Lagrangian Formulation of General Relativity

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Introduction

In 1788 Joseph Louis Lagrange suggested an alternative formulation of classical mechanics, in which one considers the variation of an integral function of the field variables of the system. The function, known as the Lagrangian, is integrated over physical space in what is known as the action. Suitable derivatives of the action are defined, and the field equations are satisfied when the action is extremized. This has led to the unfortunate term "least-action principle," which is a misnomer since the extremum need not correspond to a local minimum.

In classical mechanics, there are myriad advantages to the Lagrangian approach. One finds that extrema satisfy a general set of equations due to Euler-Lagrange, and furthermore, these equations hold in any reference frame.[1] Since the formulation allows the choice of arbitrary generalized coordinates, calculations may be simplified considerably. In particular, constraints on the system may be handled implicitly by the choice of coordinates, so that one avoids finding expressions for, for example, normal and tension forces.

In general relativity, motivation for the Lagrangian approach is more subtle. The theory necessarily handles arbitrary reference frames, so no advantages are to be gained there. However, there is an aesthetic appeal to the Lagrangian formulation, for Einstein's equation can be derived from a very natural Lagrangian density. Still, perhaps the greatest motivation comes from a desire to unify general relativity with quantum field theory. The latter relies heavily on a path integral formulation of quantum mechanics due to Feynman, in which probability amplitudes are given by integrating $e^{\frac{i}{\hbar}S}$ over all possible paths, where S is the action. The incidence of the Lagrangian in both general relativity and quantum field theory thus provides a point of contact for a potential reconciliation of the presently incompatible theories.

In this paper we present the Lagrangian formulation of general relativity and use the formulation to investigate possible extensions to Einstein's theory. For this, we review a seminal contribution by Barrow and Ottewill. [1]

Lagrangian Formulation

Following Wald[5], we give the general formulation of a Lagrangian field theory. Consider a tensor field Ψ defined on a compact manifold M, suppressing

[1] Recall that Newton's laws only hold in inertial frames, and fictitious forces such as the Coriolis and centrifugal forces need to be added when applicable.

all indices on Ψ. Let Ψ_λ be a smooth one-parameter family of field configurations on M, starting from Ψ_0 and such that $\Psi_\lambda|_{\partial M} = \Psi_0$ for all λ. Denote $d\Psi/d\lambda|_{\lambda=0}$ by $\delta\Psi$. Consider $S[\Psi] = \int_M L[\Psi]$ where L is a local function of Ψ and finitely many of its derivatives. We wish to define a suitable derivative for S. Suppose $dS/d\lambda|_{\lambda=0}$ exists for all such families Ψ_λ starting from Ψ_0, and that there exists a smooth tensor field χ, dual to Ψ, such that

$$\frac{dS}{d\lambda} = \int_M \chi\delta\Psi \tag{1}$$

where all indices in the integral are contracted. Then χ is called the *functional derivative of* S at Ψ_0, and written $\chi = \frac{\delta S}{\delta\Psi}|_{\Psi_0}$. Finally, suppose the field configurations Ψ which extremize S,

$$\frac{\delta S}{\delta\Psi}\ |_{\Psi_0} = 0 \tag{2}$$

are precisely those which are the solution to the field equation for Ψ. Then S is called the *action* and L the *Lagrangian density*, and together they constitute the Lagrangian formulation of the field theory.

As an example, consider

$$L_{EM} = -1/4F_{ab}F^{ab} = -\partial_{[a}A_{b]}\partial^{[a}A^{b]} \tag{3}$$

where F is the Maxwell tensor from electromagnetism, and A is the vector potential. It is easy to see that L_{EM} is a Lagrangian density over the field variable A^b for Maxwell's equations in a subset of flat spacetime:

$$\begin{aligned}
\frac{dS_{EM}}{d\lambda}\ |_{\lambda=0} &= -\int_M (\partial_{[a}A_{0b]}\partial^{[a}(\delta A)^{b]} + \partial_{[a}(\delta A)_{b]}\partial^{[a}A_0^{b]})d^4x && (4)\\
&= -\int_M 2\partial_{[a}A_{0b]}\partial^{[a}(\delta A)^{b]}d^4x && (5)\\
&= \int_{\partial M} 2\partial_{[a}A_{0b]}(\delta A)^{[b}n^{a]} + \int_M 2\partial^{[a}\partial_{[a}A_{0b]}(\delta A)^{b]}d^4x && (6)\\
&= \int_M 2\partial^a\partial_{[a}A_{0b]}(\delta A)^b d^4x && (7)
\end{aligned}$$

so $\frac{\delta S}{\delta A^b} = 2\partial^a\partial_{[a}A_{b]}$, where we've used the fact that δA^b vanishes on the boundary. This gives the field equation $\partial^a\partial_{[a}A_{b]} = 0$, which is Maxwell's equation in a vacuum.[5]

To produce Einstein's equation in a vacuum, we consider the Lagrangian density $L_G = \sqrt{-g}R$ where g is the determinant of the metric tensor $g_{\mu\nu}$, and R is the Ricci curvature. We wish to integrate L_G over our domain to produce what is known as the *Hilbert action*. However, a complication arises from the fact that the natural volume element for integration depends on $g^{\mu\nu}$, which will be our field variable.[2] To overcome this, we fix a volume element **e**, and perform integration with respect to **e**. Further progress will require some results about integration on manifolds.

