Abstract of "The geometry of shape recognition via the Monge-Kantorovich optimal transport problem" by najma ahmad, Ph.D., Brown University, May 2004.

A toy model for a shape recognition problem in computer vision is studied within the framework of the Monge-Kantorovich optimal transport problem with a view to understand the underlying geometry of boundary matching. This formulation generates an optimal transport problem between measures supported on the boundaries of two planar domains Ω, $\Lambda \subset \mathbb{R}^2$ — with optimality measured against a cost function $c(\mathbf{x}, \mathbf{y})$ that penalizes a convex combination of distance $|\mathbf{x} - \mathbf{y}|^2$ and a relative change in local orientation $|\mathbf{n}_\Omega(\mathbf{x}) - \mathbf{n}_\Lambda(\mathbf{y})|^2$. The questions addressed are the existence, uniqueness, smoothness and geometric characterization of the optimal solutions.

The geometry of shape recognition via the Monge-Kantorovich optimal transport problem

by
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This dissertation by najma ahmad is accepted in its present form by the Department of Physics as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

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To the

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Chapter 1

Introduction

In an optimal transportation problem one is given a distribution ρ_1 of supply and a distribution ρ_2 of demand and a cost function $c(\mathbf{x}, \mathbf{y}) \geq 0$ representing the cost to supply a unit mass from a source at $\mathbf{x}$ to a target at $\mathbf{y}$ and asked to find the most efficient way of transportation to meet the demand with the given supply. Efficiency is measured in terms of minimizing the total cost of transportation. A classic example is where ρ_1 gives the distribution of iron mines throughout the countryside and ρ_2 the distribution of factories that require iron ore, with $c(\mathbf{x}, \mathbf{y})$ giving the cost to ship one ton of iron ore from the mine at $\mathbf{x}$ to the factory located at $\mathbf{y}$. Let Ω and Λ denote the domains of this supply and demand. To model a transport problem one must choose for the cost a function $c : \Omega \times \Lambda \longrightarrow \mathbb{R}$ that accounts for all possible sources of expenses encountered — in this particular example these can be the cost for loading and unloading of iron ore, the length of trips between the mines and the factories, the cost of gasoline consumption in the transport process etc. The pairing of $\mathbf{x} \in \Omega$ with $\mathbf{y} \in \Lambda$ can be represented by a measure γ on $\Omega \times \Lambda$ with $d\gamma(\mathbf{x}, \mathbf{y})$ giving a measure of the amount of iron ore transported between the pairs $(\mathbf{x}, \mathbf{y}) \in \Omega \times \Lambda$. One can then define the *total transport cost* by the integration

$$C(\gamma) := \int_{\Omega \times \Lambda} c(\mathbf{x}, \mathbf{y})\, d\gamma(\mathbf{x}, \mathbf{y}). \tag{1.1}$$

One essential feature of γ is that summing it over all the sources $\mathbf{x} \in \Omega$ for a given $\mathbf{y}$ gives the total consumption $\rho_2(\mathbf{y})$ of iron ore at $\mathbf{y}$. Similarly, for a given $\mathbf{x} \in \Omega$

summing γ over all $\mathbf{y} \in \Lambda$ gives the total production $\rho_1(\mathbf{x})$ of iron ore at $\mathbf{x}$. In other words, γ has ρ_1 and ρ_2 for *left* and *right marginals* — defined more precisely below. Given ρ_1 and ρ_2 optimization is achieved by minimizing the total cost (1.1) over all possible ways γ of pairing the mines $\mathbf{x} \in \Omega$ with the factories $\mathbf{y} \in \Lambda$ when γ has ρ_1 and ρ_2 for marginals. These very ideas form the crux of the Monge-Kantorovich optimal transportation problem.

A precise formulation of the problem requires a bit of notation. Let $P(\mathbb{R}^d)$ denote the set of Borel probability measures on $\mathbb{R}^d$ — non-negative Borel measures for which $\rho[\mathbb{R}^d] = 1$. The *support* of $\rho \in P(\mathbb{R}^d)$, denoted $\operatorname{spt} \rho$, is defined to be the smallest closed subset of $\mathbb{R}^d$ carrying the full ρ measure, i.e. $\rho[\mathbb{R}^d \backslash \operatorname{spt} \rho] = 0$.

Definition 1.0.1 (push-forward measures). Given a measure $\rho \in P(\mathbb{R}^d)$ and a Borel map $\mathbf{u} : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$ the *push-forward* of ρ through $\mathbf{u}$ — denoted $\mathbf{u}_{\#}\rho$ — is a Borel measure on $\mathbb{R}^n$ and defined by $\mathbf{u}_{\#}\rho[V] := \rho[\mathbf{u}^{-1}(V)]$ for all Borel $V \subset \mathbb{R}^n$.

The map $\mathbf{u}$ is said to *push ρ forward* to $\mathbf{u}_{\#}\rho$, and when $\mathbf{u}$ is defined ρ-almost everywhere, the map is called *measure preserving* between ρ and $\mathbf{u}_{\#}\rho$.

Definition 1.0.2 (marginals). Given Borel probability measures $\rho_1, \rho_2 \in P(\mathbb{R}^d)$ a joint measure γ defined on the product space $\mathbb{R}^d \times \mathbb{R}^d$ is said to have ρ_1 and ρ_2 for *left* and *right marginals* if

$$\gamma[U \times \mathbb{R}^d] = \rho_1[U] \quad \text{and} \quad \gamma[\mathbb{R}^d \times V] = \rho_2[V]$$

for every Borel measurable $U, V \subset \mathbb{R}^d$.

We denote by $\Gamma(\rho_1, \rho_2)$ the set of all joint measures on $\mathbb{R}^d \times \mathbb{R}^d$ that have ρ_1 and ρ_2 for marginals.

1.1 Background and motivation

Given Borel probability measures ρ_1 and ρ_2 on bounded domains Ω and Λ in $\mathbb{R}^d$ the *Monge-Kantorovich optimal transport problem* is to find a joint measure γ on the product space $\Omega \times \Lambda$ with ρ_1 and ρ_2 as its left and right marginals. This joint measure γ is *optimal* in the sense that it minimizes the *total transport cost* $C(\gamma) := \int_{\Omega\times\Lambda} c(\mathbf{x},\mathbf{y})\, d\gamma(\mathbf{x},\mathbf{y})$ over the convex set $\Gamma(\rho_1,\rho_2)$ of all joint measures having ρ_1 and ρ_2 for marginals — here $c(\mathbf{x},\mathbf{y}) \geq 0$ is a continuous function on $\Omega\times\Lambda$ representing the cost to transport a unit mass from $\mathbf{x}\in\Omega$ to $\mathbf{y}\in\Lambda$. Formulations of the Monge-Kantorovich problem exist in more general settings — see Appendix-A and the references cited there; the above formulation is however most suited to our purpose. If the optimal γ is supported on the graph of a map $\mathbf{u}:\Omega\longrightarrow\Lambda$ that pushes ρ_1 forward to ρ_2 then it is given by $\gamma = (\mathbf{id}\times\mathbf{u})_{\#}\,\rho_1$ with $\mathbf{u}$, called the *optimal transport map*, minimizing the total transport cost $C(\mathbf{u}) := \int_\Omega c(\mathbf{x},\mathbf{u}(\mathbf{x}))\, d\rho_1(\mathbf{x})$ among all maps pushing ρ_1 forward to ρ_2 — this is the so-called *Monge optimization problem.* An example where such a map exists can be found in Gangbo and McCann [9]: (1) ρ_1 is absolutely continuous with respect to Lebesgue measure and $c(\mathbf{x},\mathbf{y}) := h(\mathbf{x}-\mathbf{y})$ with h a strictly convex function or (2) ρ_1 and ρ_2 have disjoint supports and $c(\mathbf{x}-\mathbf{y}) := l(|\mathbf{x}-\mathbf{y}|)$ with $l\geq 0$ a strictly concave function. Depending on the measures ρ_1 and ρ_2 and the cost function $c(\mathbf{x},\mathbf{y})$, this optimal transport map contains information as to how far and in what direction the mass located in a neighborhood of $\mathbf{x}\in\Omega$ is transported [9]. For measures supported on the domain boundaries $\partial\Omega$ and $\partial\Lambda$, if the cost is chosen to depend also on the relative orientation of the outward unit normals to these boundaries, Fry observed that the corresponding Monge-Kantorovich problem serves as a prototype for a shape recognition problem in computer vision that uses boundary matching as a form of comparison to identify objects [8]. With this motivation we study the following variational problem:

$$\inf_{\tilde\gamma\in\Gamma(\tilde\mu,\tilde\nu)} \int_{\partial\Omega\times\partial\Lambda} \left[(1-\beta)|\mathbf{x}-\mathbf{y}|^2 + \beta\,|\tilde{\mathbf{n}}_\Omega(\mathbf{x}) - \tilde{\mathbf{n}}_\Lambda(\mathbf{y})|^2\right] d\tilde\gamma(\mathbf{x},\mathbf{y}), \tag{1.2}$$

a variant of the optimization problem in Gangbo and McCann [10]. Here $\tilde\mu$ and $\tilde\nu$ are Borel measures on the boundaries of the planar domains $\Omega,\Lambda\subset\mathbb{R}^2$ (dimension $d=2$) with finite and equal total mass $\tilde\mu[\partial\Omega] = \tilde\nu[\partial\Lambda] < +\infty$, $\tilde{\mathbf{n}}_\Omega(\mathbf{x})$ and $\tilde{\mathbf{n}}_\Lambda(\mathbf{y})$ are the outward unit normals to $\partial\Omega$ and $\partial\Lambda$ at $\mathbf{x}$ and $\mathbf{y}$ respectively, $\tilde\gamma\in\Gamma(\tilde\mu,\tilde\nu)$ is a joint measure on the product space $\partial\Omega\times\partial\Lambda$. The cost function $(1-\beta)|\mathbf{x}-\mathbf{y}|^2 + \beta\,|\tilde{\mathbf{n}}_\Omega(\mathbf{x}) - \tilde{\mathbf{n}}_\Lambda(\mathbf{y})|^2$, correlating the points $\mathbf{x}\in\partial\Omega$ with the points $\mathbf{y}\in\partial\Lambda$, penalizes a convex combination of a pure translation $|\mathbf{x}-\mathbf{y}|^2$ that measures the extent to which the global shape of the two boundaries differs, and a pure rotation $|\tilde{\mathbf{n}}_\Omega(\mathbf{x}) - \tilde{\mathbf{n}}_\Lambda(\mathbf{y})|^2$ measuring the change in local orientation as $\mathbf{x}$ gets mapped onto $\mathbf{y}$. The parameter $\beta \in [0, 1]$ controls the relative significance of the two contributions. This formulation for boundary matching was motivated by the works of Mumford and Fry in computer vision where Fry [8] developed an algorithm that enabled a computer to identify the species of a sample leaf by comparing its boundary with a catalog of reference leaves. To gain geometric insight into this comparison we analyze a *toy model*:

