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SOME FOUNDATIONS OF ANALYSIS FOR ENGINEERING SCIENCE (MAT194F)

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'So is your intellect stripped clear, and I will now reveal a truth so radiant that it will sparkle for you like a star'

Dante, Paradise, Canto II, 109-111

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1. INTRODUCTION

Rigorous Proof in Calculus

Hints on how to read this section

This is introductory material that you should read carefully and think about.

Mathematics is a most useful tool in engineering and science. But, it can serve this purpose only if you have a good command of technique, can make good judgements about appropriate procedures and have an understanding of how its ideas fit together. Thus, good **intuition** of what is mathematically correct - something we gain from practice and experience - is an important part of this understanding. An equally important part of this understanding comes from appreciating **logical proof** and the logical connection of ideas. Sometimes, mathematics is difficult to apply or it becomes too complex for our intuition to handle; in these situations you need to be able to go back to basic ideas and reason things through, step by step. While intuition, formulae and procedures are very useful, you cannot rely only on these all the time. Logical proof also gives you a look at the beauty and intellectual power of mathematics.

In short, mathematicians require both a good **intuitive sense** of what constitutes a correct mathematical statement, as well as the **ability to rigorously prove** the correctness of the statement.

While geometrical and physical intuition play a major role in understanding and using calculus ideas, over the past 200 years, mathematicians realized that there were severe conceptual problems and difficulties involving the infinite that needed to be sorted out using rigorous logic. The key notions of **limit** and **continuity** needed to be defined in a way that made the subject self-contained and not dependent on either geometry or physics. Part of doing this required a deep understanding of the system of real numbers as a **continuum**. This is not easy to achieve, and one of your goals over the year will be to reflect on the **number system** and try to understand how its properties feed into the ideas of the calculus.

It was natural for mathematicians in the seventeenth and eighteenth centuries to think in terms of "infinitesimal," as finding tangents involved looking at secants of points infinitely close together and calculating areas involved adding the areas of infinitely small rectangles. But this notion led to many paradoxes and caused considerable controversy and confusion. If we want to find the slope of a tangent, regarded as a secant going through coincident points on a curve, we have to look at a ratio of differences of the form 0/0. What can this possibly mean? What can division by 0 signify? We define a/b to be that number which when multiplied by b gives a. With this idea, there are infinitely many candidates we could use as the definition of 0/0 (say 17), and none for $a/0$ when a is nonzero. Maybe $a/0$ is a new, "infinite number", ∞. But then what value can we assign to $\infty - \infty$ or ∞ / ∞ for example? Also, presumably $\infty + 2 = \infty + 3$, but doesn't that then mean that 2 = 3? There are many conceptual difficulties. So in the nineteenth century, mathematicians such as Cauchy and Weierstrass sought to formulate calculus in a rigorous way that avoided the use of infinitesimals. This is essential, since no one has, until recently, come up with a satisfactory definition of an "infinitesimal." Are they numbers? Not really - not like other numbers. So what exactly are they? You may think that the results of the rigorous approach are a bit arcane, but the process is necessary to ensure that the subject is on a firm footing.

To get around the problem of dealing with infinities and infinitesimals, we introduce the concept of a limit. This can be rigorously defined, but requires that we understand carefully what numbers are, along with their properties. We can be helped in doing this by thinking of instantaneous speed in physics, as a limit of average speeds over smaller and smaller time intervals.

Calculus is a challenging subject, because to use it well, you have to be able to think on different levels and to switch from one to another as appropriate. Certainly there are techniques that you will learn, because the power of calculus resides in its ability to give general procedures that can be applied to a wide variety of problems, and to provide algorithms that can be applied without much conscious thought. But if you approach the subject as a collection of formulae and processes to memorize, then you will find mastery of the subject to be so heavy that you will not be able to use it. Often problems become simple by looking at them with the right perspective and an exercise of judgment about the application of a technique may lead to a simple solution rather than one that is horribly complicated. So you will need to practise, but with your eyes open and ears attuned to nuances. You will also regularly need to refer to your geometrical or physical intuition about what is going on to help get an overview of the situation. As with all challenging and worthwhile enterprises, *practice and experience make all the difference*. Do not expect things to be clear right away. Just hang in. Thousands before you have made it. You will too!

2. THE REAL NUMBER SYSTEM

2.1 Numbers: The Terrain for Calculus

The limit concept entails knowing when numbers are close together or far apart, as well as understanding that the number system is a **continuum**, that is like a geometric line or Newtonian time.

We actually define, in effect, the number system by a **set of axioms**. These basic properties arise from our intuition, but are accepted as assumptions. All we require is that they are *consistent*, *i.e.* do not contradict each other. Upon this foundation we systematically derive, i.e. logically prove, all the propositions – **theorems** - we will need to do calculus. If you think about it for a while, you will realize that numbers do *not* exist in the physical world. They are *ideas,* '*mental invention,'* '*abstractions.'* We simply *make them up!* But they mirror the reality we want to study.

2.2 The Field Axioms of Numbers

Hints on how to read this section

*An entire course could be devoted to the undertaking of a rigorously logical development of the familiar rules of arithmetic and algebra, starting from the axioms of the real numbers. That is not the intention here, so just skim through sections 2.2 and 2.3 in order to get the 'spirit' of how such an exercise in pure logic would go. For calculus, the most important point to appreciate is that while we simply make up the axioms – i.e. the properties of real numbers - we have good reasons not to include a/0 nor ∞ as real numbers. That simple fact has **enormous** consequences for how we can logically define the **derivative** and the **integral** – the two most important types of number in calculus. It is this simple fact that forces us to come up with the concept of the **limit** - and to rigorously define it.*

The number system we construct, the **reals**, to be denoted by **R**, will have two main arithmetic operations, **addition** and **multiplication**, along with the derived operations of *subtraction and division.* Each of these **operations** will act on an **ordered pair** (a, b) of numbers to produce a number, be it $a + b$, ab, $a-b$, $a \div b$. Thus, not only are numbers, in themselves, our mental inventions but so also are operations through which they can be made to relate with one another. In this, mathematics is just a logical game like chess. We do not attempt to directly define the operations, but do so indirectly by specifying the axioms or rules that we want them to obey. This is all that is needed to "play the game" of mathematics. First we have the **field axioms**:

Axiom 1: **(Commutative law)** For each pair $x, y \in$ **R,**

$$x + y = y + x \quad \text{and} \quad xy = yx.$$

Axiom 2: **(Associative law)** For each triple $x, y, z \in$ **R,**

$$x + (y + z) = (x + y) + z \quad \text{and} \quad (xy)z = x(yz).$$

Axiom 3: **(Distributive law)** For each triple $x, y, z \in$ **R,**

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz.$$

Axiom 4: **(Existence of identities)** There exist two distinct real numbers, denoted by 0 and 1 for which

$$x + 0 = 0 + x = x \quad \text{and} \quad x \cdot 1 = 1 \cdot x = x$$

for each $x \in \mathbf{R}$.

Axiom 5: **(Existence of inverses)** For each $x \in \mathbf{R}$, there exists a unique additive inverse, which we denote by $-x$ ("minus x"), for which

$$x + (-x) = (-x) + x = 0.$$

For each $x \neq 0$ in $\mathbf{R}$, there exists a unique multiplicative inverse, which we denote by x^{-1} or $1/x$ (the reciprocal of x), for which

$$x \cdot (x^{-1}) = (x^{-1}) \cdot x = 1.$$

Note that we do not provide a reciprocal for 0. Any attempt to do so leads us to logical difficulties. Let us see why. On the basis of the axioms, we can show that $a \cdot 0 = 0$ for each $a \in \mathbf{R}$. From the axioms, we see that

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$$

Add to each side of this equation the additive inverse of $a \cdot 0$ to get $0 = a \cdot 0$.

Exercise 1: Go over this carefully and list which axioms are being used.

Since every element of $\mathbf{R}$ multiplied by 0 gives 0, there is no element we can multiply 0 by to get 1.

Having defined addition and multiplication, it is now a minor matter to define subtraction and division by:

$$a - b \equiv a + (-b) \qquad a \div b \equiv a/b \equiv a \cdot (1/b).$$

You may think that if numbers are the basic building blocks of mathematics, we should define them first. But what we really need to know are their properties and ways of dealing with them. So, in effect, it is enough to do what we have done: specify the rules - axioms - that they must obey. It's like a game, say chess, where each piece such as the pawn is effectively defined by the rules it has to obey. Any piece that obeys the same rules *is* a pawn and any piece that does not *isn't*.

The use of the word "real" as applied to numbers is historical and somewhat unfortunate, since no numbers actually exist in the physical world. It also seems to imply that other numbers satisfying other axioms which we are equally free to specify and which are useful in their own sphere, are somehow less real, such as the, also unfortunately named, **imaginary numbers**. Real numbers and imaginary numbers are equally "imaginary" in that they exist only as creations of our imagination. Each type of number has its own close parallels to certain aspects of the physical world, and so can be used to help construct our **mental models** of that world. Mental models are central to all science and engineering. It is too bad that imaginary numbers, because of this historical naming, have ended up seemingly weird or unrelated to the physical world, in comparison with the reals.

On the other hand, the word "real" does bring out an important point. While we were free to choose the axioms defining reals, we did not choose them blindly. We were guided by our observations of the physical world which often - albeit not always - behaves in ways reminiscent of these specific axioms. Real numbers tend to be the most common type of number we use for constructing our mental models of the physical world. When you come to study quantum theory, electrical circuits, etc., you will find that you will need more sophisticated mathematical artefacts (mental inventions) to do this – such as imaginary numbers.

2.3 The Order Properties of Real Numbers

If numbers are to be modelled by a line (which is a feature that we simply *want* reals to possess), then we have to have some analogue of the property of the line that provides that we can move from left to right along the line with each pair of points having one to the left and one to the right. We do this by specifying that the numbers can be ordered by a relation, < which has the following properties:

Axiom 6: (Trichotomy) For each pair (a, b) of numbers, exactly one of the following holds:

$$a < b \qquad a = b \qquad b < a.$$

We define a number a to be **positive** if and only if $0 < a$.

Axiom 7: If a and b are positive, then so are $a + b$ and ab.

There are some important consequences of these axioms. If a is a positive, then $-a < 0$. To see this, note that $-a \neq 0$ (since $a \neq 0$). If $-a$ were positive, then $0 = a + (-a)$ would also be positive, contrary to the law of trichotomy.

Exercise 2: Show that 1 is positive.

A number a is **negative** if and only if $-a$ positive. The number is **nonnegative** if and only if it is either 0 or positive. We write $a \leq b$ to mean that $a = b$ or $a < b$, $a > b$ to mean $b < a$ and $a \geq b$ to mean that $a = b$ or $a > b$.

Axiom 8: **(The archimedean property)** Given any element $r \in \mathbf{R}$ and any positive element p, we can find a positive integer n for which $np > r$.

The archimedean property is a little deep, but it basically says that the real numbers have no infinite elements, that we can reach beyond any given real number by counting up 1, 2, 3, 4, … sufficiently far. An important consequence of the archimedean property is the following:

Proposition: $x \leq 0$ if and only if $x < 1/n$ for each positive integer n.

Proof: **"If and only if" proofs require that we carry out the proof in both directions.** First, suppose we are given that $x \leq 0$. Then, clearly, $x < 1/n$ for each positive integer n. So one part is done.

Next, say we are given that $x < 1/n$ for each positive integer n. We will now construct a **proof by contradiction.** Let us suppose that somehow $x > 0$. Then, by Axiom 8, there exists a positive integer m for which $m > 1/x$, or equivalently, $x > 1/m$. This contradicts our assumption that $x < 1/n$ for each positive integer n. Hence, we must abandon the possibility that $x > 0$ and deduce that $x \leq 0$. ♠

We define various types of intervals:

$[a, b] = \{x : a \leq x \leq b\}$ (closed interval)

$(a, b) = \{x : a < x < b\}$ (open interval)

$[a, b) = \{x : a \leq x < b\}$

$(a, b] = \{x : a < x \leq b\}$

$[a, +\infty) = \{x : a \leq x\}$

$(a, +\infty) = \{x : a < x\}$

$(-\infty, b] = \{x : x \leq b\}$

$(-\infty, b) = \{x : x < b\}$

Note that the symbol ∞ does not represent a number; it is used in a conventional sense to indicate the lack of a bound for the interval being defined. We are free to define *expressions* that include the symbol ∞ provided we define the entire expression, as we have done here.

2.4 Inequalities

Hints on how to read this section

This is useful, important material that you will use in this and other courses. You should make sure you are 'comfortable' with this material.

An important part of doing mathematical analysis is dealing with inequalities. This requires a good grasp of algebraic technique and logical argument. It is often necessary to compare quantities or determine when an expression is positive or negative. Since often a judgement call is necessary, it is better to work a lot of examples (and think about what you are doing) rather than to try to remember a lot of rules. Here are some basic properties for you to keep in mind. They are all **theorems** that you can prove using the given **axioms**.