Recall that integration of a function on a manifold is defined via a volume element, a continuous non-vanishing n-form.[3] This is necessary so that the integral is well-defined with respect to changes in coordinates. We restrict attention to the case where n=4 and the metric has Lorentzian signature. The vector space of 4-forms has dimension $\frac{4!}{4!(4-0)!} = 1$, so every volume element is a scalar multiple of every other. As alluded to above, a natural choice of volume element is provided by the metric. For a given metric, there is a unique volume element ϵ_{abcd} (up to sign) specified by

$$\epsilon_{abcd}\epsilon^{abcd} = -4! \tag{8}$$

For the following, we write the volume element with suppressed indices. Assume that **e** is obtained by the above formula from the unperturbed metric. A calculation shows that if ϵ is the volume element obtained when the metric is varied, then $\epsilon = \sqrt{-g}\mathbf{e}$.[5] In light of this relation, the appearance of $\sqrt{-g}$ in the Lagrangian density is not surprising.

As mentioned earlier, we consider g^{ab} as the field variable, and so define $\delta g^{ab} = \frac{dg^{ab}}{d\lambda}|_{\lambda=0}$. For convenience we define $\delta g_{ab} = \frac{dg_{ab}}{d\lambda}|_{\lambda=0}$. $g_{ab}g^{bc} = \delta^c_a$ then implies that $\delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}$, so that the metric *cannot* be casually used to raise and lower the metric variations.

To determine the variation of the Hilbert action we must calculate

$$\frac{dL_g}{d\lambda}|_{\lambda=0} = \sqrt{-g}g^{ab}\delta R_{ab} + R\delta(\sqrt{-g}) + \sqrt{-g}R_{ab}\delta g^{ab} \tag{9}$$

Computation of $\delta R_{ab} = \frac{dR_{ab}}{d\lambda}|_{\lambda=0}$ requires knowledge of the behaviour of R_{ab} near $\lambda = 0$. It is helpful to write our perturbed metric[4] as $g_{ab}(\lambda) = g_{ab} + \gamma_{ab}$, where it is understood here and in the following that g_{ab} written without λ refers to the unperturbed metric. We may assume that $\gamma_{ab} = \frac{dg_{ab}(\lambda)}{d\lambda}|_{\lambda=0} = \delta g_{ab}$, since we will only be interested in the derivative of R_{ab} at $\lambda = 0$. Let ${}^\lambda\nabla_a$ denote the derivative operator associated with $g_{ab}(\lambda)$, and let ∇_a denote the derivative operator associated with g_{ab}. Then there exists a tensor field $C^c_{ab}(\lambda)$ such that for any ω_b, ${}^\lambda\nabla_a\omega_b = \nabla_a\omega_b - C^c_{ab}(\lambda)\omega_c$ and

[2] An alternative way of deriving Einstein's equation is to vary the Palatini action, which uses the same Lagrangian density but considered as a function of R_{ab} and varied with respect to $g^{\mu\nu}$ and ∇_a. Wald sketches this derivation.

[3] That is, a totally antisymmetric (0,n) tensor

[4] i.e. when $\lambda \neq 0$

$$C^c_{ab}(\lambda) = \frac{1}{2} g^{cd}(\lambda)\left(\nabla_a g_{bd}(\lambda) + \nabla_b g_{ad}(\lambda) - \nabla_d g_{ab}(\lambda)\right) \tag{10}$$