$$\text{toy model}: \begin{cases} \Omega,\, \Lambda \subset \mathbb{R}^2 & \text{bounded strictly convex planar domains,} \\ \partial\Omega,\, \partial\Lambda & C^4\text{- smooth boundaries,} \\ K_\Omega,\, K_\Lambda > 0 & \text{curvatures bounded away from zero,} \\ \tilde{\mu} \ll \mathcal{H}^1\lfloor_{\partial\Omega} \ll \tilde{\mu}, & \text{Borel probability measures } \tilde{\mu} \text{ on } \partial\Omega \\ \tilde{\nu} \ll \mathcal{H}^1\lfloor_{\partial\Lambda} \ll \tilde{\nu} & \text{and } \tilde{\nu} \text{ on } \partial\Lambda\text{, mutually continuous w.r.t.} \\ & \text{one-dimensional Hausdorff measure } \mathcal{H}^1 \\ & \text{restricted to the boundaries.} \end{cases} \tag{1.3}$$

1.2 Formulation of the problem

One approach to solving (1.2) for the toy model is to represent the domain boundaries by their constant speed parametrizations:

$$\begin{array}{ll} \begin{cases} \mathbf{x} : \mathbb{T}^1 \longrightarrow \partial\Omega \\ \mathbf{y} : \mathbb{T}^1 \longrightarrow \partial\Lambda \end{cases} & \text{simple closed } C^4 \text{ planar curves} \\ s,\, t & \text{constant speed parameters} \\ \begin{cases} v_\Omega := \left|\frac{d\mathbf{x}(s)}{ds}\right| \\ v_\Lambda := \left|\frac{d\mathbf{y}(t)}{dt}\right| \end{cases} & \text{constant speeds} \\ \mu, \nu << \mathcal{H}^1\lfloor_{\mathbb{T}^1} << \mu, \nu & \text{Borel probability measures on } \mathbb{T}^1\text{,} \\ & \text{mutually continuous w.r.t. } \mathcal{H}^1\lfloor_{\mathbb{T}^1}. \end{array} \tag{1.4}$$

and solve the equivalent optimization problem

$$\sup_{\gamma\in\Gamma(\mu,\nu)}\left\{\int_{\mathbb{T}^2}[(1-\beta)\mathbf{x}(s)\cdot\mathbf{y}(t)+\beta\mathbf{n}_\Omega(s)\cdot\mathbf{n}_\Lambda(t)]\,d\gamma(s,t)=:\mathcal{C}_\beta(\gamma)\right\} \tag{1.5}$$

on the flat torus $\mathbb{T}^2 := \mathbb{T}^1\times\mathbb{T}^1$ generated by the product of the parameter spaces — the one dimensional tori $\mathbb{T}^1$. For each $0\le\beta\le 1$ we call γ_o^β an *optimal solution* of the transport problem (1.5) if it maximizes the linear functional $\mathcal{C}_\beta(\gamma)$, representing the total transport cost, on the convex set $\Gamma(\mu,\nu)$:

$$\gamma_o^\beta \in \arg\max_{\gamma\in\Gamma(\mu,\nu)} \mathcal{C}_\beta(\gamma). \tag{1.6}$$

The existence of an optimizer for (1.2) and hence for (1.5) follows from the weak-$*$ lower semi-continuity on the weak-$*$ compact, convex domain of non-negative measures — see e.g. Kellerer [13]. Uniqueness, when present, is a consequence of the characteristic geometry that the support of γ_o^β must conform to. We characterize this geometry in terms of the sign of the mixed partial of the cost function that divides the flat torus, independent of β, into the disjoint subsets: Σ^+ where the mixed partials are positive — meaning the cost is *convex type* — and Σ^- where the mixed partials are negative — meaning the cost is *concave type* — in Definition-3.2.2 below. The geometric constraints (1.3) also guarantee the boundary curves $\partial\Sigma^+ = \partial\Sigma^- =: \Sigma^0$ (satisfying $\frac{\partial^2 c}{\partial s\partial t}(\beta,s,t)=0$) give homeomorphisms of $\mathbb{T}^1$ and consist of two non-intersecting curves — Σ_P^0 positively oriented and Σ_N^0 negatively oriented with respect to Σ^+. The differential characterization of the cost function is motivated by the non-decreasing or local non-increasing geometry of the optimal transport problem on the real line when the cost is a convex or a concave function of $x-y$ for $x,y\in\mathbb{R}$ through an observation by McCann — see McCann [19] and the references there — where a cost function on $\mathbb{R}$ that mimics the geometry of an optimal solution for a concave cost was characterized in terms of the sign of the mixed partial of the cost. This geometry is also characteristic of the optimal doubly stochastic measures on the unit square with uniform densities for marginals and a variable cost function that changes from convex to concave to convex as in Uckelmann [28], and with a more complex cost in a numerical study by Rüschendorf and Ucklemann [27]. A similar structure appears in a recent study by Plakhov [23] of the Newton's problem regarding the motion with minimal resistance of a (unit volume convex) body through a homogeneous medium of infinitesimal non-interacting particles that collide elastically with the body — the quantity of interest is the average resistance and the change in total energy of the body due to the impacts of the colliding particles. In a reformulation the problem reduces to minimizing the functional $F(\gamma) := \int_{I_0 \times I_0} \big[1 + \cos(\phi + \phi')\big] d\gamma(\phi, \phi)$, for $I_0 := [-\pi/2, \pi/2]$, over the convex set of joint measures on $I_0 \times I_0$ with marginals $\cos\phi d\phi$ — here ϕ, ϕ' represent the angles the incident and the reflected particle-velocities make with the normal to the surface of the body. The support of the minimizer exhibits the characteristic geometry on $I_0 \times I_0$.

The notion of this monotonicity can be adopted to the flat torus through the following definition:

Definition 1.2.1 (monotone subsets of $\mathbb{T}^2$). A subset $G \subset \mathbb{T}^2$ is *non-decreasing* if every triple of points $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in G$ can be reindexed if necessary so that (s_1, s_2, s_3) and (t_1, t_2, t_3) are both oriented positively on $\mathbb{T}^1$. The subset G is *non-increasing* if every triple of points from G can be reindexed so that (s_1, s_2, s_3) is positively oriented on $\mathbb{T}^1$ while (t_1, t_2, t_3) is negatively oriented on $\mathbb{T}^1$.

In the analysis to follow, we show for each $\beta \in [0, 1]$ that γ_o^β is supported in the graphs of two maps

$$t_\beta^\pm : \mathbb{T}^1 \longrightarrow \mathbb{T}^1 \tag{1.7}$$

called the *optimal transport maps*, with $\mathrm{graph}(t_\beta^+) \subset \Sigma^+ \cup \Sigma^0$ a non-decreasing subset of $\mathbb{T}^2$ and $\mathrm{graph}(t_\beta^-) \setminus \mathrm{graph}(t_\beta^+) \subset \Sigma^-$ a locally non-increasing subset. We identify $t_\beta^-(s) = t_\beta^+(s)$ when $(\{s\} \times \mathbb{T}^1) \cap \mathrm{spt}\, \gamma_o^\beta$ is a unique point of $(\Sigma^0 \cup \Sigma^+) \subset \mathbb{T}^2$. This geometry is a consequence of a monotonicity condition enforced by the optimal correlation of points on $\mathrm{spt}\, \gamma_o^\beta$. This is called the *c-cyclical monotonicity* — a notion introduced by Smith and Knott [14] to characterize optimal measures. Denoting the cost function $c : [0, 1] \times \mathbb{T}^1 \times \mathbb{T}^1 \longrightarrow \mathbb{R}$ by

$$c(\beta, s, t) := (1 - \beta)\mathbf{x}(s) \cdot \mathbf{y}(t) + \beta \mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t), \tag{1.8}$$

we define a c-cyclically monotone subset of $\mathbb{T}^2$ as follows:

Definition 1.2.2 (c_β-cyclical monotonicity). Given $\beta \in [0, 1]$ a subset $S \subset \mathbb{T}^2$ is said to be c_β-*cyclically monotone* if for any finite number of points $(s_k, t_k) \in \mathbb{T}^2$, $k =$

$0 \ldots n$ and all permutations α of $(n+1)$-letters,

$$\sum_{k=0}^{n} c(\beta, s_k, t_k) \geq \sum_{k=0}^{n} c(\beta, s_{\alpha(k)}, t_k). \tag{1.9}$$

In other words optimality requires that the points on $\operatorname{spt} \gamma_o^\beta \subset \mathbb{T}^2$ be paired so as to maximize the correlation $\int_{\mathbb{T}^2} c(\beta, s, t)\, d\gamma(s,t)$ — see (1.5). Thus given $\beta \in [0,1]$ a pair of points (s_1, t_1) and (s_2, t_2) on $\operatorname{spt} \gamma_o^\beta$ satisfy

$$c(\beta, s_1, t_1) + c(\beta, s_2, t_2) - c(\beta, s_1, t_2) - c(\beta, s_2, t_1) \geq 0, \tag{1.10}$$

or equivalently

$$\int_{s_1}^{s_2} \int_{t_1}^{t_2} \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t)\, ds\, dt \geq 0, \tag{1.11}$$

where the integrations on the one dimensional tori are over the positively oriented arcs $[\![s_1, s_2]\!]$ and $[\![t_1, t_2]\!]$ of $\mathbb{T}^1$ — a convention that will be followed through the entire analysis. The local geometry of the support in Σ^+ and Σ^- is therefore dictated by the non-negativity constraint in (1.11). For future reference we will call (1.10) the *c_β-monotonicity*, which is a pairwise condition $n = 1$ of (1.9). We further define local monotonicity by:

Definition 1.2.3 (local monotonicity). A set $Z \subset \mathbb{T}^2$ is *non-decreasing* at $(s,t) \in \mathbb{T}^2$ if there exists a neighborhood U of (s,t) such that $Z \cap U$ is non-decreasing. Similarly, $Z \subset \mathbb{T}^2$ is *non-increasing* at (s,t) if there exists a neighborhood U of (s,t) such that $Z \cap U$ is non-increasing.