(1) $a \leq b$ if and only if $b - a \geq 0$.

Comment: If you have to show that $a \leq b$, then it is often a good strategy to manipulate $b - a$ into a form that you can easily read off as non negative.

(2) $a = b$ if and only if $a \leq b$ and $b \leq a$.

(3) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

Comment: Note that $(b + d) - (a + c) = (b - a) + (d - c)$ and use (1).

(4) If $a \le b$, then $-b \le -a$.

(5) If $a \le b$ and $c > 0$, then $ca \le cb$.

(6) If $a \le b$ and $c < 0$, then $ca \geq cb$.

(7) If $0 < a \le b$, then $0 < 1/b \le 1/a$.

(8) $x^2 \geq 0$ for each real number x.

Definition: Let $x \geq 0$. Then, $\sqrt{x}$ is that nonnegative number u for which $u^2 = x$. (Thus, by definition, $\sqrt{x} \geq 0$ when $x \geq 0$). Note that the equation $x^2 = 4$ has two solutions, $x = \pm\sqrt{4} = \pm 2$, which again emphasizes the point that $\sqrt{x}$, e.g. $\sqrt{4}$, is nonnegative.

(9) Suppose that $0 \le x, y$. Then, $x < y$ if and only if $x^2 < y^2$. Also, $x < y$ if and only if $\sqrt{x} < \sqrt{y}$.

Comment: Note that $y^2 - x^2 = (y - x)(y + x)$ and that $\sqrt{y} - \sqrt{x} = (y - x)/(\sqrt{y} + \sqrt{x})$; now use (6).

(10) Let u and v be distinct real numbers with $u < v$. Then $(x - u)(x - v)$ is positive when $x < u$ or when $x > v$ and negative when $u < x < v$.

(11) **The Arithmetic-Geometric Means Inequality**: Let $a, b \geq 0$. Then:

$$\sqrt{ab} \le \frac{1}{2}(a + b)$$

with equality if and only if $a = b$.

First proof. Use

$$\left(\frac{a+b}{2}\right)^2-\left(\sqrt{ab}\right)^2=\frac{a^2+2ab+b^2}{4}-ab=\frac{a^2-2ab+b^2}{4}=\frac{(a-b)^2}{4}\geq 0.$$

Second proof. Observe that the quadratic equation

$$0 = (x - a)(x - b) = x^2 - (a+b)x + ab$$

has real roots. Hence, its **discriminant** $(a + b)^2 - 4ab$ is nonnegative. Thus $ab \leq \left(\frac{1}{2}(a+b)\right)^2$ and the result follows.

Third Proof. Let AB be the diameter of a circle, C a point on AB and D a point on the circumference, for which $|AC| = a$, $|CB| = b$ and $DC \perp AB$. The right triangles ACD and DCB are similar, so that $AC : CD = CD : CB$ whence $|CD|^2 = |AC|\,|CB| = ab$. The length of CD is thus $\sqrt{ab}$, and this does not exceed the radius $\frac{1}{2}(a + b)$ of the circle.

Comment on algebraic vs. geometric proofs. Geometric proofs often have the advantage over algebraic proofs of being more intuitive. They were favoured in past centuries, for example, in Newton's day. Purely algebraic proofs, on the other hand, can be more easily tested for the rigour of their argument, not least because we do not have to concern ourselves with the origins of geometric properties. Thus, they have come to be more favoured. Today, we could imagine programming a computer to be able to check a sequence of algebraic statements of a proof for correctness; each line of algebra would have to follow from previous lines according to precisely formulated rules of inference on either axioms or previously proven theorems stored in the computer's memory. We will therefore, try to prove things algebraically wherever practical, but also use geometric reasoning if this gives us more insight.

(12) **The Cauchy-Schwarz Inequality**: Let $a, b, c; u, v, w$ be six real numbers. Then

$$au + bv + cw \le \sqrt{a^2 + b^2 + c^2}\ \sqrt{u^2 + v^2 + w^2}$$

with equality if and only if $a : b : c = u : v : w$.

<u>First proof.</u> Observe that

$$(a^2 + b^2 + c^2)\,(u^2 + v^2 + w^2) - (au + bv + cw)^2$$

$$= a^2v^2 + a^2w^2 + b^2u^2 + b^2w^2 + c^2u^2 + c^2v^2 - 2abuv - 2acuw - 2bcvw$$

$$= (av - bu)^2 + (aw - cu)^2 + (bw - cv)^2 \ge 0,$$

with equality if and only if $av = bu$, $aw = cu$ and $bw = cv$.

<u>Second proof.</u> Consider the quadratic equation

$$0 = (ax - u)^2 + (bx - v)^2 + (cx - w)^2$$

$$= (a^2 + b^2 + c^2)x^2 - 2(au + bv + cw)x + (u^2 + v^2 + w^2).$$

Since the right side is always nonnegative, it does not have distinct real roots. Therefore, its discriminant is non-positive, *i.e.,*

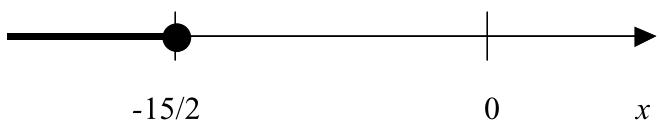
$$4(au + bv + cw)^2 - 4(a^2 + b^2 + c^2)(u^2 + v^2 + w^2) \le 0$$

from which the desired inequality follows. Equality occurs if and only if the discriminant vanishes, which in turn occurs if and only if the equation has coincident real roots, if and only if, for some $x, u = ax$, $v = bx$ and $w = cx$.

Some Examples

1. What values of x satisfy $3x - 7 \geq 5x + 8$? i.e. what interval of x does this represent? i.e. what set of x-values does this represent?

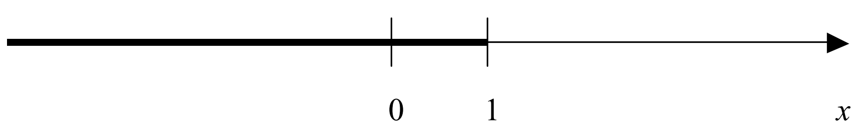
Solution: subtract 8 from each side: $3x - 7 - 8 \geq 5x + 8 - 8$. Thus $3x - 15 \geq 5x$. Next, subtract 3x from each side: $3x - 3x - 15 \geq 5x - 3x$. Thus $-15 \geq 2x$. Next, divide each side by 2: $-15/2 \geq x$. We have now answered the question: this inequality represents all values of x which are less than or equal to $-15/2$, i.e. the set $x \in (-\infty, -15/2]$. Geometrically this is indicated as:



where the solid dot indicates that the point $x = -15/2$ is included, i.e. the set is *closed* at this end. (If this point were not to have been included, the set would be *open* at this end, and an open circle would be used instead of a solid one, or the thick line simply ends without any symbol.)
In order to further convince yourself that this is the correct answer, try inserting some sample values into the original inequality, say $x = -8$, or $x = -7$, etc.

2. Solve the inequality $x^3 < x^2$, i.e. find what interval of x this represents.

Solution: divide each side by x^2, noting that since x^2 is always positive, the order of the inequality will stay unchanged: $x^3/x^2 < x^2/x^2$, thus $x < 1$. We have therefore found the answer. We can also write it as $x \in (-\infty, 1)$. Note that the interval is open at the upper end. Geometrically:



Again, try plugging sample values into the original inequality to re-assure yourself that your answer is correct.

3. Exercise 3 Solve the inequality:

$$\frac{x-2}{x-5} > \frac{x-3}{x-4}.$$

Hint: If you multiply both sides of an inequality by some factor, be careful to check the sign of the factor, since if it is negative, this will change the sign of the inequality.

4. Exercise 4 Solve the inequality:

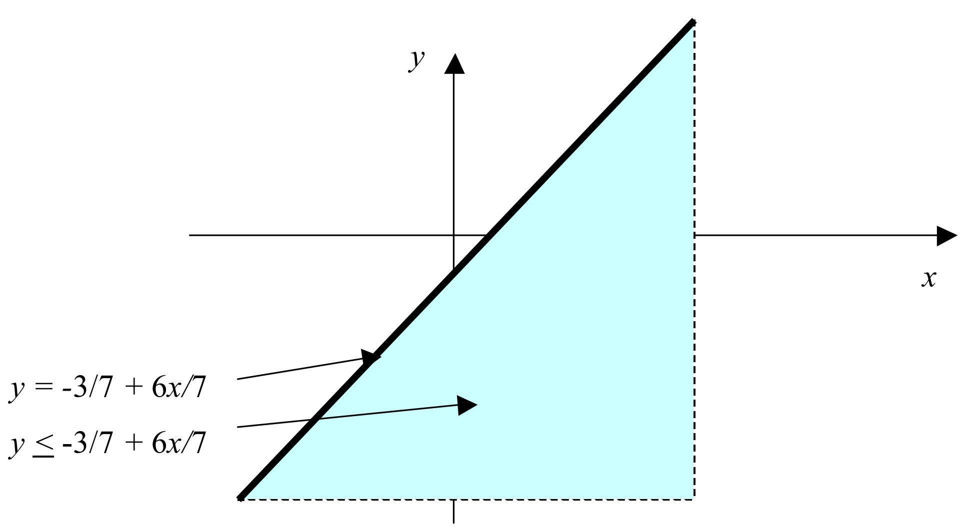
$$x^4 + 2x \geq 3x^2.$$

This is a fairly tough one. Hint No. 1: try adding some number to each side which will then result in a common factor on each side. Hint No. 2: when you then cancel that common factor, carefully consider its sign and how the cancellation effects the sign of the inequality.

5. Sketch the inequality: $6x - 7y \geq 3$.

Note that now both x and y are involved, therefore we are no longer considering a portion of the x-axis, as in the foregoing examples, but a region of the (x,y) plane.

Solution: add $7y$ to each side, etc., to finally obtain: $y \leq -3/7 + 6x/7$. In order to visualize what region of the plane is involved, suppose that the original expression had been an **equality**. In that case, what would be represented? the **line** $y = -3/7 + 6x/7$. Therefore the inequality involves the entire region in the plane lying below this line – also including the line itself (since we were given "$\geq$" not simply ">").



6. Exercise 5 Sketch the graphs of each of the following inequalities:

(a) $x^2 + y \geq 0$

(b) $(x + y - 1)(x - y + 2) \geq 0$

(c) $y^2 + 3x \leq 2$

(d) $x^2 + y^2 \leq x + y$. Hint: think about a circle.

7. Prove that:

$$1 < \frac{(x+y)^2}{x^2+xy+y^2} \leq \frac{4}{3} \text{ for any positive real numbers } x \text{ and } y.$$

Solution: The left inequality is equivalent to $x^2 + 2xy + y^2 > x^2 + xy + y^2$, which is true. The right inequality follows from

$$4(x^2 + xy + y^2) - 3(x^2 + 2xy + y^2) = x^2 - 2xy + y^2 = (x - y)^2 \geq 0.$$

Note that $x^2 + xy + y^2$ is positive when x and y are positive. **Exercise 6**: prove that.

8. Prove that $x^3 + 2 \geq 3x$ whenever $x \geq 0$.

Solution:

$$x^3 + 2 - 3x = x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)^2 (x + 2) \geq 0$$

for $x \geq 0$. The result follows.

9. Solve the inequality

$$\frac{1}{x-2}+\frac{1}{x+2}\leq\frac{1}{x-1}+\frac{1}{x+1}.$$

Solution: The difference of the two sides is

$$\left(\frac{1}{x-1}+\frac{1}{x+1}\right)-\left(\frac{1}{x-2}+\frac{1}{x-2}\right)$$

$$=\frac{2x}{x^2-1}-\frac{2x}{x^2-4}$$

$$=\frac{2x\left[(x^2-4)-(x^2-1)\right]}{(x^2-1)(x^2-4)}$$

$$\frac{-6x}{(x^2-1)(x^2-4)}.$$

This is nonnegative if and only if $1 < x < 2$, or $-1 < x \leq 0$ or $x < -2$. Thus the inequality holds if and only if one of these three conditions holds.

2.5 Absolute Value

Hints on how to read this section

This is useful, important material that you will use in this and other courses. You should make sure you are 'comfortable' with this material.

We often have to deal with absolute values of numbers, and it is important for you to become adept at this. Intuitively, the absolute value of a number is its size or magnitude, without regard to its sign. Geometrically, we can think of the absolute value as its distance from 0 on the number line.

However, these notions lack the precision necessary for rigorous analysis. The definition we present may seem unnatural, but it has two advantages. First, it involves only algebraic ideas so we do not need any reference to geometry. Secondly, it can be used in a precise way to develop other results that we can use in dealing with absolute values.