Then for any ω_c, we have

$$\begin{aligned} {}^\lambda\nabla_a {}^\lambda\nabla_b \omega_c &= {}^\lambda\nabla_a \left(\nabla_b \omega_c - C^d_{bc}\omega_d\right) && (11)\\ &= \nabla_a \left(\nabla_b \omega_c - C^d_{bc}\omega_d\right) - C^e_{ab}\left(\nabla_e \omega_c - C^d_{ec}\omega_d\right) \\ &\quad - C^e_{ac}\left(\nabla_b \omega_e - C^d_{be}\omega_d\right) && (12)\end{aligned}$$

so that

$$\begin{aligned} {}^\lambda R_{abc}{}^d \omega_d &= 2 {}^\lambda\nabla_{[a} {}^\lambda\nabla_{b]}\omega_c && (13)\\ &= 2\nabla_{[a}\nabla_{b]}\omega_c - 2\nabla_{[a}(C^d_{b]c}\omega_d) - 2C^e_{c[a}\nabla_{b]}\omega_e + 2C^e_{c[a}C^d_{b]e}\omega_d && (14)\\ &= \left(R_{abc}{}^d - 2(\nabla_{[a}C^d_{b]c}) + 2C^e_{c[a}C^d_{b]e}\right)\omega_d && (15)\end{aligned}$$

and we can drop ω_d from both sides. The Ricci tensor is

$${}^\lambda R_{ab} = R_{ab} - 2(\nabla_{[a}C^c_{c]b}) + 2C^e_{b[a}C^c_{c]e} \tag{16}$$

and we can now find its variation. The term quadratic in C^e_{ba} will vanish, because, by definition, $C^e_{ba}|_{\lambda=0} = 0$. Then

$$2\delta C^c_{cb} = g^{cd}\left(\nabla_c \delta g_{bd} + \nabla_b \delta g_{cd} - \nabla_d \delta g_{bc}\right) \tag{17}$$

implies that

$$\begin{aligned} \delta g^{ab} R_{ab} &= -2g^{ab}\nabla_{[a}\delta C^c_{c]b} && (18)\\ &= -g^{ab}g^{cd}\left(\nabla_{[a}\nabla_{c]}\delta g_{bd} + \nabla_{[a}\nabla_{|b|}\delta g_{c]d} - \nabla_{[a}\nabla_{|d|}\delta g_{c]b}\right) && (19)\\ &= -\frac{1}{2}g^{ab}g^{cd}\nabla_c\nabla_d\delta g_{ab} - \frac{1}{2}g^{ab}g^{cd}\nabla_a\nabla_b\delta g_{cd} + g^{ab}g^{cd}\nabla_c\nabla_{(b}\delta g_{a)d} && (20)\\ &= \nabla^a\left(\nabla^b(\delta g_{ab}) - g^{cd}\nabla_a(\delta g_{cd})\right) && (21)\\ &\equiv \nabla^a v_a && (22)\end{aligned}$$

where v_a has been defined as indicated. To compute $\delta(\sqrt{-g})$ we appeal to the non-singular matrix identity[5]

$$tr\left[\frac{dA}{d\lambda}A^{-1}\right] = \frac{1}{\det A}\frac{d(\det A)}{d\lambda} \tag{23}$$

which, when applied to g_{ab} yields

$$\frac{dg_{ab}(\lambda)}{d\lambda}g^{ab}(\lambda) = \frac{1}{g(\lambda)}\frac{dg(\lambda)}{d\lambda} = \frac{2}{\sqrt{-g(\lambda)}}\frac{d\sqrt{-g(\lambda)}}{d\lambda} \tag{24}$$

so that

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab} \tag{25}$$

Finally, from (9) we obtain

$$\frac{dS_G}{d\lambda} = \int_M \nabla^a v_a\sqrt{-g}\mathbf{e} + \int_M \left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab}\sqrt{-g}\mathbf{e} \tag{26}$$

The first term is the integral of a divergence with respect to ϵ, the natural volume element. We can apply Stokes' theorem to convert this to an integral over ∂M. This term is related to the extrinsic curvature of the boundary, and in general does not vanish. However, a modification of L_G can eliminate the term, so in modern work it is customary to ignore it.[1, 2] It is usually only present in textbook derivations.[5, 3][5] Thus, ignoring this contribution we have

$$\frac{\delta S_G}{\delta g^{ab}} = \left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\sqrt{-g} \tag{27}$$

which gives Einstein's equation in a vacuum when extremized. This result is easily generalized to the case where matter fields are present by the addition of matter terms to the Lagrangian density:

$$L = L_G + \alpha_M L_M \tag{28}$$

where α_M is a normalization constant. For example, electromagnetic fields can be incorporated with $L_M = -\sqrt{-g}g^{ac}g^{bd}\nabla_{[a}A_{b]}\nabla_{[c}A_{d]}$, which we note is a natural generalization of (3). A perfect fluid can be included with the Lagrangian density $L_{PF} = 16\pi\sqrt{-g}P$.[4]

It is interesting to note that 28 gives Einstein's equation $G_{ab} = 8\pi T_{ab}$ when we make the identification

$$T_{ab} = -\frac{\alpha_M}{8\pi}\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{ab}} \tag{29}$$

so that when L_M is known (or postulated) this equation can be used as the *definition* of the stress-energy tensor T_{ab}.

[5] This may not be entirely true, but it appears that way from my limited research.