The key observation upon which our analysis is predicated is summarized in the following lemma that depicts the local structure of c_β-monotone subsets of $\Sigma^\pm$ for any C^2-differentiable cost function $c_\beta : \mathbb{T}^2 \longrightarrow \mathbb{R}$ on the flat torus — for this we recall from (3.12) that Σ^+ and Σ^- are the subsets where the cost if of convex type and concave type respectively. This lemma localizes the differential characterization of cost functions given by McCann [19].

Lemma 1.2.4 (c_β-monotonicity in $\Sigma^\pm$). *Let $c_\beta : \mathbb{T}^2 \longrightarrow \mathbb{R}$ denote a C^2-differentiable function. If $Z \subset \mathbb{T}^2$ is c_β-monotone then $Z \cap \Sigma^+$ is locally non-decreasing while $Z \cap \Sigma^-$ is locally non-increasing.*

Proof. Fix $(s,t) \in \Sigma^+$. Let U be a neighborhood of (s,t) in Σ^+ containing a non-empty rectangle $]\!]s_1, s_2[\![\times]\!]t_1, t_2[\![$. We deduce local non-decreasingness of $Z \cap \Sigma^+$ by showing that the upper-left corner (s_1, t_2) and the lower-right corner (s_2, t_1) cannot both belong to $Z \cap U$. Using the C^2-differentiability and periodicity of the cost function and the fact that the cost is convex type on Σ^+ one gets for (s_1, t_2) and (s_2, t_1):

$$\begin{aligned} & c_\beta(s_1,t_2) + c_\beta(s_2,t_1) - c_\beta(s_1,t_1) - c_\beta(s_2,t_2) \\ = & \int_{s_1}^{s_2} \int_{t_2}^{t_1} \frac{\partial^2 c_\beta}{\partial s \partial t}(s,t)\, ds\, dt \\ = & -\int_{s_1}^{s_2} \int_{t_1}^{t_2} \frac{\partial^2 c_\beta}{\partial s \partial t}(s,t)\, ds\, dt \\ < & \ 0, \end{aligned} \tag{1.12}$$

which contradicts c_β-monotonicity of the points (s_1, t_2) and (s_2, t_1) — thus precluding their simultaneous occurrence in the c_β-monotone subset $Z \cap U$. The second claim can be argued similarly — with the cost concave type on Σ^-. This concludes the proof of the lemma. □

The optimization problem (1.5) is a continuum analog of the linear program

$$\sup \left\{ \sum_{i,j=1}^{n} c_{ij}\gamma_{ij} \mid \sum_{i=1}^{n} \gamma_{ij} = \nu_j, \sum_{j=1}^{n} \gamma_{ij} = \mu_i \right\}, \tag{1.13}$$

where c_{ij} and the vectors $\mu, \nu \in \mathbb{R}^n$ are given, and the problem is to find the optimal $n \times n$ matrix $\gamma_{ij} \geq 0$. Here c_{ij} represents the cost of shipping from $x_i \in \Omega$ to $y_j \in \Lambda$, and the solution can be visualized as a measure $\gamma = \sum_{i,j=1}^{n} \gamma_{ij}\delta_{(x_i,y_j)}$ on the product space $\Omega \times \Lambda$. Its marginals represent the prescribed distributions of production $\mu = \sum_{i=1}^{n} \mu_i$ on Ω and consumption $\nu = \sum_{j=1}^{n} \nu_j$ on Λ, while its support consists of the set of points $\operatorname{spt}\gamma = \{(x_i, y_j) \mid \gamma_{ij} \neq 0\}$. The dual program of this well-known problem, is to find the vectors $u, v \in \mathbb{R}^n$ which minimize

$$\inf \left\{ \sum_{i=1}^{n} u_i\mu_i + \sum_{j=1}^{n} v_j\nu_j \mid u_i + v_j \geq c_{ij} \right\}. \tag{1.14}$$

The Kantorovich duality principle [12] gives the infinite dimensional analog (4.1). For each fixed $\beta \in [0,1]$ the dual problem provides a unique c_β-cyclically monotone subset of $\mathbb{T}^2$ — denoted $\partial_{c_\beta} u^\beta$ (see (4.8) for definition and Propositions-4.2.5 and

4.3.2 for existence and uniqueness) — that contains $\operatorname{spt}\gamma_o^\beta$ for all optimal γ_o^β. The local geometry of $\partial_{c_\beta}u^\beta \cap \Sigma^\pm$ is then dictated by Lemma-1.2.4. One can therefore speculate existence of multiple optimal solutions $\gamma_1 \neq \gamma_2$ illustrated in Figures-1.1(a) and (b) in compliance with Lemma-1.2.4 — refer to Remark-2.0.2 for the symbols on the diagram. The convex combination given by Figure-1.2 would then also be optimal and satisfy Lemma-1.2.4.

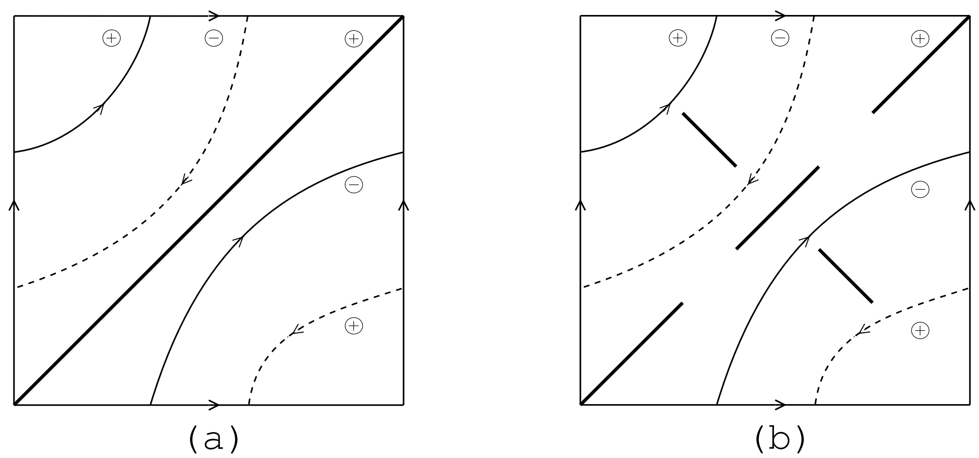

Figure 1.1: (a) $\operatorname{spt}\gamma_1$ and (b) $\operatorname{spt}\gamma_2$ both conform to the local geometry dictated by Lemma-1.2.4 — see Remark-2.0.2 for legend.

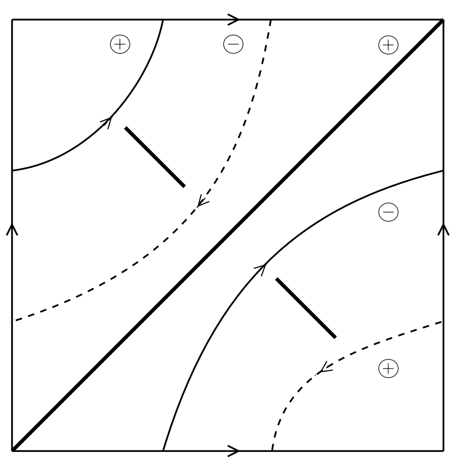

Figure 1.2: the support of the convex combination $(1-t)\gamma_1 + t\,\gamma_2$, $t \in]0,1[$.

Recall that for arc length measures $\mu = \mathcal{H}^1\lfloor_{T^1} = \nu$, the set $\Gamma(\mu,\nu)$ represents the convex set of doubly stochastic measures on the torus, which is a continuous analog of the convex set of doubly stochastic matrices. By Birkhoff [3] the extreme points of the latter set are permutation matrices. One possible continuum analog is given by the graphs of bijective, measure preserving mappings. The study of such extreme doubly stochastic measures on the unit square $I \times I := [0,1] \times [0,1]$ has a vast literature: a functional analytic characterization of these measures has been given by Douglas [6], Lindenstrauss [15], Losert [16] and others. In [15] a conjecture due to Phelps that every such extreme measure is singular with respect to the Lebesgue measure on $I \times I$ has been proven; while in [16] an extreme measure is constructed which is not concentrated on graphs. The solution to our variational problem (1.3)-(1.5) gives an example of an extreme measure concentrated on two graphs. This follows from a counter example due to Gangbo and McCann [10] where a transport problem between two triangles Ω and Λ — reflections of each other and made strictly convex by slight perturbation along the sides — shows that, even for a convex cost $c(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2$ with arc length measures $\tilde{\mu} = \mathcal{H}^1\lfloor_{\partial\Omega}$ and $\tilde{\nu} = \mathcal{H}^1\lfloor_{\partial\Lambda}$, the optimal solution fails to concentrate on the graph of a single map. The same conclusion holds for two isosceles triangles with different side lengths but the same perimeter giving at least two optimal maps [10]. Moreover, each point on the source curve need not necessarily have a unique destination on the target — as evident from the numerical simulations, due to Fry [8], with a convex pentagon evolved optimally by the convex cost $c(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2$ onto a non-convex pentagon. This constitutes a key difference between the optimal transport problem for boundary measures and for measures supported on domain interior. The preferred direction in which the residual mass at any such point flows conforms to a unique geometry that singles out the optimal γ_o^β among all joint measures in $\Gamma(\mu,\nu)$. This geometry can be described as: the μ-mass located at each $s_0 \in \mathbb{T}^1$ is transported under $t_\beta^+ : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ to a primary destination $t_\beta^+(s_0) \in \mathbb{T}^1$ — the excess mass if any at s_0 — i.e. when $\frac{d\mu}{ds}(s_0) > \frac{dt_\beta^+}{ds}(s_0)\,\frac{d\nu}{dt}(t_\beta^+(s_0))$ — then flows to a secondary destination $t_\beta^-(s_0) \in \mathbb{T}^1$ under $t_\beta^- : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ so that if $(s_0, t_\beta^-(s_0))$ is in $\operatorname{spt}\gamma_o^\beta$ then $t_\beta^+(s_0) \in \mathbb{T}^1$ is supplied by s_0 alone. This geometry — proved in Lemma-5.4.4 — is also a consequence of c_β-monotonicity and imposes a global constraint on the c_β-cyclically monotone set $\partial_{c_\beta} u^\beta$ containing $\operatorname{spt}\gamma_o^\beta$ by forbidding the simultaneous occurrence of points satisfying $(s,t) \in \Sigma^+ \cap \partial_{c_\beta} u^\beta$ and $(s,t_1),(s_1,t) \in \Sigma^- \cap \partial_{c_\beta} u^\beta$ — with (s,t) a *convex type* hinge — see Definition-5.4.3. Figure-1.2 therefore represents a forbidden pattern for optimal solutions in this context, because it exhibits convex type hinges. Consequently $\partial_{c_\beta} u^\beta$ can support exactly one solution $\gamma_o^\beta \in \Gamma(\mu, \nu)$ for prescribed μ and ν — making γ_o^β unique. When rotations go unpenalized, this geometry was established for $\beta = 0$ by Gangbo and McCann [10]. The current study consists of finding uniqueness of γ_o^β and investigating the smoothness of the optimal maps in the opposite regime $\beta = 1$, and also when $1 - \epsilon < \beta \leq 1$ and $0 \leq \beta < \epsilon$ for some $\epsilon > 0$. Much of the subtlety of the problem boils down to ruling out convex type hinges in this more general situation. Whether this geometry prevails to achieve uniqueness for arbitrary β is still unresolved. The principal tools in the sequel are dualization by Kantorovich [12] and stability of non-degenerate critical points under small perturbations.