Definition: Let x be a real number.

$$|x| \equiv \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Here are a few basic properties:

(1) $|x| \geq 0$ for each real x.

Comment: This comes straight from the definition. If $x \geq 0$, then $|x| = x \geq 0$; if $x < 0$, then $|x| = -x \geq 0$.

(2) $-|x| \leq x \leq |x|$ for each real x.

(3) Let $c > 0$. Then $|x| \leq c$ if and only if $-c \leq x \leq c$.

Comment. This is a very important result, as it allows us to find a formulation of an absolute value relation, which does not involve the absolute value sign, and so may be easier to manipulate. The proof is straightforward, but should be studied closely as it illustrates some important ideas about the nature of proof. As the result postulates the equivalence of two statements, there are two distinct parts, in which we assume one statement and deduce the other from it. For each part, all we have is the definition of absolute value, and so we look at various cases for x.

Proof. First assume that $|x| \leq c$. Then, if $x \geq 0$, then $-c \leq 0 \leq x = |x| \leq c$. On the other hand, if $x < 0$, then $-c \leq 0 < -x = |x| \leq c$. Multiplying this inequality by -1 yields that $c \geq x = -|x| \geq -c$.

Now we prove the implication in the other direction. Assume that $-c \leq x \leq c$, from which

$c \geq - x \geq - c$. If $x \geq 0$, then $|x| = x \leq c$, while if $x < 0$, then $|x| = - x \leq c$.

(4A) For any real numbers x and y, we have the **Triangle Inequality**:

$$|x + y| \leq |x| + |y|$$

Comment: To prove this, we could go back to the definition and consider the cases for the different signs of x and y. But with (2) and (3) in hand, we can save some work.

Proof: Note that

$$-|x| \leq x \leq |x|$$

and

$$-|y| \leq y \leq |y|.$$

Adding these inequalities yields

$$-\left(|x| + |y|\right) \leq x + y \leq |x| + |y|.$$

From (3), we find that $|x + y| \leq |x| + |y|$.

(4B) For any real numbers x and y,

$$|x - y| \geq |x| - |y|$$

Hint: Use the triangle inequality on $x - y$ and y.

$|xy| = |x|\ |y|$ for each real x and y.

Comment: We can look at cases. For example, suppose that $x \geq 0$ and $y < 0$. Then $xy < 0$, so that $|xy|$ $= -(xy) = x(-y) = |x|\ |y|$.

One reason for the importance of the absolute value is that it provides us with a tool for describing when numbers are close together or far apart. We can define the *distance* between two numbers a and b by

$$\text{dist}(a, b) = |a - b|.$$

This allows us to look at so called *topological* properties of the line, which we will need for describing such concepts as limit and continuity:

Some Examples

1. Solve the inequality $|x| \geq 3$, i.e. find the set of x-values this represents.

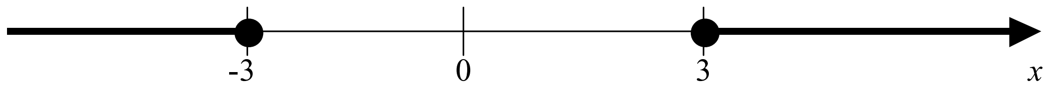
Solution: There are not very many people who are able to work directly with the absolute value expression, and most people have to "open it up" by going back to its definition, i.e replacing it with expressions that do not involve the absolute value signs. We can do this by noting that there are just 2 possibilities: (a) $x \geq 0$, (b) $x < 0$.

If (a) then $|x| = x \geq 3$ **and at the same time** $x \geq 0$, if that is actually *possible*. Well, *is* that possible? i.e. are there any values of x which make this possible? Yes, certainly: each and every value of $x \geq 3$ satisfies *both* these conditions at the same time. Think about it. Thus, it turns out that possibility (a) reduces to simply $x \geq 3$.

If (b) then $|x| = -x \geq 3$, *i.e.* $x \leq -3$ **and at the same time** $x < 0$, if that is actually *possible*. Well, is that possible? i.e. are there any values of x which make this possible? Yes, certainly: every single value of $x \leq -3$ satisfies *both* these conditions at the same time. Think about it. Thus, it turns out that possibility (a) reduces to simply $x \leq -3$.

Thus all x satisfying **either** $x \geq 3$ **or** $x \leq -3$ are represented by $|x| \geq 3$.

Geometrically:



In order to satisfy yourself that you have found the correct answer, try plugging some sample values into the original expression, such as $x = 2$ and $x = 5$, etc.

Note the two important concepts involved here, and which need to be carefully distinguished – and not muddled or confused:

First, there's the concept that every **single** value of x in some particular set of x-values may be required to satisfy **two** conditions **at the same time** (if that's actually possible).

Second, a set of x-values may consist of more than one subset, which is where the **either/or** gets involved.

Just to be clear, and to state the obvious: in the latter case it is not true that a single value of x is supposed to be in both subsets; for example, we are not trying to say that a single value of x would satisfy both $x \geq 3$ and $x \leq -3$ at the same time – which would be ridiculous. You need to think about these important concepts for a while to get them clear in your head.

2. Sketch the graph of $y = |x - 3|$.

Solution: Notice that now both x and y are involved. What could this possibly mean? If you think about it for a while, you will realize that this must represent a set of points in the xy-plane – in contrast with the first example, where the only variable involved was x, implying a set of points on the x-line. A set of points in the xy-plane might constitute a line – not necessarily straight – or a whole region, like the interior of a circle.

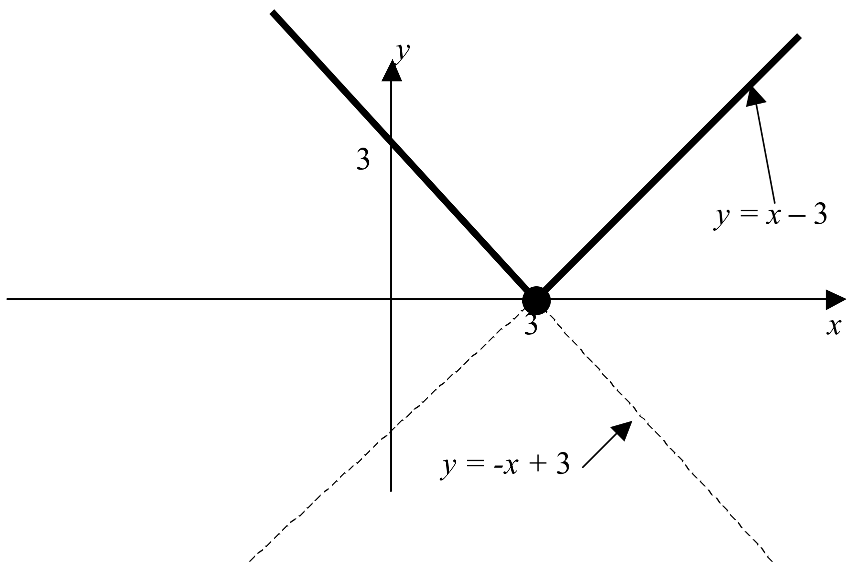
Again we open up the absolute value expression by noting that there are just 2 possibilities: (a) $x - 3 \geq 0$, (b) $x - 3 < 0$.

If (a) then $y = x - 3$ **and at the same time** $x - 3 \geq 0$, i.e. $x \geq 3$. Thus the points in the plane that are represented here are the points on the line $y = x - 3$ for values of $x \geq 3$.

If (b) then $y = -(x - 3) = -x + 3$ **and at the same time** $x - 3 < 0$, i.e. $x < 3$. Thus the points in the plane that are represented here are the points on the line $y = -x + 3$ for values of $x < 3$.

Thus any x which satisfies **either** (a) **or** (b) is included, i.e. is represented by $y = |x - 3|$.

Geometrically:



Again to satisfy yourself that this is the correct answer, try plugging sample values into the original equation, including both points that you think should be included, like (4,1), and ones that you think should not, such as (5,1).

3. Solve the inequality: $|x - 2| > |x + 4|$.

Solution: We want to know if there is any value of x which can satisfy this expression, and to identify that value. Or perhaps there's a subset of all x-values that will satisfy it, or perhaps more than one subset.

Here there are four possibilities that we have to consider. It is conceivable that for one and the same value of x it will be possible to simultaneously satisfy :

(a) $(x - 2) \geq 0$ and at the same time $(x + 4) \geq 0$ and at the same time $+(x - 2) > +(x + 4)$, if that is actually possible.

(b) $(x - 2) \geq 0$ and at the same time $(x + 4) < 0$ and at the same time $x - 2 > -(x + 4)$, if that is actually possible.

(c) $(x - 2) < 0$ and at the same time $(x + 4) \geq 0$ and at the same time $-(x - 2) > x + 4$, if that is actually possible.

(d) $(x - 2) < 0$ and at the same time $(x + 4) < 0$ and at the same time $-(x - 2) > -(x + 4)$, if that is actually possible.

OK, let's start with the hypothetical possibility (a) and see if this actually "yields" any "fruit", i.e. does this describe any possible x-values? The first condition gives $x > 2$. The second gives $x > -4$. The third gives $-2 > 4$. There certainly are x-values which could satisfy the first and second condition simultaneously, namely each and every $x > 2$; however, the third requirement is absolutely impossible – no matter what the value of x considered. Thus hypothetical possibility (a) turns out to have no x-values satisfying all the requirements.

OK, possibility (b) next, where we are looking for any x-value that simultaneously satisfies: $x \geq 2$, $x < -4$, $x > -1$. There are no such x-values.

OK, possibility (c) next, where we are looking for any x-value that simultaneously satisfies: $x < 2$, $x \geq -4$, $x < -1$. Score! Yes, any value of x satisfying $-4 \leq x < -1$, satisfies all three conditions simultaneously. Note that we have found a whole subset of x-values here. Try a few sample values within this subset in the original expression to convince yourself.

Finally, possibility (d) next, where we are looking for any x-value that simultaneously satisfies: $x < 2$, $x < -4$, $2 > -4$. Score! Yes, any value of x satisfying $x < -4$, satisfies all three conditions simultaneously.

Try a few sample values within this subset in the original expression to convince yourself.

So, gathering all out "fruit" together, we find that each and every x satisfying $x < -1$ satisfies the original expression.

4. **Exercise 7** Solve the following inequalities:

(a) $0 < |x| \le 1$

(b) $0 < |x - 2| < \frac{1}{2}$

(c) $0 < |x - 3| \le 8$

(d) $|2x + 1| < \frac{1}{4}$

(e) $\left|2|x - 3| - 5|x + 4|\right| > 4$

5. **Exercise 8** Sketch (approximately) the graphs of the following equations:

(a) $y = 2|x + 1| + |2x - 3|$

(b) $|x + y| + |x - y| = 1$

(c) $y = \sqrt{x + 2} - \sqrt{x - 2}$ (no calculus)

(d) $y^2 = |x^2 - x|$.

6. **Exercise 9** Find an inequality of the form $|x - c| < \delta$, the solution of which is the open interval:

(a) (-3,3)
(b) (-3,7)
(c) (-7,3)

that is, find the appropriate values of c and δ.

7. **Exercise 10** Determine all values of $A > 0$ for which the statement is true:

(a) If $|x-2| < A$, then $|2x-4| < 3$.

(b) If $|x+1| < 2$, then $|3x+3| < A$.

(c) If $|x-1| < 5$, then $|2x-3| < A$.

8. **Exercise 11** Sketch (a rough sketch is adequate) some function $f(x)$ to satisfy each of the following:

(a) for $|x-2| < 1$, $f(x) > 0$.

(b) for $|x+1| < ½$, $f(x)$ is increasing.

(c) for $0 < |x+1| < 3$, $(f(x) + 1) > 0$.

(d) for $|x-1| > 1$, $|f(x)| < 1$.

(e) for $|x+2| < 3$, $|f(x) - 1| < 0$.

(f) for $|x-c| < ½$, $|f(x) - L| < 1$, where c and L are given constants.

(g) for $|x-c| < \delta$, $|f(x) - L| < \varepsilon$, where c, L, δ and ε are given constants.

9. **Exercise 12** Sketch (a rough sketch is adequate) some function *f(x)* which does not satisfy each of the constraints in the last exercise.

2.6 Functions

Hints on how to read this section

This is useful, important material that you will use in this and other courses. You should make sure you are 'comfortable' with this material.

In mathematical discussion, the terminology and notion attached to the function notion is often used a bit loosely, but usually the situation is quite clear from the context and if you are alert, there should be no difficulty for you. The basic idea of a function is that some kind of **input** is related to a corresponding **output**. It may be useful to think of it as a **rule** that assigns to each element of a set *X* (domain) exactly one element of a set *Y,* as long as we do not have too restricted an idea of what sort of "rules" are allowed. The formal definition is that a **function is any set of ordered pairs** (*a, b*) where *a* belongs to a set *A* and *b* belongs to a set *B* where, given any *a,* there is exactly one *b* that corresponds to it. While the elements involved in functions do not have to be **numbers**, in this course they will be.

