega) > 0 \), connected and non-empty interior \( \Omega^0 \) and \( m(\partial \Omega) = 0 \). We assume that \( K \in \{N-1, 0, -(N-1)\} \), \( \partial \Omega = S \neq \{p\} \) and \( S \) has backward mean curvature bounded below by \( \kappa(N-1) \in \mathbb{R} \). Then, there exists \( x \in X \) such that

\[
d_S(x) = \text{InRad} \Omega = \tau_{K, \kappa(N-1), N}^\Omega
\]

if and only if \((K, \kappa) \in \{-(N-1), 0\} \times (0, \infty) \cup \{N-1\} \times \mathbb{R} \) and there exists an \( RCD(N-2, N-1) \) space \( Y \) such that \((\Omega^0, d_{\Omega})\) is isometric to \((B_{K, \kappa(N-1), N}(0), \tilde{d})\) in \( L^{\frac{N}{N-1}}(\sin K/(N-1)) \times Y \), where \( \tilde{d}_{\Omega} \) and \( \tilde{d} \) are the induced intrinsic distances of \( \Omega^0 \) and \( B_{K, \kappa(N-1), N}(0) \), respectively.

**Proof.** We observe that the inequality (23) implies \( H \leq \frac{d}{dt} \bigg|_{t=0} \log h_\alpha \circ g(\alpha, t) \) for \( \alpha \)-a.e. \( \alpha \in V_0 \), so in particular for \( \alpha \in \Omega(B_{in}) \). Then using the Riccatti comparison and the maximum principle we can follow verbatim the same proof as in Section 4. \( \square \)

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