The persistence of uniqueness for values of the control parameter β close to zero or one is proved under an additional hypothesis that $\operatorname{spt} \gamma_o^\beta$ does not intersect the nodal lines Σ^0 of the mixed partial of the cost function when $\beta = 0$ or 1:

$$\frac{\partial^2 c_\beta}{\partial s \partial t}(s,t) \neq 0 \qquad \text{for all } (s,t) \in \operatorname{spt} \gamma_o^\beta \text{ and } \beta = 0, 1. \tag{1.15}$$

We call this hypothesis a *geometrical non-degeneracy condition*, analogous to the non-vanishing of $\frac{\partial^2 f}{\partial x^2}(x_0, 0) \neq 0$ at a local minimum x_0 for $f(x,0)$, that ensures $f(x,\epsilon)$ has a local minimum near x_0 for ϵ small. In higher dimension ($d > 1$) the non-degeneracy condition on the cost function $c : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ gives

$$\det D^2_{xy}\, c \neq 0, \tag{1.16}$$

which plays a role in the uniqueness argument of Ma, Trudinger and Wang [17] concerning solutions of the dual (A.3) to the Kantorovich's optimal transportation problem (A.1).

1.3 Organization of the manuscript

When only rotation is penalized, a proof for the existence of a unique solution for the transport problem (1.5) under strict convexity and C^2-differentiability of the boundaries is presented in Chapter-3 — with an explicit geometry of the support sketched out on $\mathbb{T}^2$ for the toy model (1.3)-(1.4). Chapter-4 gives a characterization of $\operatorname{spt} \gamma_o^\beta$ on $\mathbb{T}^2$ invoking Kantorovich's duality principle: a potential $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ is defined whose differentials determine the destinations on the target for the mass $d\mu(s)$ located in a neighborhood of each $s \in \mathbb{T}^1$. This chapter also includes a compactness result for the space of dual solutions in the topology of uniform convergence. Chapter-5 is a perturbative argument to develop the necessary tools to rule out convex type hinges, thereby proving uniqueness of optimal solutions for values of β close to zero or one. Appendix-A gives a very brief account of the Monge-Kantorovich optimal transportation problem. Appendix-B establishes some differentiability properties of the dual potential u^β.

Chapter 2

Notations and definitions

notation	**meaning**	**definition**
toy model	—	equation (1.3)
$\mathbf{x} : \mathbb{T}^1 \longrightarrow \partial\Omega$	constant speed	equation(1.4)
$\mathbf{y} : \mathbb{T}^1 \longrightarrow \partial\Lambda$	parametrizations	
M_Ω, M_Λ	bounds on the domains	$M_\Omega := \sup_{\mathbf{x}\in\overline{\Omega}} \lvert\mathbf{x}\rvert$, $M_\Lambda := \sup_{\mathbf{y}\in\overline{\Lambda}} \lvert\mathbf{y}\rvert$
v_Ω, v_Λ	constant speeds	$v_\Omega = \left\lvert \frac{d\mathbf{x}(s)}{ds} \right\rvert$, $v_\Lambda = \left\lvert \frac{d\mathbf{y}(t)}{dt} \right\rvert$
$\mathbf{T}_\Omega(s)$, $\mathbf{T}_\Lambda(t)$	tangent vectors to the boundaries $\partial\Omega$, $\partial\Lambda$	$\dot{\mathbf{x}}(s) = v_\Omega \mathbf{T}_\Omega(s)$, $\dot{\mathbf{y}}(t) = v_\Lambda \mathbf{T}_\Lambda(t)$
$K_\Omega(s)$, $K_\Lambda(t)$	curvatures	$\dot{\mathbf{T}}_\Omega(s) = -v_\Omega\, K_\Omega(s)\mathbf{n}_\Omega(s)$ $\dot{\mathbf{T}}_\Lambda(t) = -v_\Lambda\, K_\Lambda(t)\mathbf{n}_\Lambda(t)$
$\mathbf{n}_\Omega(s)$, $\mathbf{n}_\Lambda(t)$	unit outward normals to the boundaries $\partial\Omega$, $\partial\Lambda$	$\ddot{\mathbf{x}}(s) = -{v_\Omega}^2 K_\Omega(s)\mathbf{n}_\Omega(s)$, $\ddot{\mathbf{y}}(t) = -{v_\Lambda}^2 K_\Lambda(t)\mathbf{n}_\Lambda(t)$
$\lVert K_\Omega\rVert_\infty$, $\lVert K_\Lambda\rVert_\infty$	bounds on curvatures	$\lVert K_\Omega\rVert_\infty := \sup_{s\in\mathbb{T}^1} \lvert K_\Omega(s)\rvert$, $\lVert K_\Lambda\rVert_\infty := \sup_{t\in\mathbb{T}^1} \lvert K_\Lambda(t)\rvert$
(s,t)	points on $\mathbb{T}^1 \times \mathbb{T}^1$	
$\partial_{c_\beta} u$	c_β-subdifferential of $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$	Definition-4.1.3
$\partial.u$	subdifferential of convex $u : \mathbb{R} \longrightarrow \mathbb{R}$	equation (4.13)

notation	**meaning**	**definition**
u'_- and u'_+	left and right derivatives of u	—
id	the identity map	$id : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$
$]\!]s_1, s_2[\![$, $[\![s_1, s_2]\!]$	positively oriented open and closed arcs of $\mathbb{T}^1$ from s_1 to s_2	—
$\lvert[\![s_1, s_2]\!]\rvert$	length of the positively oriented arc $[\![s_1, s_2]\!] \subset \mathbb{T}^1$	$\lvert[\![s_1, s_2]\!]\rvert := \left\lvert\int_{s_1}^{s_2} ds\right\rvert$
$\operatorname{spt}\mu$	support of a Borel measure μ on $\mathbb{T}^1$	smallest closed subset $K \subset \mathbb{T}^1$ with $\mu[\mathbb{T}^1 \backslash K] = 0$
$\Gamma(\mu, \nu)$	set of joint measures with μ and ν for marginals	$\gamma \in \Gamma(\mu, \nu) \implies$ $\gamma[B \times \mathbb{T}^1] = \mu[B]$ $\gamma[\mathbb{T}^1 \times B] = \nu[B]$ for all Borel $B \subset \mathbb{T}^1$
$g_\# \mu$	push forward of the measure μ by the map $g : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$	$g_\#\mu[B] = \mu[g^{-1}(B)]$ for all Borel $B \subset \mathbb{T}^1$

Lemma 2.0.1 (change of variable formula). *For a Borel measure ρ on $\mathbb{R}^d$ if the map $\mathbf{u} : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ pushes ρ forward to $\mathbf{u}_\#\rho$ then the change of variable theorem states: for every Borel function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$,*

$$\int_{\mathbb{R}^n} h \, d\mathbf{u}_\#\rho = \int_{\mathbb{R}^d} h \circ \mathbf{u} \, d\rho, \tag{2.1}$$

meaning both integrals exist and are equal if either integral can be defined.

Remark 2.0.2 (a remark on the schematics). In all the diagrams of the flat torus $\mathbb{T}^2$, the disjoint components Σ^+ and Σ^- — defined by (3.12) — are represented by $\oplus$ and $\ominus$ respectively; while the solid curve is Σ^0_P and the dashed curve is Σ^0_N which represent their boundaries $\Sigma^0 := \partial\Sigma^+ = \partial\Sigma^-$ oriented with respect to Σ^+ — see (3.21). The heavy solid curves, when present, are used to indicate the hypothetical support of a measure on $\mathbb{T}^2$.

Chapter 3

Purely rotational cost

3.1 Existence and uniqueness of solution

Consider the variational problem (1.2). When $\beta = 0$ the cost is purely translational and solving (1.2) is equivalent to computing the Wasserstein L^2 metric between measures $\tilde{\mu}$ and $\tilde{\nu}$ supported on the domain boundaries. The optimal solution $\tilde{\gamma}_o^0$ — the superscript standing for $\beta = 0$ — is unique provided $\Omega \subset \mathbb{R}^2$ is strictly convex and $\tilde{\mu}$ is absolutely continuous with respect to the boundary measure $\mathcal{H}^1\lfloor_{\partial\Omega}$. This uniqueness result has been proven by Gangbo and McCann in [10] where they also established the structure of the corresponding optimal transport maps $\tilde{\mathbf{t}}_0^{\pm} : \partial\Omega \longrightarrow \partial\Lambda$ (the subscript zero for $\beta = 0$) under strict convexity of both the domains Ω and Λ.