A little more informally, we can think of a function as a pairing *f* which associates to each element *a* of a **domain** set a uniquely determined element $b = f(a)$ of a **range** set. *b* is called the **image** of *a* under *f*. This pairing can be described as a list (the telephone book is an example, where each subscriber is mapped to his telephone number) or by a written out description (map each number to its square). When the elements are numbers then, of course, "$f(a)$" represents a number. Frequently, we specify the rule by writing a simple algebraic statement, such as $f(x) = x^3 + 5x$ for real *x,* which is straightforward to interpret: *x* is the name of an element (number) in the domain set while $f(x)$ is the name (number) of the corresponding element in the range set, and the rule that spells out the correspondence is clear. In practice we often say that $f(x)$ is "the function", which isn't the strictly correct way to put it; when speaking that way, then "the function" is a number (in this course).

In some cases, we need to use less compact forms, using more than one algebraic expression; but this makes them no less valid as functions as long as the value of the function for each element of its domain is unambiguously determined. For example,

$$h(x) = \begin{cases} x^2 + 5x, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

is a perfectly valid function (although it is not *continuous,* a concept we will discuss below).

Note that, to **define a function** properly, you must specify:

1. the **domain** of definition,
2. the **rule** that implements the relation between elements of the domain and of the range.

Note that any set can serve as a domain; the domain need not be the set of all reals. For example, the domain of $f(x) = x^{1/2}$ is only $x \geq 0$. Two functions are equal (the same) if and only if they have the same domain and map each domain element to the same element of the range.

When we write the equation defining the function f in the form $y = f(x)$, then x is the **independent** variable and y is the **dependent** variable.

2.7 Estimates on Functions

Hints on how to read this section

This is useful, important material that you will use in this and other courses. You should make sure you are 'comfortable' with this material.

We say that a function $f(x)$ is **bounded above** on its domain D if and only if there is a number M for which $f(x) \leq M$ for every $x \in$ D. Such a number M is an **upper bound**. If there is one upper bound, then there are many; each number larger than an upper bound M is also an upper bound. A function $f(x)$ is **bounded below** on its domain D if and only if there is a number m for which $f(x) \geq m$ for every $x \in$ D. Such a number m is a **lower bound**. The function $f(x)$ is **bounded** on D if and only if it is bounded above and bounded below. Equivalently, $f(x)$ is bounded on D if and only if there is a real number N for which $|f(x)| \leq N$ for each $x \in$ D.

Note that nothing is said in these definitions about these bounds being the 'tightest' possible. Often it is necessary to know whether or not a function is bounded and immaterial how close the bounds are to its actual values. In this case, we can make our estimates quite coarse if it will save a lot of work.

Examples

1. Prove that:

$$f(x) = \frac{x+3}{x-2} \quad \text{is bounded for } x \text{ in } (3, 5)$$

Solution: We are given that $3 < x < 5$. Adding 3 to each term gives $6 < x + 3 < 8$. Thus, replacing $x + 3$ by either 6 or 8 yields $\frac{6}{x-2} < f(x) < \frac{8}{x-2}$. (Note, that $x - 2 > 0$, so the inequality is preserved. Think about why it's important to note that point.) Halfway there! Similarly, $1 < x - 2 < 3$ hence $\frac{1}{3} < \frac{1}{x-2} < 1$. Thus, we get $\frac{6}{3} < \frac{6}{x-2} < f(x) < \frac{8}{x-2} < 8 \cdot 1$ i.e. $2 < f(x) < 8$. We have therefore shown that $f(x)$ is bounded both below and above.

Exercise 11: To convince yourself, show some sample values. Do the values you find "cover" all the way down to 2 and up to 8? No, they do not need to! We were required to find only some bounds, not the tightest ones.

2. Prove that the function:

$$f(x) = \frac{x^3 - 7x^2 + 6x + 4}{x^2 - 5x + 4}$$

is bounded on the domain $D = \{x : 2 \le x \le 3\}$.

Comment: Later in the course, we will have a theorem that will tell us that this is true without any additional work. Here we will appeal to basic results. There is no set method of doing such problems, and the general strategy is to try to minimize the amount of work to be done. We can deal separately with the numerator and denominator, finding an upper bound for the numerator and a lower bound for the denominator. For example,

$$|x^3 - 7x^2 + 6x + 4| \le |x^3 - 7x^2 + 6x| + |4| \le |x||x-1||x-6| + 4.$$

When $2 \le x \le 3$, we have $|x| < 3$, $1 \le x - 1 \le 2$ and $-4 \le x - 6 \le -3$, so that $|x-1| \le 2$ and $|x-6| \le 4$.

Hence,

$$|x^3 - 7x^2 + 6x + 4| \le 3 \cdot 2 \cdot 4 + 4 = 28.$$

As for the denominator, $|x^2 - 5x + 4| = |x-1||x-4|$. When $2 \le x \le 3$, both $|x-1| \ge 1$ and $|x-4| \ge 1$, so that $|x-1||x-4| \ge 1$. Putting this together, we find that

$$|f(x)| = \frac{|x^3 - 7x^2 + 6x + 4|}{|x^2 - 5x + 4|} \le \frac{28}{1} = 28$$

when $2 \le x \le 3$.

Exercise 12: find another way of showing that the function is bounded.

2.8 Completeness of the Reals Axiom

Hints on how to read this section

You should try to understand these ideas.

The final axiom we need for the real numbers is the one that guarantees that it will be a **continuum**. To understand the situation better, let us do a little classification of the numbers in **R**.

The **integers** are the numbers $\{0, \pm 1, \pm 2, \ldots\}$.

A real number is **rational** if and only if it can be written in the form p/q where p and q are integers and q is nonzero. Every rational number can be written in *lowest terms,* for which the greatest common divisor of the numerator (p) and denominator (q) is equal to 1. The sum, product, difference and quotient of pairs of rationals are also rational.

A real number is **algebraic** if and only if it is the solution of a **polynomial equation** of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

whose coefficients a_i are all integers.

A real number is **irrational** or **non-rational** if and only if it is not rational.

The set of all rational numbers, represented as a line, looks as though it covers the line pretty thickly, as we can find rational numbers as close to each other as we like. But it turns out that there are some gaps. Consider the question: is there a positive number whose square is 3? Since the larger the number, the larger the square, we might feel that the square root of 3 should lie between 1 and 2, as the square of 1 is 1 and that of 2 is 4. But it turns out that no rational number will fill the bill. (Can you prove this? Your high school text probably contained a proof.) There is a gap in the rationals where we feel the square root of 3 ought to be.

To get around this, we impose a **Completeness axiom** which we simply *want* the real numbers to obey, and which the set of rationals will fail to satisfy. To formulate this, we make some definitions about real sets. (This will be fairly dry, but hang in and things will improve later on.)

A **set** S of real numbers is **bounded above** if and only if there exists some number M for which $x \leq M$ for each $x \in S$. We call M an **upper bound** of the set. It is **bounded below** if and only if there exists some number $m \leq x$ for each $x \in S$. We call m **lower bound** of the set.

<u>Definition:</u> The **least upper bound** *or* **supremum** of a nonempty set is the smallest of the upper bounds. The least upper bound of a set S is denoted by sup ***S*** *or lub* ***S.***

<u>Definition:</u> The **greatest lower bound** or **infimum** of a nonempty set is the largest of the lower bounds. It is denoted by inf ***S*** or glb ***S***.

Axiom 9: (Completeness)

Any nonempty subset of the real numbers that is bounded above has a least upper bound.

Although it is not usually stated explicitly, the point of the Completeness Axiom is that any nonempty subset of the real numbers that is bounded above has a least upper bound *that is also a real number*. One could not make the equivalent statement about merely rational numbers. Think about it.

An equivalent statement to Axiom 9 is that every nonempty subset of the real numbers that is bounded below has a greatest lower bound.

It turns out that can be quite challenging to use Axiom 9 directly to prove such important things as the existence of a number whose square is equal to 3. We will shortly show that there is a much easier way, but this requires that we first prove a general theorem – the **Intermediate Value Theorem**, **IVT** - using Axiom 9, see page 51 . Now you are probably thinking that Axiom 9 (or later the IVT) looks like an odd, clumsy and roundabout way to "force" reals to include such rationals as $\sqrt{3}$. Perhaps, you think, it would have been simpler and neater to simply add an axiom that says that $x^2 = 3$ has a real solution or, equivalently that $3^{1/2}$ is that real number with the property that $3^{1/2}$ x $3^{1/2} = 3$, similarly to the way we "forced" the reals to include 0 or 1. But such roots are not the only type of irrational we want to include. There are many others and quite different ones, such as π and the values of trigonometric functions, etc. We will soon show that the very powerful IVT can be easily used to show the existence of the square root of 3. It also turns out that with the IVT we can readily establish the existence of many other irrationals as reals. So Axiom 9 is indeed a powerful and elegant tool, even if its statement seems hardly useful or interesting at first sight.

3. LIMITS and CONTINUITY

3.1 Motivation for Developing a Rigorous Definition of Limit

Hints on how to read this section

This material contains important ideas which you should try to understand.

It has to be admitted that a lot of fairly hard work is required to develop a rigorous definition of the **limit**. Most students find the concepts involved are among the most challenging that they have encountered so far in their studies of mathematics and science. Why should we be bothered with such a challenging task? for some very good reasons. Without the limit, we do not have a logically solid definition for the two most important quantities in calculus -- and in much of science and engineering to boot -- the **derivative** and the **integral**.

No matter how complicated a line of reasoning we come up with in trying to define the **derivative**, and no matter how clever we try to be, unless we define the limit first, then we cannot avoid being faced with the conundrum of 0/0 and hitting a dead end.

Now you may think that it should not be such a big problem to just accept 0/0 as a real, since, after all, we just make numbers up anyway. That is, we just make up the axioms, which collectively *define* numbers, in effect. So why cannot we just add another axiom - if that is what is needed - that 0/0 is a number? If we do this, however, it turns out that we can easily prove that 17 = 18 for example, or that 0 = 1, so that there is only one real number. **Exercise 13**: Make such a proof. There is nothing logically wrong with having a number system with just one element in it; it is just that we do not *want* this to be a property of the system. It is boring. Like playing poker with every card wild. And it is useless for any kind of physical application.

We are confronted with essentially the same problem when we try to define the **integral** without first defining the limit. Intuitively, the integral is the sum of an infinitely large number of infinitely small areas, so ultimately we would not be able to avoid treating $0 \cdot \infty$ as a real, and thus ∞ as well. The obvious definition of ∞ involves division by 0 and so we are back at the same logical dead end as before.

Unfortunately, the obvious and intuitively attractive ways of defining the derivative and integral are, in the end, just not strictly logical. The derivative and integral are too important to accept this state of affairs. How could we dare to proceed to construct the rest of the edifice of calculus - and thus of much of all science and engineering - on such an uncertain foundation? This constitutes the strong motivation we need to persevere in the daunting task of defining the limit in a rigorously logical way.

Now, the good news. Most of the hard work is done once we have the rigorous definition of the limit in hand, and the further work needed to define the derivative and integral in terms of limits is relatively easy. You will be rewarded for your effort in having mastered one of mankind's greatest intellectual achievements. Newton, one of history's greatest geniuses, was unsatisfied with the logical basis of his definition of the derivative and integral. But, he was unable to penetrate through to the final resolution of this major intellectual challenge (and was twitted by Bishop Berkeley for having a faith, no less irrational than that of religion which was under attack by the freethinkers of the time, in the so called "ghosts of departed quantities" - infinitesimals). It was not until the nineteenth century that Cauchy and Weierstrass, standing on the shoulders of giants such as Newton, were finally able to secure the logical foundation of calculus. While all people are the heirs of this monumental achievement, it is given to only a few in each generation to appreciate and understand it. You are one of them.

3.2 Limits

Hints on how to read this section

This material covers what is probably the single most important theoretical concept in calculus. All students find this material pretty challenging. You should not expect to understand the ideas involved right away. You will probably find that you have to return again and again to this material. Gradually, a bit of dawn will break and you will start to make sense out of some aspects. With further visits to the material, you will understand more aspects. Even by the time of the final exam, however, your understanding will probably only be partial. But don't despair! The concept of the limit is arguably the greatest intellectual creation of the human race. Many of the greatest geniuses of the last 2500 years have contributed to the creation of this concept.

Suppose that a function $f(x)$ is defined for x belonging to some open interval (a, b) except possibly at a single point c in the interval. Sometimes, we want to describe the behaviour of the function at points near c, and one concept for doing this is that of the limit. Consider the following statement:

The limit of $f(x)$ is L, as x approaches c.