A final note that we present without rigorous proof is that invariance of S_M under diffeomorphisms implies conservation of T_{ab}. Consider a 1-parameter family of diffeomorphisms, $f_\lambda : R \to R$. If the matter field is specified by Ψ, then the equation $\frac{dS_M}{d\lambda} = 0$ leads to

$$0 = \int_M \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab} + \int_M \frac{\delta S_M}{\delta \Psi} \delta \Psi \tag{30}$$

Since Ψ satisfies the matter field equations, $\frac{\delta S_M}{\delta \Psi}|_\Psi = 0$. Wald shows that Lie derivatives are related to variations in the sense that $\mathcal{L}_w g^{ab} = \delta g^{ab}$ when w^a generates f_λ.[5] It is also shown that $\mathcal{L}_w g^{ab} = 2\nabla^{(a} w^{b)}$, so that if w^a is smooth and compactly supported,

$$0 = \int_M \sqrt{-g} T_{ab} \nabla^{(a} w^{b)} \mathbf{e} \tag{31}$$

$$= \int_M T_{ab} \nabla^{(a} w^{b)} \epsilon \tag{32}$$

$$= -\int_M (\nabla^a T_{ab}) w^b \epsilon \tag{33}$$

and therefore $\nabla^a T_{ab} = 0$. The same argument when applied to S_G will show that, independent of Einstein's equation, $\nabla^a G_{ab} = 0$. Thus the contracted Bianchi identity arises from the invariance of the Hilbert action under diffeomorphisms. These results further supports the idea of an action principle as fundamental in theoretical physics.

Extensions to General Relativity

Because of the simple form of the Lagrangian density, the Lagrangian formulation of general relativity provides a natural framework to consider extensions to general relativity. Since Hilbert's original derivation of Einstein's equation by an action principle in 1915, numerous variations of the Hilbert action have been proposed and investigated.[1] A primary motivation was the undesirable singular behaviour that is allowed by Einstein's equation at areas of large curvature. With Lagrangian densities that include higher orders of curvature, perhaps this would be avoided. Also, some quantum corrections to general relativity are equivalent to adding higher order curvature terms.[1] We consider a contribution by Barrow and Ottewill from 1983, in which modifications to the Lagrangian density of the form $f(R)$ are considered, where f is an arbitrary analytic function. This ignores higher order terms of forms like $R_{abcd}R^{abcd}$, $R_{ab}R^{ab}$, etc., but is otherwise quite general.

Consider the gravitational action

$$S_G = -\frac{1}{2}\int f(R)\sqrt{-g}d^4x \tag{34}$$

We consider a homogeneous and isotropic Friedman universe for which

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\sigma r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{35}$$

The Friedman model has proven to provide an excellent fit to observable astronomical and cosmological data.[1, 5] However, it predicts a spacetime singularity in the finite past. For this reason, it is desirable to determine the effect of possible corrections to general relativity on this prediction.

Barrow and Ottewill's analysis shows that in general, there is a trade-off between pathological behaviour in the past and in the future. Models that avoid a big-bang singularity by having a non-zero minimum value of a blow up in the future, with $R \to \infty$ as $t \to \infty$. This behaviour can be traced back to the fact that the extremum of the action that yields the field equations is not minimal. That is, $\delta S_G = 0$ but $\delta^2 S_G < 0$. This suggests that a guiding principle for non-pathological theories of gravity could be to seek a theory such that $\delta S_G = 0$, $\delta^2 S_G > 0$, and of course such that solutions are admitted that match the observable universe.

In recent decades, hope for a unification of quantum mechanics with general relativity has lain in string theory, which posits, among other things, that the universe may be 10 or 26-dimensional, and that the building blocks of reality are 1 dimensional strings of energy of the Planck length ($10^{-35}m$). Despite its power, many cosmological observations remain a mystery. Chief among these is the elusive dark matter, which according to measurements of the expansion of the universe, must account for over seventy percent of the mass of the universe. A recent suggestion by Carroll et al. is that instead of dark matter, the expansion of the universe may be attributed to a modification of general relativity in which a term of the form R^{-1} is added to the Lagrangian.[2] Specifically, the modified Hilbert action is taken to be

$$S_G = -\frac{1}{2}\sqrt{-g}\left(R - \frac{\mu^4}{R}\right)d^4x \tag{36}$$

where μ is a new constant with units of mass. Carroll et al. show that inclusion of the new curvature term can indeed explain the present expansion of the universe without introducing any pathological behaviour.

The Lagrangian formulation of general relativity has proven to be more than an academic curiosity. The underlying structure of the action principle and the simplicity of general relativity when cast in this form has made it a natural framework in which to extend Einstein's theory. So long as cosmological mysteries remain to be solved, Lagrange's principle will continue to be a useful tool.

References

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