In this section we prove uniqueness of the optimal solution $\tilde{\gamma}_o^1$ for (1.2) when $\beta = 1$ and study the geometric properties of its support. By a suitable choice of coordinates we first reduce the $\beta = 1$ case into a special $\beta = 0$ problem — which is a Monge-Kantorovich optimal transport problem between measures supported on the gauss circles corresponding to the domain boundaries. Uniqueness is then derived exploiting the results of Gangbo and McCann [10]. We therefore invoke the notion of suitable measures introduced in [10] and borrow the following definitions:

Definition 3.1.1 (generalized normals and gauss maps). For the convex set $\Omega \subset \mathbb{R}^2$ we say that $\mathbf{n} \in \mathbb{R}^2$ is a *generalized outward normal* to $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$ if $0 \geq \mathbf{n} \cdot (\mathbf{z} - \mathbf{x})$ for all $\mathbf{z} \in \overline{\Omega}$. Thus the generalized outward normal coincides with the classical normal at points where $\partial\Omega$ is differentiable. The convexity of Ω yields a non-zero normal at every point of $\partial\Omega$. The set $N_\Omega(\mathbf{x})$ of *unit outward normals* at $\mathbf{x} \in \partial\Omega$ (which is non-empty) then consists of a single element, denoted $\mathbf{n}_\Omega(\mathbf{x})$, if and only if $\mathbf{x}$ is a point of differentiability for $\partial\Omega$. We use this unit outward normal to define the *gauss map* $\tilde{\mathbf{n}}_\Omega : \partial\Omega \longrightarrow \mathbf{S}^1$ from the domain boundary $\partial\Omega$ into the unit circle $\mathbf{S}^1$ in $\mathbb{R}^2$. And similarly $\tilde{\mathbf{n}}_\Lambda : \partial\Lambda \longrightarrow \mathbf{S}^1$ for $\partial\Lambda$.

Definition 3.1.2 (suitable boundary measures). A pair of measures (μ, ν) on $\mathbb{R}^2$ is said to be *suitable* if there exists bounded strictly convex domains Ω and $\Lambda \subset \mathbb{R}^2$ such that (i) $\operatorname{spt}\mu \subset \partial\Omega$ and, (ii) $\operatorname{spt}\nu = \partial\Lambda$ and (iii) μ has no atoms. If also (ν, μ) is suitable then the pair (μ, ν) is called *symmetrically suitable.*

notations: In the following analysis we denote by $\mathbf{S}^1$ the gauss circles generated by the gauss maps $\tilde{\mathbf{n}}_\Omega : \partial\Omega \longrightarrow \mathbf{S}^1$ and $\tilde{\mathbf{n}}_\Lambda : \partial\Omega \longrightarrow \mathbf{S}^1$ and adopt the notation whereby points $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} \in \partial\Lambda$ on the domain boundaries are represented under gauss maps by $\hat{\mathbf{x}} := \tilde{\mathbf{n}}_\Omega(\mathbf{x})$ and $\hat{\mathbf{y}} := \tilde{\mathbf{n}}_\Lambda(\mathbf{y})$. Also any mapping defined on $\partial\Omega$ (or $\partial\Lambda$) will be denoted by tildes while the corresponding quantities defined on the respective gauss circles by hats. The subscript 1 on any symbol indicates restriction to $\beta = 1$ — while in $\tilde{\gamma}_o^1$ the superscript stands for $\beta = 1$ and the subscript o for optimal.

Theorem 3.1.3 (uniqueness of optimal $\tilde{\gamma}$ for $\beta = 1$). *Let $\Omega, \Lambda \subset \mathbb{R}^2$ be bounded strictly convex domains in $\mathbb{R}^2$ with C^2 boundaries and symmetrically suitable measures $\tilde{\mu}$ on $\partial\Omega$ and $\tilde{\nu}$ on $\partial\Lambda$. Denoting the outward unit normals to $\partial\Omega$ and $\partial\Lambda$ by $\tilde{\mathbf{n}}_\Omega$ and $\tilde{\mathbf{n}}_\Lambda$, rewrite (1.2) for $\beta = 1$ as*

$$\inf_{\tilde{\gamma} \in \Gamma(\tilde{\mu}, \tilde{\nu})} \int_{\partial\Omega \times \partial\Lambda} |\tilde{\mathbf{n}}_\Omega(\mathbf{x}) - \tilde{\mathbf{n}}_\Lambda(\mathbf{y})|^2 \, d\tilde{\gamma}(\mathbf{x}, \mathbf{y}) \,. \tag{3.1}$$

Then there exists a unique joint measure $\tilde{\gamma} \in \Gamma(\tilde{\mu}, \tilde{\nu})$, denoted $\tilde{\gamma}_o^1$, for which the minimum is achieved.

Proof. Because $\partial\Omega$ and $\partial\Lambda$ are C^2 smooth and Ω, Λ are strictly convex the gauss maps $\tilde{\mathbf{n}}_\Omega : \partial\Omega \longrightarrow \mathbf{S}^1$ and $\tilde{\mathbf{n}}_\Lambda : \partial\Lambda \longrightarrow \mathbf{S}^1$ from the boundaries into the unit circles $\mathbf{S}^1$ are homeomorphisms.

Claim: The push forward measures $\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}$ and $\tilde{\mathbf{n}}_{\Lambda\#}\tilde{\nu}$, obtained by pushing $\tilde{\mu}$ and $\tilde{\nu}$ to the unit circles through the gauss maps, are symmetrically suitable measures on $\mathbf{S}^1$.

Proof of Claim: The proof consists in showing by contradiction that (i) $\operatorname{spt}\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu} = \mathbf{S}^1$ and (ii) $\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}$ has no atoms and using the symmetry under the interchange $(\partial\Omega, \tilde{\mu}) \leftrightarrow (\partial\Lambda, \tilde{\nu})$ to conclude the same for $\tilde{\mathbf{n}}_{\Lambda\#}\tilde{\nu}$.

(i). Assume $\operatorname{spt}\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu} \subset \mathbf{S}^1$ with strict containment. Then there exists $\hat{\mathbf{x}}_0 \in \mathbf{S}^1$ such that $\hat{\mathbf{x}}_0 \notin \operatorname{spt}\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}$. This implies there exists some open arc $\widehat{A} \subset \mathbf{S}^1$ containing the point $\hat{\mathbf{x}}_0$ with $\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}\,[\,\widehat{A}\,] = 0$. Then

$$0 = \tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}\,[\widehat{A}\,] = \tilde{\mu}\,[\tilde{\mathbf{n}}_{\Omega}^{-1}(\widehat{A}\,)] \geq \tilde{\mu}\,[A] > 0\,,$$

where A is some open arc of $\partial\Omega$ containing the point $\mathbf{x}_0 \in \partial\Omega$ such that $\tilde{\mathbf{n}}_{\Omega}(\mathbf{x}_0) = \hat{\mathbf{x}}_0 \in \mathbf{S}^1$ and $\tilde{\mathbf{n}}_{\Omega}(A) \subseteq \widehat{A}$. Since $(\tilde{\mu}, \tilde{\nu})$ are symmetrically suitable $\operatorname{spt}\tilde{\mu} = \partial\Omega$ — which gives the above strict inequality.

(ii). Assume $\tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}$ has an atom at $\hat{\mathbf{x}}_0 \in \mathbf{S}^1$. Then

$$0 < \tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}[\{\hat{\mathbf{x}}_0\}] = \tilde{\mu}[\{\tilde{\mathbf{n}}_{\Omega}^{-1}(\hat{\mathbf{x}}_0)\}] = \tilde{\mu}[\{\mathbf{x}_0\}] = 0\,,$$

where $\mathbf{x}_0 \in \partial\Omega$ such that $\tilde{\mathbf{n}}_{\Omega}(\mathbf{x}_0) = \hat{\mathbf{x}}_0$. The last equality follows from the fact that since $\tilde{\mathbf{n}}_{\Omega}$ is a homeomorphism, the subset $\{\tilde{\mathbf{n}}_{\Omega}^{-1}(\hat{\mathbf{x}}_0)\} = \{\mathbf{x}_0\} \subset \partial\Omega$ is a singleton set and by the hypothesis of symmetrically suitable $\tilde{\mu}$ has no atom.

The contradictions in (i) and (ii) and the symmetry under $(\partial\Omega, \tilde{\mu}) \leftrightarrow (\partial\Lambda, \tilde{\nu})$ then confirm the claim.

back to the proof of the theorem: denote the pushed forward measures on the gauss circles by $\hat{\mu} := \tilde{\mathbf{n}}_{\Omega\#}\tilde{\mu}$ and $\hat{\nu} := \tilde{\mathbf{n}}_{\Lambda\#}\tilde{\nu}$. Then $\hat{\gamma} := (\tilde{\mathbf{n}}_{\Omega} \times \tilde{\mathbf{n}}_{\Lambda})_{\#}\tilde{\gamma}$ is a joint measure on $\mathbf{S}^1 \times \mathbf{S}^1$ with marginals $\hat{\mu}$ and $\hat{\nu}$. Using the change of variable formula

(2.1) one can check that solving (3.1) is equivalent to finding

$$\inf_{\hat{\gamma} \in \Gamma(\hat{\mu}, \hat{\nu})} \int_{\mathbf{S}^1 \times \mathbf{S}^1} |\hat{\mathbf{x}} - \hat{\mathbf{y}}|^2 \, d\hat{\gamma}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \,. \tag{3.2}$$

Now, $\operatorname{conv}(\mathbf{S}^1) \subset \mathbb{R}^2$ is a strictly convex, bounded domain on the plane and by the claim $(\hat{\mu}, \, \hat{\nu})$ are symmetrically suitable measures on $\mathbf{S}^1 \times \mathbf{S}^1$. The Uniqueness Theorem-2.6 (and the regularity results [section-**3**]) of Gangbo and McCann [10] then asserts that the optimization problem (3.2) has a unique minimizer thus ensuring a unique solution $\tilde{\gamma}_o^1$ for (3.1) through the inverse gauss maps. $\square$

Proposition 3.1.4 (optimal maps and smoothness). *Let Ω, Λ denote strictly convex bounded planar domains with C^2-smooth boundaries and symmetrically suitable measures $\tilde{\mu}$ on $\partial\Omega$ and $\tilde{\nu}$ on $\partial\Lambda$. Then there exist two maps $\tilde{\mathbf{t}}_1^{\pm} : \partial\Omega \longrightarrow \partial\Lambda$ whose graphs contain the support of the unique optimal solution $\tilde{\gamma}_o^1$ of Theorem-3.1.3 for (3.1) with*

$$\{(\mathbf{x}, \tilde{\mathbf{t}}_1^+(\mathbf{x}))\}_{\mathbf{x} \in \partial\Omega} \subset \operatorname{spt} \tilde{\gamma}_o^1 \subset \{(\mathbf{x}, \tilde{\mathbf{t}}_1^+(\mathbf{x}))\}_{\mathbf{x} \in \partial\Omega} \cup \{(\mathbf{x}, \tilde{\mathbf{t}}_1^-(\mathbf{x}))\}_{\mathbf{x} \in \widetilde{S}^2} \,, \tag{3.3}$$

where $\widetilde{S}^2 \subset \partial\Omega$ is the subset of $\partial\Omega$ on which $(\{\mathbf{x}\} \times \partial\Lambda) \cap \operatorname{spt} \tilde{\gamma}_o^1$ may consist of more than one point. Moreover, the map $\tilde{\mathbf{t}}_1^+ : \partial\Omega \longrightarrow \partial\Lambda$ is a homeomorphism while the restriction of $\tilde{\mathbf{t}}_1^- : \partial\Omega \longrightarrow \partial\Lambda$ to $\widetilde{S}^2$ is continuous with continuous inverse.