Notationally, this is written:

$$\lim_{x \to c} f(x) = L$$

Informally, this means that as we take values of *x* closer and closer to *c*, the corresponding values *f*(*x*) get closer and closer to *L*. Let us focus on the number *L* and specify a degree of closeness to *L;* then the statement signifies that we can make *f*(*x*) achieve that degree of closeness to *L* if we make *x* sufficiently close to *c*.

Note: since *L* is a number, then clearly "$\lim_{x \to c} f(x)$"must be a number also.

Exercise 14: you are given that $f(x) = x^2$, $c = 15$ and $L = 225$. Suppose that 50 has been specified as the degree of closeness to 225. How close do you have to make *x* to 15 in order to ensure that *f*(*x*) achieves the specified degree of closeness? Would $14 < x < 16$ be one correct answer? Give some other examples of correct answers. How many correct answers are there?

Let us formalize things. We specify the degree of closeness to *L* by a positive number ε. Then, the degree of closeness that *x* has to get to *c* can be signified by another positive number δ . So we make the formal definition:

$$\lim_{x \to c} f(x) = L$$

if and only if,

given any number $\varepsilon > 0$,

we can find a number $\delta > 0$, which will depend on ε,

for which $|f(x) - L| < \varepsilon$

whenever $|x - c| < \delta$.

Most students find this a difficult definition to assimilate, and you will need to think about it and probably use it for a while before its import becomes clear.

Here is one way to think about it that may be helpful:

1. "$\lim_{x \to c} f(x)$" is a **number**, as already pointed out.

2. What number?

3. That we have to **guess**! More or less. It is usually not hard to guess the number, actually. Don't worry. That is not generally the challenging part. Anyhow, one way or another we have to come up with a **candidate number** - call it L.

4. How do we know L will do the job?

5. It has to pass a **test**!

6. What test?

7. Imagine that you have an adversary (the εnemy!) who is free to impose any degree of closeness to the limit, any $\varepsilon > 0$, he wants on you (although it only gets challenging when he makes it *small*).

8. *Your* job is to defeat the εnemy by finding a set of x-values sufficiently close to c, *i.e.* find a value of $\delta > 0$ such that when $c - \delta < x < c + \delta$, ...

9. ……then the corresponding f-values will satisfy $L - \varepsilon < f(x) < L + \varepsilon$, *i.e.* the f-values will not be further than the imposed ε away from candidate L.

10. For some specific and *given* $f(x)$ and c (say, $f(x) = x^3$ and $c = 2$), and for your candidate number L (in this case, $L = 8$; now you would not have a hard time guessing this value, would you?) and for a particular value of imposed ε (say $\varepsilon = 0.1$), let us say that you do manage to find a value of $\delta > 0$ (say $\delta = 0.001$) that will do the job. (Obviously, your choice for δ will be guided by

the particular values of $f(x)$, *c, L* and ε - which as far as you are concerned at this point are all *given*, and so you really found your value of δ in terms of them). Success!

11. But not so quick! Your εnemy is free to impose any $\varepsilon > 0$ upon you. So, it is not enough for you to find a δ that will do the job for only one particular value of ε. It is tougher than that. You will have to find δ that expressed *in terms of* ε, *e.g.* by an algebraic relation between δ and ε, so that the job gets done once and for all, and you have totally and permanently defeated your εnemy.

12. When you have achieved that, then your candidate number *L* is declared to have passed the test and we now know the value of the number $\lim_{x \to c} f(x)$.

Limit Examples

1. Prove that:

$$\lim_{x \to 0} x^3 = 0$$

Comment: It was not hard to guess the limit, was it?

It is useful to take the following steps.

(1) Note that some $\varepsilon > 0$ has been imposed on us. We treat ε as given

(2) It is required that $|f(x) - L| < \varepsilon$, or $|x^3| < \varepsilon$....

.

(3) ...when *x* satisfies $|x - c| < \delta$ or $|x| < \delta$. We need to come up with a prescription that makes (2) hold.

(4) Take the left side of (2) and using (3) manipulate the left side of (2) to put it "under δ-control" *i.e.* less than some expression that involves only δ and perhaps some specific constant number, like 12. The left side is $|x|^3$ and so it will be less then δ^3.

(5) Now we have to choose our δ; it's pretty obvious $\delta = \varepsilon^{1/3}$ will do the job because then the left side of 2 will be less than ε.

We can reduce this to just the bare-bones logic:

Given $\varepsilon > 0$, let $\delta = \varepsilon^{1/3}$. Then $|x - 0| < \delta$ implies $|x^3 - 0| < \varepsilon$ as required.

2. Prove that:

$$\lim_{x \to 2} x^3 = 8$$

Comment: Again it was not hard to guess the limit, was it?

It is useful to take the following steps.

(1) Note that some $\varepsilon > 0$ has been imposed on us. We treat ε as given

(2) It is required that $|f(x) - L| < \varepsilon$, or $|x^3 - 8| < \varepsilon$ ....

.

(3) ...when x satisfies $|x - c| < \delta$ or $|x - 2| < \delta$. We need to come up with a prescription that makes (2) hold.

(6) Take the left side of (2) and using (3) manipulate the left side of (2) to get it "under δ-control" *i.e.* less than some expression that involves only δ and perhaps some specific constant number, like 12.

It can often be a useful thing to try to factor the expression on the LHS(2) in terms of a factor $|x-c|$, i.e. here $|x-2|$, i.e. getting LHS(2) = $|x-2|\cdot$(*another factor)* If one can achieve this, then one will have an excellent start to getting the LHS (2) "under δ-control", since then by using (3) one gets LHS(2) < $\delta\cdot$*(another factor)*. We can often expect to be able to achieve this, since after all, we are expecting that LHS(2) will get small as x approaches 2, which implies that LHS(2) probably contains a factor $|x-2|$. So we divide the LHS(2) by (x –c), here (x –2), to obtain LHS(2) = $|x^3-8| = |x-2||x^2+2x+4|$. Thus, using (3) we have LHS(2) < $|x^3-8| = \delta|x^2+2x+4|$.

Next, we need to get this additional factor "under control". Usually this is not a very "interesting" factor. That is, it doesn't do anything "exciting" like approach infinity. That's all we need for our purposes. We'll be quite happy if we can bracket it between two specific positive numbers, such as 5 and 31. Note the important observation that since we are calculating the limit of the function as x tends to 2, we are (i) not interested in the value of the function <u>at</u> x = 2, and (ii) not interested in the value of the function when x is <u>a long way</u> from 2, but only when it is in arbitrarily small intervals about 2. So in bounding the second factor, we can <u>decide in advance</u> that we are going to restrict the domain of the function to a convenient interval that contains 2, say (1, 3). After all, <u>we</u> are in the driver's seat here – it is <u>we</u> who have to come up with the specification of what the δ-band is to be. OK, well then we are saying that whatever else we will specify about δ, in addition it is not ever to be bigger than some specific size – here 1, i.e we will insist that $\delta < 1$. Thus $|x-2| < \delta \Rightarrow 2-\delta < x < 2+\delta \Rightarrow 1 < x < 3$.

Now if $1 < x < 3$, then $0 < 7 < x^2 + 2x + 4 < 19$, and so $|x^2+2x+4| < 19$ (note the importance of having shown that $x^2 + 2x + 4$ is positive.) Therefore LHS(2) = 19δ. Success! We have got LHS(2) "under δ-control".

7. Note that we are still free to specify δ in terms of ε, provided we carry along this over-riding condition that $\delta < 1$. OK, then let's specify $\delta = \varepsilon/19$. How can we then state our specification for δ so as to include both these features? As follows: we specify:

$$\delta = \min(1, \ \varepsilon/19)$$

which simply means that δ is equal to whichever is the smaller, 1 or $\varepsilon/19$. Therefore two things are true at the same time: $|x-2|<\varepsilon/19$ and $|x-2|<1$. We find it useful to use the first result to get the first factor of the LHS(2) "under δ-control", i.e. the factor $|x-2|$, while it is useful to use the second result to get the second factor of the LHS(2) "under δ-control", i.e. the factor $|x^2+2x+4|$.

Finally, we reduce all this to just the bare-bones logic:
Given $\varepsilon > 0$, let $\delta = \min(1, \varepsilon/19)$. Then $|x-2|<\delta$ gives $|x-2|<\varepsilon/19$ and $|x-2|<1$ which implies $|x^3-8| = |x-2||x^2+2x+4| < (\varepsilon/19)\cdot 19 = \varepsilon$, as required.

You are probably wondering if this idea of $\delta = \min(1, \varepsilon/19)$ is perhaps a kind of 'cheat', or that perhaps it's not strictly logical. In order to satisfy yourself that this is a perfectly legitimate, unambiguous and complete way to specify δ in terms of ε, let's go through all the values of ε that could possibly have been imposed upon you by the εnemy. There are only three possibilities:
(a) $0<\varepsilon<19$, (b) $\varepsilon>19$, (c) $\varepsilon=19$. So:
(a) if $0<\varepsilon<19$ then $\varepsilon/19<1$ so $\delta=\varepsilon/19$. Then $|x-2|<\delta$ gives $|x-2|<\varepsilon/19$ and $|x-2|<1$ which implies $|x^3-8| = |x-2||x^2+2x+4| < (\varepsilon/19)\cdot 19 = \varepsilon$, as required.
(b) if $\varepsilon>19$ then $\varepsilon/19>1$ so $\delta=1$. Then…. **Exercise 15** : complete this as an exercise.
(c)**Exercise 16**: do this as an exercise.

Isn't there a simpler way to do this?

At this point, many students ask: "Isn't there a simpler way to do this?" Why don't we simply solve for the values of x – let's call them α and β - setting $f(\alpha) = L-\varepsilon$ and $f(\beta) = L+\varepsilon$? Then no matter how small a value of ε is imposed, we have found an interval (α, β) containing c such that for all x in (α, β) the values of $f(x)$ satisfy $|f(x)-L| < \varepsilon$. This approach, in fact, is perfectly logical – for the simple cases where it can be made to work algebraically – but often it can't. Consider, for example, $\lim_{x\to 0} x^2 = 0$ where there simply is no value of α that will make $f(\alpha) = L-\varepsilon = 0-\varepsilon = -\varepsilon$. Or consider $\lim_{x\to\pi}\sin x = 0$, where there are no values of α that satisfy $f(\alpha) = L-\varepsilon = 0-\varepsilon = -\varepsilon$ when $\varepsilon > 1$, and infinitely many when $\varepsilon < 1$. Therefore we need a more general and powerful method – such as the one we used in the last example. Although it is harder to understand it at first, it's worth perservering.

3. Ok, now for a tougher one:

$$\lim_{x\to 2}\frac{x^5-32}{x^3-8} = ?$$

Exercise 17: Experiment with a pocket calculator, taking values of x close to 2 to show that it seems as though the limit might be 20/3.

To test this, let us look at the difference between this value and the function for values of x unequal to 2:

$$\frac{x^5-32}{x^3-8}-\frac{20}{3}=\frac{3x^5-20x^3+64}{3(x^3-8)}=\frac{(x-2)^2(3x^3+12x^2+16x+16)}{3(x-2)(x^2+2x+4)}$$

$$= (x-2)\left[\frac{3x^3+12x^2+16x+16}{3(x^2+2x+4)}\right].$$

Before we proceed, let us consider why we might have anticipated the appearance of the factor x - 2. When we put the difference over a common denominator $3(x^3 - 8)$, we get a rational function whose numerator and denominators are both polynomials. We expect that this function is small when x is close to 2 and would thus vanish when $x = 2$. By the **Factor Theorem**, a polynomial that vanishes when $x = 2$ has x - 2 as a factor. Because x - 2 is a factor of the denominator, we would expect x - 2 to divide the numerator to a higher power (*i.e.,* at least to $(x - 2)^2$). The end result is that the difference is a product of x - 2 and some other function of x.

The difference will be close to zero when x is close to 2, *as long as* the other factor does not "blow up" near 2. So we need to be sure that the second factor, the one in square brackets is bounded. The situation is therefore essentially the same as in the 2^{nd} example, above. As before, we note the important observation that as we calculate the limit of the function as x tends to 2, we are (i) not interested in the value of the function at $x = 2$, and (ii) not interested in the value of the function when x is a long way from 2, but only when it is in arbitrarily small intervals about 2. So in bounding the second factor, we can decide in advance that we are going to restrict the domain of the function to a convenient interval that contains 2, say (1, 3).

You can think of the description of δ in terms of ε as a prescription for a program for a subroutine that will spit out a numerical value for the closeness that x should be to 2 once you enter in the degree of closeness that you want the function to have to its limit.

