

MAT 477H — Fall Semester Syllabus 2007-08 (revised)
PRIMARY SOURCES

Weeks

- 1 Introduction and Historical Overview of Optimal Transportation [4].
- 2 Continue Orientation [5]. Finish deciding who presents what.
- 3 L. Kantorovich. On the translocation of masses. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **37**, 199–201 (1942).
T.C. Koopmans. Optimum utilization of the transportation system. *Econometrica (Supplement)*, 17:136–146, 1949.
Kantorovich and Koopmans shared the 1975 Nobel Prize in Economics for these. The mathematics is in the Kantorovich paper, the economics in Koopmans, but they fit together nicely. Both were trained as mathematicians, though Koopmans also had a degree in physics. Any background in real and functional analysis or linear programming would be a plus.
- 4 R.T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pacific J. Math.* **17**, 497–502 (1966).
Along with Theorem 1, this presenter needs to cover some basic background on Legendre transforms of convex functions [7, Chapter 6] and help prepare for Brenier.
- 5 (ex 6) L. Caffarelli. Allocation maps with general cost functions. In P. Marcellini et al, editors, *Partial Differential Equations and Applications*, number 177 in Lecture Notes in Pure and Appl. Math., pages 29–35. Dekker, New York, 1996.
- 6 (ex 7) A.D. Aleksandrov. Existence and uniqueness of a convex surface with a given integral curvature. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **35**, 131–134 (1942).
Spectacular, but not easy. Some background in geometry (the Gauss map, curvature) is a plus. This presentation will probably take place on Oct 18. The uniqueness argument is spelled out in more detail in McCann.

- 7 (ex 5) Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C.R. Acad. Sci. Paris Sér. I Math.* **305**, 805–808 (1987).

This short though difficult French paper proves a nice theorem with a lot of nice connections, to e.g. Hodge theory. The basic argument got expanded in English [8] but it is probably easier to follow in one of [9, 10] or the following paper by Caffarelli. I recommend these two presenters coordinate to divide up the proofs and the applications.

- 8 R.J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* **80**, 309–323 (1995).

Brenier’s theorem put into perspective by Aleksandrov and Rockafellar.

- 9 H. Tanaka. An inequality for a functional of probability distributions and its application to Kac’s one-dimensional model of a Maxwellian gas. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **27** (1973) 47–52.

Used of a transportation distance to proof central limit theorem and convergence to equilibrium of the velocity profile in a model for a one-dimensional gas.

- 10 S. Alesker, S. Dar and V. Milman. A remarkable measure preserving diffeomorphism between two convex bodies in \mathbf{R}^n . *Geom. Dedicata* **74** (1999) 201–212.

More than you ever wanted to know about inequalities governing vector sums of convex bodies. The basic inequality is Brunn-Minkowski [11, §3.2.41]: domination of the geometric by the arithmetic mean gone bananas. See also [12].

- 11 A. Aleksandrov. Smoothness of the convex surface of bounded Gaussian curvature. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **36** (1942) 195–199.

Regularity (smoothness) theorems are hard, but rewarding. This paper was ahead of its time, and may be the most challenging work we cover this semester. Uses a geometrical argument to show the surface constructed in 6. above is actually differentiable.

- 12 D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.* **182** (2004) 307–332.

Combines entropy and transportation to recover this fundamental inequality in analysis. Although the paper is long, the key theorem and its proof are more succinctly covered on pages 200-203 of [1]. This presentation may take place on Nov 29.

- 13 F. Otto. Viscous fingering: an optimal bound on the growth rate of the mixing zone. *SIAM J. Appl. Math.* **57** (1997) 982–990.

Derives an inequality which limits the rate of mixing between two immiscible fluids at the unstable interface as the more mobile one displaces the other under pressure but forms fingers as the displacement occurs in a porous medium (or a channel between two glass plates). The presenter should have a basic comfort level with partial differential equations, ideally those which arise in the study of fluids. This presentation may take place on Dec 6.

MAT 477H — Fall Semester Supplementary

References

- [1] C. Villani. *Topics in Optimal Transportation*. American Mathematical Society (2003) Providence.
- [2] C. Villani. *Optimal Transport, Old and New*. To appear www.umpa.ens-lyon.fr/~cvillani/Cedrif/surveys.html#oldnew.
- [3] L.C. Evans. Partial differential equations and Monge-Kantorovich mass transfer. In R. Bott et al., editors, *Current Developments in Mathematics*, pages 26–78. International Press, Cambridge, 1997.
- [4] W. Gangbo and R.J. McCann. The geometry of optimal transportation. *Acta Math.*, 177(2):113–161, 1996.
- [5] R.J. McCann. Exact Solutions to the Transportation Problem on the Line. *Proc. Roy. Soc. London Ser. A* **455** (1999) 1341–1380.
- [6] G.G. Lorentz. An Inequality for Rearrangements, *Amer. Math. Monthly* **60** (1953) 176–179
- [7] R.A. Adams. “Sobolev Spaces.” Academic Press, New York, 1975.
- [8] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44**, 375–417 (1991).
- [9] W. Gangbo and R.J. McCann. Optimal maps in Monge’s mass transport problem. *C.R. Acad. Sci. Paris Sér. I Math.* **321**, 1653–1658 (1995).
- [10] R.J. McCann. Polar Factorization of Maps on Riemannian Manifolds. Submitted to *J. Differential Geom.*
- [11] H. Federer. *Geometric Measure Theory*. Springer-Verlag, New York, 1969.
- [12] R.J. McCann. A convexity principle for interacting gases. *Adv. Math.* **128** (1997) 153–179.