Proof. We recall from the proof of Theorem-3.1.3 that by a change of variable under the gauss maps $\tilde{\mathbf{n}}_\Omega$ and $\tilde{\mathbf{n}}_\Lambda$, problem (3.1) can be rewritten as (3.2); the latter represents the $\beta = 1$ case of the original problem (1.2) reduced to a $\beta = 0$ problem on gauss circles. Theorems-2.6 and -3.8 of Gangbo and McCann [10] then yield two unique optimal maps $\hat{\mathbf{t}}_1^{\pm} : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ that are characterized by the inequalities

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{t}}_1^+(\hat{\mathbf{x}}) \geq 0 \quad \text{and} \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{t}}_1^-(\hat{\mathbf{x}}) \leq 0, \tag{3.4}$$

with $\hat{\mathbf{t}}_1^+ = \hat{\mathbf{t}}_1^-$ identified except on a subset $\widehat{S}^2 \subset \mathbf{S}^1$ where the $\hat{\mu}$-mass can split as it is pushed forward through $\tilde{\mathbf{n}}_\Omega$ — i.e. the subset where $(\{\hat{\mathbf{x}}\} \times \mathbf{S}^1) \cap \operatorname{spt} \hat{\gamma}_o$ may fail to be a singleton set of $\mathbf{S}^1 \times \mathbf{S}^1$. Moreover, the graphs of these maps cover the support $\operatorname{spt} \hat{\gamma}_o$ of the unique optimal solution $\hat{\gamma}_o$ for (3.2), with

$$\{(\hat{\mathbf{x}}, \hat{\mathbf{t}}_1^+(\hat{\mathbf{x}}))\}_{\hat{\mathbf{x}} \in \mathbf{S}^1} \subset \operatorname{spt} \hat{\gamma}_o \subset \{(\hat{\mathbf{x}}, \hat{\mathbf{t}}_1^+(\hat{\mathbf{x}}))\}_{\hat{\mathbf{x}} \in \mathbf{S}^1} \cup \{(\hat{\mathbf{x}}, \hat{\mathbf{t}}_1^-(\hat{\mathbf{x}}))\}_{\hat{\mathbf{x}} \in \widehat{S}^2 \subset \mathbf{S}^1}. \tag{3.5}$$

One can use the symmetry $(\partial\Omega, \tilde{\mu}) \leftrightarrow (\partial\Lambda, \tilde{\nu})$ to similarly define the inverse optimal maps $\hat{\mathbf{s}}_1^{\pm} : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$. Identifying the points on the gauss circles with the outward unit normals, i.e. $\hat{\mathbf{n}}(\hat{\mathbf{x}}) = \hat{\mathbf{x}}$ and $\hat{\mathbf{n}}(\hat{\mathbf{y}}) = \hat{\mathbf{y}}$, we follow Gangbo and McCann [10] to define the decompositions $\mathbf{S}^1 = \tilde{\mathbf{n}}_\Omega(\partial\Omega) = \widehat{S}^0 \cup \widehat{S}^1 \cup \widehat{S}^2$:

$$
\begin{array}{rrlcl}
\text{(o).} & \widehat{S}^0 := & \{\hat{\mathbf{x}} \in \mathbf{S}^1 & | & \hat{\mathbf{t}}_1^+(\hat{\mathbf{x}}) = \hat{\mathbf{t}}_1^-(\hat{\mathbf{x}}) \text{ with } \hat{\mathbf{n}}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{t}}^+(\hat{\mathbf{x}})) = 0\}, \\
\text{(i).} & \widehat{S}^1 := & \{\hat{\mathbf{x}} \in \mathbf{S}^1 & | & \hat{\mathbf{t}}_1^+(\hat{\mathbf{x}}) = \hat{\mathbf{t}}_1^-(\hat{\mathbf{x}}) \text{ with } \hat{\mathbf{n}}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{t}}_1^+(\hat{\mathbf{x}})) > 0\}, \\
\text{(ii).} & \widehat{S}^2 := & \{\hat{\mathbf{x}} \in \mathbf{S}^1 & | & \hat{\mathbf{t}}_1^+(\hat{\mathbf{x}}) \neq \hat{\mathbf{t}}_1^-(\hat{\mathbf{x}}) \text{ with } \hat{\mathbf{n}}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{t}}_1^+(\hat{\mathbf{x}})) > 0 \\
& & & & \quad \text{and } \hat{\mathbf{n}}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{t}}_1^-(\hat{\mathbf{x}})) < 0\},
\end{array}
\tag{3.6}
$$

and $\mathbf{S}^1 = \tilde{\mathbf{n}}_\Lambda(\partial\Lambda) = \widehat{T}^0 \cup \widehat{T}^1 \cup \widehat{T}^2$:

$$
\begin{array}{rrlcl}
\text{(iv).} & \widehat{T}^0 := & \{\hat{\mathbf{y}} \in \mathbf{S}^1 & | & \hat{\mathbf{s}}_1^+(\hat{\mathbf{y}}) = \hat{\mathbf{s}}_1^-(\hat{\mathbf{y}}) \text{ with } \hat{\mathbf{n}}(\hat{\mathbf{y}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{s}}^+(\hat{\mathbf{y}})) = 0\}, \\
\text{(v).} & \widehat{T}^1 := & \{\hat{\mathbf{y}} \in \mathbf{S}^1 & | & \hat{\mathbf{s}}_1^+(\hat{\mathbf{y}}) = \hat{\mathbf{s}}_1^-(\hat{\mathbf{y}}) \text{ with } \hat{\mathbf{n}}(\hat{\mathbf{y}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{s}}_1^+(\hat{\mathbf{y}})) > 0\}, \\
\text{(vi).} & \widehat{T}^2 := & \{\hat{\mathbf{y}} \in \mathbf{S}^1 & | & \hat{\mathbf{s}}_1^+(\hat{\mathbf{y}}) \neq \hat{\mathbf{s}}_1^-(\hat{\mathbf{y}}) \text{ with } \hat{\mathbf{n}}(\hat{\mathbf{y}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{s}}_1^+(\hat{\mathbf{y}})) > 0 \\
& & & & \quad \text{and } \hat{\mathbf{n}}(\hat{\mathbf{y}}) \cdot \hat{\mathbf{n}}(\hat{\mathbf{s}}_1^-(\hat{\mathbf{y}})) < 0\}.
\end{array}
\tag{3.7}
$$

The continuity results in Proposition-3.7 of Gangbo and McCann [10] then assert that

(vii). $\hat{\mathbf{t}}_1^+ : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ is a homeomorphism with $(\hat{\mathbf{t}}_1^+)^{-1} = \hat{\mathbf{s}}_1^+$; and
(viii). $\hat{\mathbf{t}}_1^- : \widehat{S}^2 \longrightarrow \widehat{T}^2$ is a homeomorphism with inverse map $\hat{\mathbf{s}}_1^- \lfloor_{\widehat{T}^2}$.

Use the gauss maps to define the transport maps $\tilde{\mathbf{t}}_1^{\pm} := \tilde{\mathbf{n}}_\Lambda^{-1} \circ \hat{\mathbf{t}}_1^{\pm} \circ \tilde{\mathbf{n}}_\Omega : \partial\Omega \longrightarrow \partial\Lambda$ and $\tilde{\mathbf{s}}_1^{\pm} := \tilde{\mathbf{n}}_\Omega^{-1} \circ \hat{\mathbf{s}}_1^{\pm} \circ \tilde{\mathbf{n}}_\Lambda : \partial\Lambda \longrightarrow \partial\Omega$. Replacing the hats by tildes in (o) through (vi) we can define similar decompositions $\partial\Omega = \widetilde{S}^0 \cup \widetilde{S}^1 \cup \widetilde{S}^2$ and $\partial\Lambda = \widetilde{T}^0 \cup \widetilde{T}^1 \cup \widetilde{T}^2$ of the domain boundaries as their respective gauss circles since $\hat{\mathbf{n}}(\hat{\mathbf{x}}) = \tilde{\mathbf{n}}_\Omega(\mathbf{x})$ and $\hat{\mathbf{n}}(\hat{\mathbf{y}}) = \tilde{\mathbf{n}}_\Lambda(\mathbf{y})$. Strict convexity and C^2-smoothness of $\partial\Omega$ and $\partial\Lambda$ make the gauss maps diffeomorphisms. This enables one to extend the above conclusions (3.5), (vii) and (viii) for the maps $\hat{\mathbf{t}}_1^{\pm}$ and $\hat{\mathbf{s}}_1^{\pm}$ on the gauss circles to the corresponding maps $\tilde{\mathbf{t}}_1^{\pm} : \partial\Omega \longrightarrow \partial\Lambda$ and $\tilde{\mathbf{s}}_1^{\pm} : \partial\Lambda \longrightarrow \partial\Omega$ between the domain boundaries to complete the proof of the proposition. □