When $1 \leq x \leq 3$ *(but $x \neq 2$)*, we have $x^2 + 2x + 4 \geq 4$ and $0 < 3x^3 + 12x^2 + 16x + 16 \leq 253$, so that

$$0 \leq \frac{3x^3 + 12x^2 + 16x + 16}{3(x^2 + 2x + 4)} \leq \frac{253}{12} \leq 22$$

and so

$$\left| \frac{x^5 - 32}{x^3 - 8} - \frac{20}{3} \right| \leq 22|x - 2|.$$

Now we want to show that the difference on the left side does not exceed a given positive ε when x is sufficiently close to 2. This can be achieved by making $22|x-2| < \varepsilon$ *or* $|x-2| < \varepsilon / 22$. Now that we have cased the situation, let us put the whole business together.

Let $\varepsilon > 0$ be given. (Think of this as being *imposed* on you; your task is to find the value of δ that will "deliver the goods".) We first select δ so that $|x-2| < \delta$ implies $1 < x < 3$, so that we have access to the bound that we have just established for the term in square brackets. So whatever δ we pick should not exceed 1. Next, we want to ensure that $|x-2| < \delta$ implies $22|x-2| < \varepsilon$. So define

$$\delta = \min(1, \varepsilon/22)$$

which is just a useful symbol we define that means that δ is the smaller of 1 and ε /22.

For example, if ε = 0.12 is imposed, we select δ= 0.12/22 ~ 0.0052. **Exercise 18**: check on a pocket calculator that taking x between 1.9948 and 2.0052 puts $(x^5 - 32) / (x^3 - 8)$ within 0.12 of 20/3.) As another example, suppose that ε = 30 is imposed. Then select δ = 1. **Exercise 19**: check, for values of x satisfying $1 < x < 3$, that $(x^5 - 32) / (x^3 - 8)$ is within 30 of 20/3.

Thus, given $\varepsilon > 0$, we choose δ = min (1, ε /22). Let us prove that this delivers the goods, writing out the argument systematically in the proper logical order:

$$|x-2| < \delta \Rightarrow |x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow 0 < \left[\frac{3x^3 + 32x^2 + 16x + 16}{3(x^2 + 2x + 4}\right] \leq 22.$$

But $$|x-2| < \delta \Rightarrow |x-2| < \varepsilon/22 \Rightarrow 22|x-2| < \varepsilon \Rightarrow \left|\frac{x^5 - 32}{x^3 - 8}\right| \leq 22|x-2| < \varepsilon. \spadesuit$$

4. Sometimes it can be illuminating to prove a statement is <u>not</u> true. Let's see if we can prove that:

$$\lim_{x \to 2} x^3 \neq 10$$

Let us try a **proof by contradiction**. Let us assume that $\lim_{x \to 2} x^3 = 10$ and show that this results in a contradiction. If 10 is the limit, then it must be possible for <u>us</u> to impose any ε that <u>we</u> want, let's say ε = 1, and it is supposed to be guaranteed that some $\delta > 0$ will exist such that for all x in $(2 - \delta, 2 + \delta)$, the f-values will lie somewhere within $(10 - \varepsilon, 10 + \varepsilon)$, i.e. within (9, 11). We immediately have a contradiction since for all x in $(2 - \delta, 2)$, no matter what the value of δ, $f(x) = x^3 < 8$, which does not lie in (9, 11).

Exercise 20 Prove that:

$$\lim_{x\to 2} x^3 \neq 8 + 10^{-100}$$

There are a number of basic structural properties of limits that we will need; **Exercise 21** you should try to prove them without recourse to a text.

(1) Suppose that $\lim_{x\to c} f(x) = L$ and that $f(x) \leq M$ for all x not equal to c in some open interval that contains c. Then $L \leq M$. (An analogous result is true for $\geq$.)

(2) Let $h(x) = k$ for all values of x, so that $h(x)$ is a constant function. Then $\lim_{x\to c} h(x) = \lim_{x\to c} k = k$.

(3) $\lim_{x\to c} x = c$.

(4) If $\lim_{x\to c} f(x) = u$ and $\lim_{x\to c} g(x) = v$, then $\lim_{x\to c}(f \pm g)(x) = u \pm v$ and $\lim_{x\to c}(fg)(x) = uv$.

(5) If k is any real number and $\lim_{x\to c} f(x) = u$, then $\lim_{x\to c}(kf)(x) = ku$.

Now you are in a position to prove the following fact. Recall that a **polynomial** is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and each coefficient a_i is a real number.

(6) Let $p(x)$ be a polynomial. Then $\lim_{x\to c} p(x) = p(c)$.

We can also define one-sided limits: $\lim_{x \to c^+} f(x) = L \Leftrightarrow$ given $\varepsilon > 0$, there exists $\delta > 0$ for which $|f(x) - L| < \varepsilon$ whenever $c < x < c + \delta$. Also, $\lim_{x \to c^-} f(x) = L \Leftrightarrow$ given $\varepsilon > 0$, there exists $\delta > 0$ for which $|f(x) - L| < \varepsilon$ whenever $c - \delta < x < c$.

A comment on the definition of the limit.

Intuitively we have no difficulty with the concept of "*x* approaching *c*", etc. – i.e. with the idea of "a number increasing" i.e. of "one number turning into another one". We employ such concepts all the time in physics, chemistry etc, and it is an indispensable idea. But from the viewpoint of *strict mathematical logic* this raises conceptual difficulties. *Logically*, we would prefer to consider numbers as "static things", each having their own specific value – and not as "dynamic things" which can change from one value into another. If numbers can change their value, then - if *x* starts at 2 and approaches 3, say - we may ask what the "next number" is after 2? Etc. There are a number of conceptual difficulties. When proceeding in a strictly logical way we would prefer to think in terms of sets of "static" numbers. Note that this is precisely how we proceed when we use the $\delta\,\varepsilon$ definition of the limit. There is no talk about numbers moving around or changing values. We simply speak of sets of *x*-values in $(c - \delta, c + \delta)$ and sets of *f*-values in $(L - \varepsilon, L + \varepsilon)$. Nevertheless – and rather amazingly – this method perfectly encompasses our intuitive sense of "*f* approaches *L* as *x* approaches *c*." This is one of the reasons why this concept is considered to be so brilliant.

3.3 Continuity

Hints on how to read this section
This is useful, important material.

In many physical situations, we are looking at relationships between variables in which one depends on the other in a continuous or smooth way. Small changes of input lead to small changes of output. For example, the position of a projectile will be a continuous function of time; the projectile will not simply vanish from one point and appear simultaneously at some distant point (at least if we are living in a Newtonian world). This means that as the independent variable x gets closer and closer to some number c, the corresponding function values $f(x)$ get closer and closer to $f(c)$. We formalize this into a definition.

> **Definition:** Let $f(x)$ be a function defined on some interval containing a number c. Then $f(x)$ is **continuous** at c if and only if both:
>
> 1. $f(x)$ is defined at c
> 2. $\lim_{x \to c} f(x) = f(c)$.

We can reformulate this definition-using $\varepsilon - \delta$ notation. Condition (2) can be rewritten as: given any $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Note that continuity is initially defined at individual points; we will say that a function is **continuous on a set** if and only if it is continuous at each point of the set.

Here are some basic properties of continuous functions:

(1) Suppose f and g are functions defined on an interval containing a point c and continuous at c. Then $f \pm g$, and fg are continuous at c. If $g(c) \neq 0$, then f/g is also continuous at c.

(2) Suppose that f is a function with domain A and range B and g is a function whose domain contains the image $f(A) \equiv \{f(x) : x \in A\}$ of f. Suppose also that f is continuous at a, and g is continuous at $f(a)$. Then the **composite function** $g \circ f$ (defined by $(g \circ f)(x) = g(f(x))$ is continuous at a.

(3) All polynomials are continuous at every value in their domains. (This includes constant functions and the function x itself.)

(4) Every rational function (*i.e.,* one of the form $p(x)/q(x)$ where p and q are polynomials) is continuous at every value for which q does not vanish.

(5) Every power function x^r for real exponent r is a continuous function of x on its domain of definition.

3.4 Calculating Limits Using Continuity

Hints on how to read this section

This is useful, important material.

We have seen that, as a matter of definition, a function is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$. Suppose that we know **in advance**, for some reason, that the function f is continuous at c. Then we can use the continuity to find the limit of f at c simply by evaluating it there. Thus $\lim_{x \to c} f(x) = f(c)$ provided that f is continuous. This observation may not be such a big deal, but in a moment we will look at an example where it has some weight.

Another important tool is the **Squeezing** or **Pinching Principle**: If for $x \neq c$ in an open interval that contains c, we have $g(x) \leq f(x) \leq h(x)$ for functions defined on the interval for which $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then the limit $\lim_{x \to c} f(x)$ exists and is equal to L.

Examples

1. Evaluate:

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}.$$

The function whose limit we are being asked to compute is a rational function, and thus continuous at all points of its domain. However, because the denominator vanishes when $x = 2$, the function is not defined at this point, i.e. 2 is not in its domain, and so there is no question of computing its limit by evaluating the function.

We make an important observation about limits: when we calculate the limit, we are considering the values of the function not *at* the point in question, but only at points *near* the limiting point. As a result, if we replace the function by a second function equal to the given function *except at* the limit point, then we do not change the value of the limit. Note that, when $x \neq 2$,

$$\frac{x^3 - 8}{x - 2} = x^2 + 2x + 4.$$

The left and right sides of this equation represent two functions which happen to agree as long as $x \neq 2$. Because of this

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 4).$$

Have we made some progress? While the left side is not defined at $x = 2$, the right side is. Moreover, $x^2 + 2x + 4$ is a polynomial, and we know that all polynomials are continuous everywhere. Therefore, we can find the limit of the polynomial by evaluating it at $x = 2$:

$$\lim_{x \to 2} (x^2 + 2x + 4) = (2^2 + 2(2) + 4) = 12.$$

This is the answer to the exercise.

2. Evaluate:

$$\lim_{x \to -8} \frac{x^{1/3} + 2}{x + 8}.$$

Again, substitution is out, since the numerator and denominator vanish and we get a 0/0 form. So we try to isolate a factor $x+8$ to cancel. Using the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, we get $x + 8 = (x^{1/3} + 2)(x^{2/3} - 2x^{1/3} + 4)$. So as long as $x \neq -8$, we have that

$$\frac{x^{1/3} + 2}{x + 8} = \frac{1}{x^{2/3} - 2x^{1/3} + 4}.$$

The right side is continuous for all x, so we can calculate its limit as $x \to -8$ by evaluation. The answer is 1/12.

3.5 Examples from Trigonometric Functions

Hints on how to read this section

This is useful, important material.

As noted earlier, we prefer purely algebraic proofs to geometric ones. However, in getting started with the trigonometric functions, it is not really practical to proceed in a purely algebraic way. For a start, this requires a purely algebraic definition of the functions. Later, we can do this based on **series**, and then, if we wish, re-derive all the previous results we are now obtaining geometrically. Then, we could define π purely algebraically, for example, as the smallest positive solution of $\sin x = 0$. Historically, of course, the trigonometric functions arose from geometry, and we do not want to neglect this important and powerful connection. So we proceed, for now, geometrically.

We will make considerable use of trigonometric functions in this course. If you are not familiar with the basic (non-calculus) properties of these functions, you should carefully work through Appendix

D of the Textbook. Some of the problems from the Exercises, page A32-A33, will be used on the Quizzes.

We begin with the basic property: $|\sin x| < |x|$ for each nonzero value of *x*. You can see why this is true by drawing a diagram showing the arc *x* of a unit circle and the semi-chord, which represents sin *x* for example; see figure on page 170 of the Textbook. Note that $-1 \le \sin x \le 1$ *and* $-1 \le \cos x \le 1$ for all *x*.

(1) For all real *x* and *a,*

$$\sin x - \sin a = 2\left(\sin\frac{x-a}{2}\right)\left(\cos\frac{x+a}{2}\right)$$

so that, using the fact that $\left|\cos\frac{x+a}{2}\right| \le 1$,

$$|\sin x - \sin a| \le 2\left|\sin\frac{x-a}{2}\right| \le |x-a|.$$

It follows from this that sin is continuous everywhere.

(2) For all real *x* and *a,*

$$\cos x - \cos a = 2\left(\sin\frac{x-a}{2}\right)\left(\sin\frac{x+a}{2}\right)$$

Exercise 22: Complete the argument to show that it follows from this that cos is continuous everywhere.

4. THREE KEY THEOREMS OF DIFFERENTIAL CALCULUS

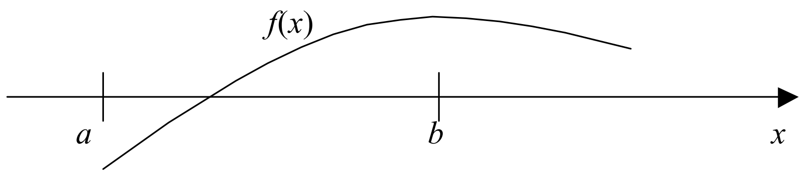
Hints on how to read this section

*These are 3 very important calculus theorems. You should, at a minimum, know the **statement** of each of these theorems – i.e. you should be quite clear about **what they say** is true. You should also make some effort to understand the proofs. These proofs may not be as challenging as the material in Sec. 3.2, on the basic concept and definition of the limit, but you will probably still find the proofs are not trivial. As with the material in Sec. 3.2, don't expect everything to become clear right away. If, after having taken a few shots at these proofs, you find that you don't seem to be making much headway, then 'cut your losses' and just focus on being sure you understand (a) what each theorem says is true, (b) how to apply these theorems – which are the most important things.*

4.1 The Intermediate Value Theorem

Intuitively, or by sketching its graph, we can see that a function continuous on a closed interval $[a,b]$ must assume every value in between the values $f(a)$ and $f(b)$ that it assumes at its endpoints. We give a rigorous proof of this, beginning with a special case.

Lemma. Given: f is a real-valued function continuous on a closed interval $[a,b]$ and $f(a) < 0 < f(b)$.



Prove: There exists a number c between a and b for which $f(c) = 0$.

Comment: To motivate the proof, think of a fisherman landing a fish, where the function represents the position of the fish above the surface of the water. At the beginning, the fish is submerged, while at the end (assuming success) the fish is resting on the dock. At some point, it must break the surface. When? At the point when it ceases to be underwater.

Proof: Let S be the set of all numbers x that lie in the interval $[a,b]$ for which $f(x) < 0$. Think about this for a moment: S is a set of x-numbers in the domain of f, but an element is included in S only if its associated f-number $f(x)$ is negative.

Exercise 23: although, our algebraic proof does not use any geometric argument, you will find it helpful to draw a diagram to keep things clear. Draw this figure for the specific example of $f(x) = x^2 - 100$, $[a,b] = [-9,11]$. What is S for this example?

Since, $f(a) < 0$, S is nonempty and contains a. In fact, it actually contains x's in some interval $[a, a+\delta]$. How do we know that? well, since f is continuous from the right at a, there exists a positive number δ such that, for all x in $a \le x < a+\delta$, it must be true that:

$$|f(x) - f(a)| < |f(a)| = -f(a). \qquad \ldots(4.1)$$

Why? note that $|f(a)|$ plays the role of ε in the definition of continuity being applied here and that a δ is guaranteed to exist for *any* positive ε that might be specified, and $|f(a)|$ is certainly a positive number. Thus:

$$f(a) - |f(a)| < f(x) < f(a) + |f(a)|$$

and so:

$$2f(a) < f(x) < 0$$

Therefore, since $f(x) < 0$ then these $x \in S$.

Exercise 24: For the same example as in Ex. 23 give a few examples of $[a, a+\delta]$ that satisfy eqn. (4.1).

Exercise 25: similarly, show that S is bounded above by all the numbers in an interval $(b-\eta, b]$ for some $\eta > 0$.

Since S is nonempty and bounded above, it has a least upper bound c, by the Completeness of the Reals Axiom. Since, $a < c < b$, the function f is defined at c. By process of elimination, we will show that $f(c) = 0$.

First, suppose, if possible, that $f(c) > 0$. Then, f should be positive some way to the left of c; why? Since, f is continuous at c, there exists a number $\delta > 0$ for which $|f(x) - f(c)| < f(c)$ for all values of x in the interval $|x - c| < \delta$. (Notice how we are taking ε to be $f(c)$ here). In particular, when

$c-\delta < x \leq c,$ we have that $f(c)$ - $f(c) < f(x)$ i.e. $0 < f(x)$. Hence, no number in the interval $(c-\delta, c]$ is a member of S. Now c is an upper bound for S, so it follows that each number in the interval $(c-\delta, c]$ is also an upper bound. But, then this contradicts the fact that c is the *least* upper bound. So $f(c) > 0$ is not possible.

Exercise 26: suppose, if possible, that $f(c) < 0$. Then (and here you are on your own!), there exists a positive number δ such that $f(x) < 0$ whenever $c \leq x < c+\delta$. But then the interval $[c, c+\delta) \subseteq S$ and this contradicts c being an upper bound of S. Write out the proof to the same level of detail as for $f(c) > 0$.

The only possibility remaining is that $f(c) = 0$.

Intermediate Value Theorem: Given: f is a continuous real-valued function defined on the closed interval $[a, b]$ and C is a number for which $f(a) < C < f(b)$.

Conclusion: There exists a number c in (a, b) for which $f(c) = C$.

Comment: You can apply the lemma to $g(x) \equiv f(x) - C$.

Exercise 27: show that $g(x)$ satisfies the hypothesis of the lemma.

Exercise 28: formulate a theorem and proof for the case that $f(a) > f(b)$.

Now you may think that the IVT is trivial. But, it depends for its validity on the completeness of the reals. It fails to hold on the rationals: we could, if we wanted to, use exactly the same definition of continuity on the rationals as on the reals, just adapting the definition so that the only numbers we refer to are rationals, and according to this definition the function is x^2 is continuous. This function assumes the value 1 at $x = 1$ and the value 4 at $x = 2$, but it never assumes the intermediate value 3. But, it is a different story on the reals. Note how extremely powerful the Intermediate Value Theorem is. Although, it comes almost directly from Axiom 9, it is enormously easier to use it to show that there is a real square root of 3.

Exercise 29: prove this.

4.2 The Extreme Value Theorem

Another existence theorem of paramount importance in calculus is the Extreme Value Theorem, which tells us under what conditions a function $f(x)$ is *guaranteed* to have a maximum or minimum value on some interval $[a,b]$. Finding a maximum or minimum is one of the most important *practical* applications of calculus. It sometimes also takes some effort. How valuable to establish first that what we are looking for actually exists!

Recall that a function is bounded on a closed interval $[a, b]$ if there are some numbers k and K for which $k < f(x) < K$ for each $x \in [a, b]$. We say that a function $f(x)$ has a **maximum** value M on $[a,b]$ if there exists some number d in $[a,b]$ such that $f(d) = M \geq f(x)$ for all $x \in [a, b]$. **Exercise 30**: draw a diagram to illustrate this.

Do "being bounded" and "having a maximum" sound about the same? Well, they are not!

Example: Let
$$f(x) \equiv \frac{\sin x}{x} \quad \textit{for } x \neq 0.$$

Is this function **bounded**? We have already noted that $-x \leq \sin x \leq x$ for each nonzero value of x, so the answer is yes: $-1 < f(x) < 1$ for all $x \neq 0$. Does it have a **maximum**? Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we see that $f(x)$ takes values arbitrary close to 1. But, it never takes the value 1 itself. You might be tempted to say that $f(0)$ is the maximum; but the function is not defined at 0. And don't say "the maximum is 1" because f *has to take on this value* somewhere in order for 1 to be the maximum. That's just part of our *definition* of the maximum.

On the other hand
$$f(x) \equiv \begin{cases} \sin x / x, & x \neq 0 \\ 3, & x = 0 \end{cases}$$

is both bounded and has a maximum. Similarly for $f(x) \equiv \begin{cases} \sin x / x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

but *not* for $$f(x) \equiv \begin{cases} \sin x / x, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

Exercise 31: If a function is continuous on [*a, b*] then it is necessarily defined for all *x* in [*a, b*] since, otherwise, it would be meaningless to state that $\lim_{x \to c} f(x) = f(c)$ for all *c* in [*a, b*]. Merely being defined for all points in a closed interval, however, is not enough to guarantee boundedness. Provide an example of a function which is defined, but not continuous, for all points in a closed interval and which is not bounded on that interval.

Lemma 1. Given: f is a function defined and continuous on the closed interval [*a,b*], and let $a < c < b$.

Prove: There exists an open interval $(c-\delta,\ c+\delta)$ with $\delta > 0$ contained in [*a,b*] upon which *f* is bounded.

Proof: Taking $\varepsilon = 1$ in the $\delta\varepsilon$ definition of continuity, we know that we can always find some $\delta > 0$ such that $|x-c| < \delta$ implies that $|f(x) - f(c)| < 1$. This implies that $-1 < f(x) - f(c) < 1$. Hence, when $c - \delta < x < c + \delta$, $f(c) - 1 < f(x) < f(c) + 1$, and the result follows. ♠

Lemma 2. Given: *f* is a function defined and continuous on the closed interval [*a,b*]

Prove: *f* is bounded on [*a,b*].

Proof: Let *u* be in [*a, b*]. Consider the interval [*a, u*] ("*u*" for upper end of the interval). If *f*(*x*) is bounded for all *x* in [*a, b*] then we will include *u* in a set, *S*. We have thus defined the set *S* by this 'test' of the *u*-values. We will then carry out the following steps:

1. We will prove that the set S is itself bounded above and so has a least upper bound, lub, call it c. You will want to keep clear the distinction between $f(x)$ being bounded and the set S being bounded – two different things. Geometrically, the boundedness of $f(x)$ has to do with the y-axis, while the boundedness of S has to do with the x-axis.

2. We will then prove that $c = b$.

3. This will almost complete our task of proving that $f(x)$ is bounded on $[a, b]$, but not quite since we will only have proven that $f(x)$ is bounded on $[a, c)$, i.e. on $[a, b)$. Why is b not yet included? Answer: the lub of a set is not necessarily a member of the set; simple example: let Q be the set of all $x < 7$; then lub $S = 7$ but 7 is not in Q. Therefore, our last step will be to prove that $f(x)$ is bounded on $[a, b]$, i.e. including the end point b.

OK, then:

1. By adapting Lemma 1 to the endpoint a, we see that f is bounded on some interval of the form $[a,u]$ with $u > a$, so the set S is nonempty. The set S is bounded above by b. Hence set S has a least upper bound, c, by the completeness of the real numbers axiom.

$$\text{So: } c = \text{lub}\{u: f \text{ is bounded on } [a,u]\}$$

2. We now prove that $c = b$. Note that it must be true that $c \leq b$. Suppose, if possible, that $c < b$. By Lemma 1, there exists a number $\delta > 0$ such that f is bounded on $(c - \delta, c + \delta)$. Being bounded on $[a, c - \delta]$ and on $(c - \delta, c + \delta)$ then by Lemma 1, f is bounded on $[a, c + \delta]$. Contradiction! c is the least upper bound of S. Therefore $c = b$.

3. Next we consider the continuity if f at $x = b$, from which we know – again by adapting Lemma 1 to an end point - that f is bounded on some interval of the form $[b - \delta, b]$. Since $b - \delta < b$ we know from what we just proved that f is bounded on $[a, b - \delta]$. f is therefore bounded on $[a, b]$. ♠

The Extreme Value Theorem: Given: f is defined and continuous on $[a,b]$.

Prove: g assumes a maximum and a minimum value on $[a,b]$.

Proof: By Lemma 3, f is bounded on $[a,b]$. Let M be the least upper bound of the set $f([a,b]) \equiv \{f(x): a \leq x \leq b\}$ (M exists by the least upper bound axiom). We now show that there exists some number c in $[a,b]$ for which $f(c) = M$.

Suppose if possible, that f never assumes the value M. Although it may not be immediately obvious why we do so, let us define a new function $g(x) \equiv 1/(M - f(x))$. The function $g(x)$ is nonnegative and continuous on $[a,b]$, and so by Lemma 3, $g(x)$ is bounded on $[a,b]$. Hence, there is some number K for which

$$0 < g(x) \leq K$$

for all x in $[a,b]$. Thus, for all x,

$$\frac{1}{M - f(x)} \leq K \Rightarrow \frac{1}{K} \leq M - f(x) \Rightarrow f(x) \leq M - \frac{1}{K},$$

This makes $M - (1/K)$ an upper bound of the set of values of $f(x)$, contradicting the selection of M as the least upper bound.

Therefore, f must assume the value of M somewhere.

Exercise 32: in a similar way show that f attains its minimum value. ♠

4.3 The Mean Value Theorem

4.3.1 Introduction

We now show some additional results for the case of functions that are not only continuous, but actually possess a derivative. In physical terms, differentiability means that a moving particle has an instantaneous speed at each point. In physical terms, the results that we are about to prove amount to the following intuitively reasonable statements:

(a) If I throw a body into the air and gravity brings it down again, then at some stage the instantaneous speed of the body is 0.

(b) If I travel between two points and work out the average speed over the time taken, then at some point on my journey, my instantaneous speed will be equal to the average speed.

We can also look at things geometrically. Differentiability of real functions means that we can construct a tangent at each point of its graph. The results can be interpreted as follows:

(c) If a smooth curve is drawn between two points in the plane that have the same distance from the x-axis, then at some point on the curve, its tangent will be horizontal. **Exercise 33**: draw some examples.