3.2 Geometry of optimal solution

Remark 3.2.1 (optimal joint measure). From the Uniqueness Theorem-2.6 of Gangbo and McCann [10] one can readily read off the unique optimal solution $\tilde{\gamma}_o^1$ when $\beta = 1$ as $\tilde{\gamma}_o^1 = \tilde{\gamma}_{o1}^1 + \tilde{\gamma}_{o2}^1$, where

$$\tilde{\gamma}_{o1}^1 = (\tilde{\mathbf{s}}_1^+ \times \widetilde{\mathbf{id}})_\# \, \tilde{\nu}_1 \quad \text{and} \quad \tilde{\gamma}_{o2}^1 = (\widetilde{\mathbf{id}} \times \tilde{\mathbf{t}}_1^-)_\# \, \tilde{\mu}_2 \,, \tag{3.8}$$

with $\tilde{\mathbf{t}}_1^\pm : \partial\Omega \longrightarrow \partial\Lambda$ and $\tilde{\mathbf{s}}_1^\pm : \partial\Lambda \longrightarrow \partial\Omega$ the optimal maps defined in Proposition-3.1.4 and $\widetilde{\mathbf{id}}$ the identity maps on $\partial\Omega$ and $\partial\Lambda$. The measure $\tilde{\nu}_1 := \tilde{\nu}\lfloor_{\widetilde{T}^0 \cup \widetilde{T}^1}$ in (3.8) is the restriction of $\tilde{\nu}$ to the subset $\widetilde{T}^0 \cup \widetilde{T}^1$ of $\partial\Lambda$, where the optimal maps $\tilde{\mathbf{s}}_1^+ = \tilde{\mathbf{s}}_1^-$ are equal with $\mathbf{x} = \tilde{\mathbf{s}}_1^+(\mathbf{y}) = \tilde{\mathbf{s}}_1^-(\mathbf{y})$ satisfying $\tilde{\mathbf{n}}_\Omega(\mathbf{x}) \cdot \tilde{\mathbf{n}}_\Lambda(\mathbf{y}) \geq 0$; while $\tilde{\mu}_2 := \tilde{\mu} - \tilde{\mu}_1$ for $\tilde{\mu}_1 := \tilde{\mathbf{s}}_{1\,\#}^+ \tilde{\nu}_1$. Furthermore, by Proposition-3.2 of [10] when $\mathbf{x}_0$ belongs to the subset $\widetilde{S}^2 \subset \partial\Omega$ one has

$$\tilde{\mathbf{n}}_\Omega(\mathbf{x}_0) = \lambda \, [\tilde{\mathbf{n}}_\Lambda(\tilde{\mathbf{t}}_1^+(\mathbf{x}_0)) - \tilde{\mathbf{n}}_\Lambda(\tilde{\mathbf{t}}_1^-(\mathbf{x}_0))]$$

for some $0 < \lambda \in \mathbb{R}$ making the gauss images of the points $\tilde{\mathbf{t}}_1^+(\mathbf{x}_0) \neq \tilde{\mathbf{t}}_1^-(\mathbf{x}_0) \in \partial\Lambda$ lie on a line parallel the normal $\tilde{\mathbf{n}}_\Omega(\mathbf{x}_0)$ to $\partial\Omega$ at $\mathbf{x}_0$.

However the geometry of $\operatorname{spt} \tilde{\gamma}_o^1$ becomes more comprehensible when it is portrayed on the product of the parameter spaces for the constant speed parametrizations of $\partial\Omega$ and $\partial\Lambda$. For that we resort to the slightly more restrictive assumptions of the toy model (1.3) and the constant speed parametrizations (1.4) that were introduced in the introduction. Notice that the above theorem and proposition regarding uniqueness of the optimal solution and smoothness of the transport maps hold equally for the toy model in which case they will be denoted by $\gamma_o^1 \in \Gamma(\mu, \nu)$ and $t_1^\pm : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ respectively — the superscript on γ_o^1 and the subscript on $t_1^\pm$ emphasizing $\beta = 1$ (see (1.6) and (1.7) above) — with the transport maps characterized by

$$\begin{aligned} \mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t_1^+(s)) &\geq 0 \\ \mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t_1^-(s)) &\leq 0, \end{aligned} \tag{3.9}$$

with t_1^+ and t_1^- identified outside the subset $S_2(\beta = 1) \subset \mathbb{T}^1$ where $(\{s\} \times \mathbb{T}^1) \cap \operatorname{spt} \gamma_o^1$ may fail to be a singleton set — cf. (ii) in (3.6). Before exploring the geometry of $\operatorname{spt} \gamma_o^1$ on $\mathbb{T}^2$, we give a characterization of the cost function (1.8) on the flat torus into two distinct classes which we call the *convex type* and the *concave type*; this characterization is enforced by the geometric constraints (1.3) of the toy model, the optimal correlation (1.9) of points on spt γ_o^β and periodicity of functions on torus.

geometric characterization of the cost: It follows from (1.11) that the geometry of the optimal solution on $\mathbb{T}^2$ is dictated — through c_β-cyclical monotonicity (1.9) — by the sign of the mixed partial of the cost function:

$$\frac{\partial^2 c}{\partial s \partial t} = [(1-\beta) + \beta K_\Omega(s) K_\Lambda(t)]\, v_\Omega\, v_\Lambda\, \mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t), \tag{3.10}$$

where we have used $\mathbf{T}_\Omega(s) \cdot \mathbf{T}_\Lambda(t) = \mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t)$ — for the other notations we refer to Chapter-2. By (1.3) the quantity in the square bracket is strictly positive — the sign of the mixed partial is therefore given by that of the dot product of the outward unit normals. For each fixed $s \in \mathbb{T}^1$, strict convexity of Λ forces the dot product to change sign twice on $\mathbb{T}^1$ giving a decomposition of $\mathbb{T}^2$ — independent of β — into three disjoint subsets:

$$\mathbb{T}^2 = \Sigma^+ \cup \Sigma^0 \cup \Sigma^-, \tag{3.11}$$

with

$$\begin{array}{rcl} \Sigma^+ & := & \{(s,t) \in \mathbb{T}^2 \mid \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t) \;>\; 0\}, \\ \Sigma^0 & := & \{(s,t) \in \mathbb{T}^2 \mid \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t) \;=\; 0\}, \\ \Sigma^- & := & \{(s,t) \in \mathbb{T}^2 \mid \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t) \;<\; 0\}. \end{array} \tag{3.12}$$

We denote

$$\Sigma^k(s) := \{t \in \mathbb{T}^1 \mid (s,t) \in \Sigma^k\} \quad \text{for} \quad k = +, 0, -\,. \tag{3.13}$$

Accordingly we define:

Definition 3.2.2 (convex type vs. concave type). A C^2-differentiable function $c : \mathbb{T}^2 \longrightarrow \mathbb{R}$ is said to be of *convex type* if its mixed partial is non-negative, i.e. $\frac{\partial^2 c}{\partial s \partial t}(s,t) \geq 0$. The function is of *concave type* if it satisfies $\frac{\partial^2 c}{\partial s \partial t}(s,t) \leq 0$.

For each fixed $\beta \in [0,1]$, this makes the cost function $c(\beta, s, t)$ of (1.8) convex type on $\Sigma^+ \cup \Sigma^0$ and concave type on $\Sigma^- \cup \Sigma^0$ so that the graphs of the optimal maps $t_1^\pm : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ (covering spt γ_o^1) are contained in the subsets $\Sigma^+ \cup \Sigma^0$ and $\Sigma^- \cup \Sigma^0$ of $\mathbb{T}^2$ respectively — compare (3.9) with (3.10), (3.12). Proposition-3.2.5 explains the rationale for this classification by exploring the geometry of these graphs and hence of $\operatorname{spt}\gamma_o^1$ on $\mathbb{T}^2$. We further note that when the optimal solution is a minimizer instead of a maximizer, the inequalities in Definition-3.2.2 will reverse due to the reversal of the inequality defining c_β-cyclical monotonicity (1.9).

It is convenient at this point to introduce the *angular parametrization* or the *inverse gauss parametrization* of $\partial\Omega$ and $\partial\Lambda$ for the toy model and their respective gauss circles $\tilde{\mathbf{n}}_\Omega(\partial\Omega) = \mathbf{S}^1$ and $\tilde{\mathbf{n}}_\Lambda(\partial\Lambda) = \mathbf{S}^1$:

Definition 3.2.3 (angular parametrization). Let ϕ (or θ) — called the *angular parameter* — denote points on $[0, 2\pi] \equiv \mathbb{R}/2\pi\mathbb{Z} \equiv \mathbf{T}^1$ parametrizing the gauss circle $\mathbf{S}^1$ so that $\hat{\mathbf{n}}(\phi) := (\cos\phi, \sin\phi) \in \mathbf{S}^1$. Under this parametrization the points on the domain boundaries, $\partial\Omega$ and $\partial\Lambda$, can be represented by

$$\mathbf{x}(\phi) \in \arg\max_{\mathbf{x}\in\partial\Omega} \mathbf{x}\cdot\hat{\mathbf{n}}(\phi) \quad \text{and} \quad \mathbf{y}(\theta) \in \arg\max_{\mathbf{y}\in\partial\Lambda} \mathbf{y}\cdot\hat{\mathbf{n}}(\theta). \tag{3.14}$$

One can check using Definition-3.1.1 that $\tilde{\mathbf{n}}_\Omega(\mathbf{x}(\phi)) = \hat{\mathbf{n}}(\phi)$ and $\tilde{\mathbf{n}}_\Lambda(\mathbf{y}(\theta)) = \hat{\mathbf{n}}(\theta)$ — giving a one to one correspondence between the constant speed parameters $(s,t) \in \mathbb{T}^1 \times \mathbb{T}^1 =: \mathbb{T}^2$ and the angular parameters $(\phi, \theta) \in \mathbf{T}^1 \times \mathbf{T}^1 =: \mathbf{T}^2$ for $\partial\Omega$ and $\partial\Lambda$ of the toy model (1.3).