(d) If a smooth curve is drawn between two points in the plane, along with the segment joining these two points, then at some point on the curve, its tangent will be parallel to the segment joining its endpoints.

Lemma. Given: f is a function which is defined, continuous and differentiable on an open interval (u, v) and f assumes its maximum value at some point w in the interval.

Prove: $f'(w) = 0$.

Comment: The same conclusion holds at minima as well as maxima.

Proof: Consider the quotient

$$\frac{f(w+h)-f(w)}{h}.$$

For small positive values of h, this quotient is non-positive, and so the limit $f'(w)$ *as* $h \to 0^+$ must be non-positive. For small negative values of h, this quotient is non-negative, and so the limit $f'(w)$ *as* $h \to 0^-$ must be non-negative. The only way $f'(w)$ can be both non-positive and non-negative is for it to vanish. ♠

4.3.2 Rolle's Theorem

Given: f is a real-valued function defined on the closed interval, $[a,b]$ such that:

(i) f is continuous on $[a,b]$

(ii) f has a derivative at each point of (a, b) and

(iii) $f(a) = f(b)$.

Prove: There exists a point c in the open interval (a, b) for which $f'(c) = 0$.

Comment: This theorem is **non-constructive** in the sense that it does not specify *how* one is supposed to actually find out which c does the job. It just tells you that it exists somewhere in the open interval (a, b).

Proof: Since f is continuous on $[a,b]$, it assumes both its maximum value M and its minimum value m. If the function is constant, then its derivative vanishes everywhere and any c will do. If the function is not constant, its maximum and minimum values must be different. Since the function has the same value at the endpoints, it cannot assume both its extreme values there, and so either its maximum or its minimum must be assumed inside the open interval (a, b) at some point c. From the lemma $f'(c) = 0$. ♠

4.3.3 Proof of Mean Value Theorem

Given: f is a real-valued function, defined and continuous on and having a derivative at every point of the open interval (a, b).

Prove: There exists some point $c \in (a, b)$ for which

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}.}$$

Exercise 34: draw a picture of the situation, and also draw the secant joining the points $(a, f(a))$ and $(b, f(b))$.

This secant has the equation

$$y = f(a) + \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

The function $g(x)$ that we will next define in the proof measures the vertical distance between a point on the curve $y = f(x)$ and a point on the secant. This distance is 0 at a and b, and so suggests that we can use Rolle's Theorem.

Proof: It is not immediately apparent that it will be helpful to define the following new function, but let's see what happens if we do:

$$g(x) \equiv f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

for $a \le x \le b$. The function g is defined and continuous on $[a,b]$ and has a derivative at each point in (a, b). Since, also $g(a) = g(b) = 0$, Rolle's theorem provides that there exists a point $c \in (a, b)$ for which $g'(c) = 0$. **Exercise 35**: now reformulate this conclusion in terms of f and get the desired result. ♠

This theorem has a very important corollary. It provides that if a function has a derivative that vanishes everywhere, then the function must be a constant.

4.3.4 Corollary to MVT

Given: f is a function defined and continuous on the closed interval $[a,b]$ and whose derivative $f'(x)$ exists and takes the value 0 at each point $x \in (a, b)$.

Prove: f is constant on the interval $[a,b]$.

Proof: It is enough to show that f assumes exactly the same value at any two given points u and v in the interval. **Exercise 36**: apply the Mean Value Theorem on the interval $[u,v]$ to obtain the result. ♠

4.3.5 The Extended Mean Value Theorem (Cauchy's Mean Value Theorem)

Given: f and g are two functions defined and continuous on $[a,b]$ and each possessing a derivative at every point of (a, b). Suppose further that $g(a) \neq g(b)$.

Prove: There exists a point $c \in (a, b)$ for which

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.}$$

Proof: Again, it is not immediately obvious that it will be helpful to define the following function, but let's see what happens when we do and then apply Rolle's Theorem to the function:

$$h(x) \equiv f(x)\,(g(b)-g(a)) - g(x)\,(f(b)-f(a)).$$

Exercise 37: do so. ♠

One of the important uses of Cauchy's Mean Value Theorem is to prove **l'Hospital's Rule**, see page 500 of the Text.

5. COMPLEX NUMBERS

> *Hints on how to read this section*
> *The Text contains a good, brief review of the basic aspects of complex numbers in Appendix G. Since you will use all of this material at some point, in this and other courses, you should make sure you are familiar with all of this material. The following material here is aimed at making you feel more comfortable with the concept of imaginary and complex numbers – which can often 'spook' students.*

Most of us have probably had the thought: "I don't know what a complex number actually *means*." Yes, we may know all the rules of how to manipulate them – all the formal mathematical rules - but that can still leave us unsatisfied, as if there's something more, something that is still eluding us. After a course in chemistry we may also be inclined to say: "I don't know what entropy actually *means*." These two problems seem similar. But surely that can't be true: entropy relates to the behaviour of the physical world, while mathematics is 100% a human creation. It should not be surprising that some aspects of the physical world are difficult to understand and may, to some degree, always elude us. But *we* human beings have created mathematics – *all* of it – so how can *it* possibly elude human understanding? Surely we must be just tripping ourselves up psychologically in some way when we say we don't know what complex numbers mean. We need to take a fresh look at this.

It can be helpful to go over the historical origins of an idea. If we can see the path by which human thinking reached its present view of something, then that can help us move along our own personal path to understanding of the thing. We know that long ago the only numbers that people accepted as 'real' or legitimate were the positive integers. In a world where almost no one had any use for numbers beyond the counting of sheep and such, it is not surprising that people would reject the concept of, say, negative integers as being unreal – even ridiculous. (Never mind fractions. Or irrationals.) If someone insisted on including negative numbers in the collection of 'real numbers' -

pointing out that one could manipulate them in a formal way, following strict logical rules, and all would work out ok – most everybody else would say: "Well, yes, it does all hang together as a pure exercise in *logic* – but what do negative integers actually *mean*?"

From our advantage of historical perspective we can perhaps see what their problem was. It has to do with what we mean by 'mean'! What was causing difficulties for those people long ago was that they had a specific *application* for numbers in their minds – e.g. counting sheep – and negative numbers have no place in such applications of numbers. We can see that what they meant by 'mean' is that there *be an application of the numbers.*

This is the key to resolving this whole business. We need to distinguish between mathematics in itself and its applications. Mathematics in itself is a logic game, like chess, and there it makes no sense to ask what something 'means'. What does a pawn *mean*? What does it *mean* that the bishop can move diagonally across the board? We know it's pointless to ask such questions. The word 'mean' is inappropriate in such logic games as mathematics. When we ask what a negative (or irrational or imaginary or complex) number 'means' we are not really using that word properly. What we actually have in mind is: "What application is there for such a number?" Well, it's perfectly possible there is *no* application. So what? There's no application for chess either. Mathematics doesn't require any application at all to make it a perfectly worthwhile human pursuit. It may happen, however, that within some particular context there may be some applications for some of the numbers we have invented. If the context is counting (live) sheep then we can't use very many of the types of numbers we have invented. But if we are keeping books on credits and debts then there is an application for negative numbers (to indicate debt) and also for zero, fractions and decimals (although not for irrationals, imaginaries or complex numbers). For keeping books it isn't actually *necessary* to use negative numbers since we could keep separate books for credits and debts, using only positive numbers in each – but it's more *convenient* to use one book and therefore to use negative numbers also. The expansion of the original, limited idea of what constituted a number was expanded time and again over the centuries – because we found it to be both possible and convenient.

A different kind of 'application' is entirely internal to mathematics – 'applying', or we might better say, 'relating' – algebra to geometry. Numbers are algebraic quantities. They are pure inventions of the human mind – as also are the concepts of geometry. (So far as we know, there are no (perfectly) straight lines in the physical world, for example.) Nevertheless, most people seem to find algebraic quantities more abstract than geometric ones – and they tend to consider the relating of algebra to geometry as a kind of application. They also feel this gives 'meaning' to the algebraic quantity – e.g. numbers. Unquestionably the fact that the collection of all integers, rationals and irrationals (which for historical, but otherwise no very good, reasons we call 'reals') can be related to the points of a line is psychologically reassuring to most people and makes them feel that they "know what real numbers mean". This, however, is really just psychological, not logical. It also, unfortunately, has the bad consequence of setting ourselves up for a problem with complex numbers. Because the latter cannot be related to the points on a line, then by contrast with 'real' numbers, we are bound to feel they are somehow not legitimate numbers and that we don't know what they 'mean'. We need to remind ourselves that we don't actually know what *any* number 'means'!

Another aspect of this psychological trick we play on ourselves has to do with the fact that we are more comfortable with things that are more common in our experience. We feel that we 'know them', that we 'know what they mean'. There are more applications of positive integers than of irrationals, say – and certainly more than for complex numbers. So we end up confusing 'meaning' and legitimacy with whether something is common or not – a different thing. As to applications, complex numbers come into their own when one is trying to keep track of two quantities, rather than just one. This is not all that common a situation – but not utterly exotic either. One of the most natural 'applications' of complex numbers is to consider them to be an **ordered pair** of ordinary numbers and to associate each complex number with a point in the plane. An important application to the physical world is to analyze electrical circuits containing capacitors and inductances. One wants to know what the circuit current is for a given applied, sinusoidal voltage, $V_{applied}(t) = V_o\cos(\omega t)$, where V_o and ω are specified constants and t is time. As you know, it turns out that one needs to find two quantities describing the current – its amplitude I_o and its phase relative to the voltage, ϕ : $I(t) = I_o\cos(\omega t + \phi)$. Because of this aspect it is more convenient and tidy to analyze such circuits using complex numbers. Of course most of us have far more occasion to count things like money than to analyze electrical circuits so most of us never get to feel all that familiar with complex numbers and can tend therefore to feel that we don't know what they 'mean'. It's a mistake to make much out of this lack of a familiar and reassuring psychological feeling. It's irrelevant. We should remind ourselves that it's pointless to ask about the ’meaning’ of any number – regardless of how familiar. We just need to be sure we know how to reliably manipulate numbers, following all the rules of logic. In the end that’s all there really is. That’s all that really counts.

No doubt another aspect of complex numbers that tends too make us feel uncomfortable is the common practice of denoting them in the form $a + bi$, e.g. $3 + 5i$. We are bound to feel somewhat puzzled and uneasy about the use of “+” in this context. Clearly the use of “+” here means something quite different than in an expression like 3 + 5. In the latter case, an *operation* is indicated, which causes the two numbers (3 and 5) to “disappear” while creating a new one (8) in their place. We can also consider “3 + 5” as, itself, constituting a single number. By contrast, the 3 and 5 most definitely do not disappear in the case of $3 + 5i$. And no operation is involved here. When all the dust has settled, 3 + 5 just contains *one* piece of information – namely, *8*. By contrast, $3 + 5i$ contains *two* pieces of information. In that case, it is logically preferable to denote complex numbers directly as an **ordered pair** (a,b), for example, (3,5). This avoids any confusion about an operation being involved. It also treats the 3 and the 5 in a symmetrical way – rather than implying there is something fundamentally more ‘legitimate’ about the first member of the pair compared with the second. This also avoids the rather lurid – if not downright ‘spooky’ – symbol i. It also leads directly to the obvious geometrical application, where complex numbers are associated with points in the plane.

Using the ordered pair approach, we entirely avoid the ‘spooky’ i. For example, consider the problem of finding a solution to the equation $x^2 + 1 = 0$, i.e. of $x^2 = -1$. Let us show that the number (0,1) is a solution:

$$(0,1)^2 = (0,1)(0,1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1,0)$$

Of course (0,1) represents i and (-1,0) represents -1. You will notice that we have to use a different definition of the operation of multiplication for complex numbers than we used for reals, namely: $(a,b)(c,d) = (ac - bd,\ ad + bc)$. If you want to see more about how the ordered pair approach works, you can look at the book ‘Calculus. Vol. 1. by T.M. Apostol, Wiley, 1967, page 358.

Answers for Some of the Problems

Ex 3. $x > 5$ or $7/2 < x < 4$.

Ex 4. $x \leq -2$ or $x \geq 0$.

Ex 7(a). $-1 \leq x < 0$ or $0 < x \leq 1$.

Ex 7(b). $1.5 < x < 2$ or $2 < x < 2.5$.

Ex 7(c). $-5 \leq x < 3$ or $3 < x \leq 11$.

Ex 7(d). $-5/8 < x < -3/8$.

Ex 7(e). $x < -10$ or $-22/3 < x < -18/7$ or $x > -10/7$.

Ex 9(a). $|x| < 3$.

Ex 9(b). $|x - 2| < 5$.

Ex 9(c). $|x + 2| < 5$.

Ex 10(a). $A \geq 3/2$.

Ex 10(b). $A \geq 6$.

Ex 10(c). $A \geq 11$.