By a change of variable the corresponding cost function $\bar{c}(\beta, \phi, \theta)$ on $[0,1] \times \mathbf{T}^2$ is defined according to the formula

$$\bar{c}(\beta, \phi, \theta) = (1-\beta)\mathbf{x}(\phi)\cdot\mathbf{y}(\phi) + \beta\cos(\phi - \theta) \tag{3.15}$$

where we wrote (using the above definition) $\tilde{\mathbf{n}}_\Omega(\mathbf{x}(\phi)) \cdot \tilde{\mathbf{n}}_\Lambda(\mathbf{y}(\theta)) = \cos(\phi - \theta)$ in the term multiplying β. For $\beta = 1$ the mixed partial of the cost function with respect to the angular parameters is given by

$$\frac{\partial^2 \bar{c}}{\partial\phi\partial\theta}(1, \phi, \theta) = \cos(\phi - \theta), \tag{3.16}$$

whose sign gives a similar decomposition $\mathbf{T}^2 = \bar{\Sigma}^+ \cup \bar{\Sigma}^0 \cup \bar{\Sigma}^-$ as (3.11), with the cost $\bar{c}(\beta, \phi, \theta)$ convex type on $\bar{\Sigma}^+ \cup \bar{\Sigma}^0$ and concave type on $\bar{\Sigma}^- \cup \bar{\Sigma}^0$; while the nodal lines are given by $\bar{\Sigma}^0 = \{(\phi, \theta) \mid \cos(\phi - \theta) = 0\}$ or equivalently by

$$\bar{\Sigma}^0 = \{\phi - \theta = (2n-1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}. \tag{3.17}$$

For each fixed ϕ, $\cos(\phi-\theta)$ has two zeros on $\mathbf{T}^1$ — this allows a further decomposition

$$\bar{\Sigma}^0 = \bar{\Sigma}^0_P \cup \bar{\Sigma}^0_N, \tag{3.18}$$

with

$$\bar{\Sigma}^0_P := \{\phi - \theta = (4n-1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}, \tag{3.19}$$

$$\bar{\Sigma}^0_N := \{\phi - \theta = (4n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\} \tag{3.20}$$

giving two non-intersecting strictly increasing components oriented positively and negatively with respect to the set $\bar{\Sigma}^+$ — see e.g. Figure-3.2 for the schematics, the symbols are defined in Remark-2.0.2.

Lemma 3.2.4 (Σ^0 locally strictly increasing). *Under the hypotheses (1.3)-(1.4) of the toy model the set Σ^0 decomposes as a disjoint union $\Sigma^0_P \cup \Sigma^0_N$ of two locally non-decreasing subsets of $\mathbb{T}^2$. Each of these subsets is given by the graph of an orientation preserving homeomorphism of $\mathbb{T}^1$.*

Proof. The lemma follows directly from (3.17), (3.19) and (3.20) through the homeomorphism $(s,t) \mapsto (\phi,\psi)$ of $(\mathbb{R}/\mathbb{Z})^2$ onto $(\mathbb{R}/2\pi\mathbb{Z})^2$ giving

$$\Sigma^0 = \Sigma^0_P \cup \Sigma^0_N, \tag{3.21}$$

with Σ^0_P positively oriented and Σ^0_N negatively oriented with respect to Σ^+ — see Figure-3.1. □

The next proposition demonstrates that $\operatorname{spt}\gamma^1_o$ is non-decreasing on Σ^+ and locally non-increasing on Σ^- by establishing monotonicity of the optimal maps whose graphs on $\mathbb{T}^2$ contain the support. Here we denote the counterpart of the set $\widetilde{S}^2 \subset \partial\Omega$ by $S_2(\beta=1) \subset \mathbb{T}^1$ on the parameter space $\mathbb{T}^1$, i.e. $s \in S_2(\beta=1)$ implies that the μ-mass at s can split into two potential destinations $t^+_1(s) \neq t^-_1(s)$ on $\operatorname{spt}\nu$.

Proposition 3.2.5 (monotonicity of the optimal maps $t^{\pm}_1$). *Consider the constant speed parametrizations (1.4) for the boundaries of the bounded strictly convex domains $\Omega, \Lambda \subset \mathbb{R}^2$ from the toy model (1.3). Then for $\beta = 1$, the optimal maps $t^{\pm}_1 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ satisfy*

(i) $t^+_1 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ *is a non-decreasing map,*

(ii) $t^-_1 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ *restricted to $S_2(\beta=1)$ is locally non-increasing.*

Proof. (i) We first argue local non-decreasingness. The claim then becomes global by the fact that $t_1^+ : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ is a homeomorphism by Proposition-3.1.4. We also recall from (3.9), (3.10) and (3.12) that graph(t_1^+) is contained in $\Sigma^+ \cup \Sigma^0 \subset \mathbb{T}^2$. To produce a contradiction we now assume that the optimal map $t_1^+ : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ fails to be locally non-decreasing somewhere. Then there exist a sufficiently small subset of Σ^+ containing the distinct points $(s^k, t^k := t_1^+(s^k))$, for $k = 1, 2$, so that — after reindexing if necessary — the points (s^1, t^1) and (s^2, t^2) constitute the upper-left and lower-right corners respectively of the rectangle $\rrbracket s^1, s^2 \llbracket \times \rrbracket t^2, t^1 \llbracket$ contained entirely in Σ^+. By Proposition-3.1.4 graph(t_1^+) is contained in $\operatorname{spt}\gamma_o^1$ — which makes $\operatorname{spt}\gamma_o^1 \cap \Sigma^+$ a locally decreasing subset, but this contradicts Lemma-1.2.4 — since by optimality $\operatorname{spt}\gamma_o^1$ is a c_1-cyclically monotone subset of $\mathbb{T}^2$ — Smith and Knott [26] — and hence a c_1-monotone subset. This precludes t_1^+ from being locally orientation reversing. Since any homeomorphism of $\mathbb{T}^1$ either preserves orientation globally, or reverses it, the map t_1^+ must be globally increasing — as asserted by claim-(i).

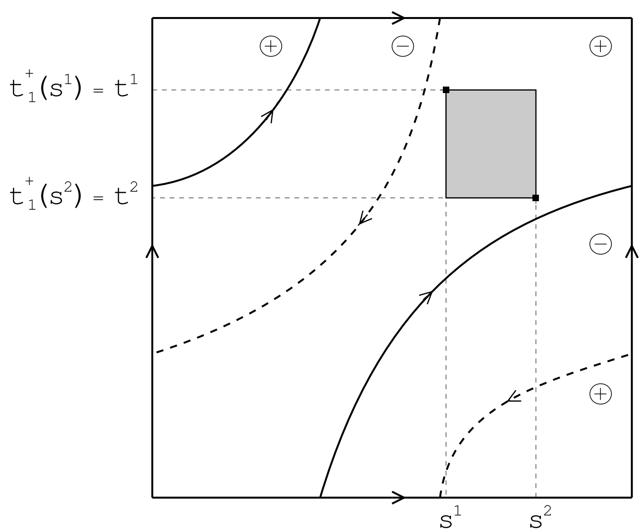

Figure 3.1: t_1^+ is locally orientation preserving: c_β-monotonicity precludes simultaneous occurrence of (s^1, t^1) and (s^2, t^2) in $\operatorname{spt}\gamma_o^1$.

(ii) The local non-increasingness of the map $t_1^- \lfloor_{S_2(\beta=1)}$ can be similarly argued.

This concludes the proof of the proposition. □

In the next lemma we establish a fact to be used later in Proposition-3.2.7 to study the structure of the subset $\operatorname{spt}\gamma_o^1 \cap \Sigma^0$.

Lemma 3.2.6. *For the cost function $c(1,s,t)$ from (1.8) the integral of its mixed partial over any subset of $\mathbb{T}^2$ of the form $Q :=]\!]s^1, s^2[\![\times]\!]t^1, t^2[\![$ is zero when Q has the diagonally opposite vertices (s^1,t^1) and (s^2,t^2) both on Σ^0_P (or both on Σ^0_N).*

Proof. Recall that all points (s,t) on Σ^0_P or Σ^0_N satisfy $\mathbf{n}_\Omega(s)\cdot\mathbf{n}_\Lambda(t) = 0$. Treating the unit normals as points on the gauss circle $\mathbf{S}^1$ and having both (s^1,t^1) and (s^2,t^2) on Σ^0_P (or on Σ^0_N), this forces

$$\mathbf{n}_\Omega(s^1)\cdot\mathbf{n}_\Omega(s^2) = \mathbf{n}_\Lambda(t^1)\cdot\mathbf{n}_\Lambda(t^2) =: \cos\phi$$

for some $0 < \phi < 2\pi$, so that

$$\begin{aligned}
\int_Q \frac{\partial^2 c}{\partial s \partial t}(1,s,t)\,ds\,dt &= \int_{s^1}^{s^2}\int_{t^1}^{t^2} \frac{\partial^2 c}{\partial s \partial t}(1,s,t)\,ds\,dt \\
&= \mathbf{n}_\Omega(s^1)\cdot\mathbf{n}_\Lambda(t^1) + \mathbf{n}_\Omega(s^2)\cdot\mathbf{n}_\Lambda(t^2) \\
&- \mathbf{n}_\Omega(s^1)\cdot\mathbf{n}_\Lambda(t^2) - \mathbf{n}_\Omega(s^2)\cdot\mathbf{n}_\Lambda(t^1) \\
&= 0 + 0 - \cos(\pi/2+\phi) - \cos(\pi/2-\phi) \\
&= 0\,,
\end{aligned} \tag{3.22}$$

thus proving the claim in the lemma. $\square$

One can readily check that the claim (3.22) of Lemma-3.2.6 holds equally on $\mathbf{T}^2$ under the angular parameters.

Proposition 3.2.7 (geometry of $\operatorname{spt}\gamma_o^1$ on $\mathbb{T}^2$). *Consider the toy model (1.3)-(1.4). For $\beta = 1$ the support of the optimal solution γ_o^1 may intersect Σ^0 at at most two points: $\Sigma^0 \cap \operatorname{spt}\gamma_o^1 \subseteq \{(s^1,t^1),\,(s^2,t^2)\} \subset \mathbb{T}^2$ with $(s^1,t^1) \in \Sigma^0_P$ and $(s^2,t^2) \in \Sigma^0_N$.*

Proof. We prove the proposition by contradiction using Lemma-3.2.6. Consider the subset $\Sigma^0_P \subset \mathbb{T}^2$ and assume that $\operatorname{spt}\gamma_o^1$ intersects it at two points — $(s^0,t^0),(s^2,t^2) \in \operatorname{spt}\gamma_o^1 \cap \Sigma^0_P$ for $s^0 \neq s^2$, $t^0 \neq t^2$. Reindex if necessary to make the set $]\!]s^0,s^2[\![\subset \mathbb{T}^1$ positively oriented — then so is $]\!]t^0,t^2[\![\subset \mathbb{T}^1$ by the relation $\mathbf{n}_\Omega(s)\cdot\mathbf{n}_\Lambda(t) = 0$ on Σ^0_P. This assumption combines with Definition-3.2.3 to give the points $(\phi^0,\theta^0) \neq (\phi^2,\theta^2)$ on $\bar{\Sigma}^0_P$ of (3.19) and some $n \in \mathbb{Z}$ so that

$$\theta^i = \phi^i - (4n-1)(\pi/2) \quad \text{for} \quad i = 0,2.$$

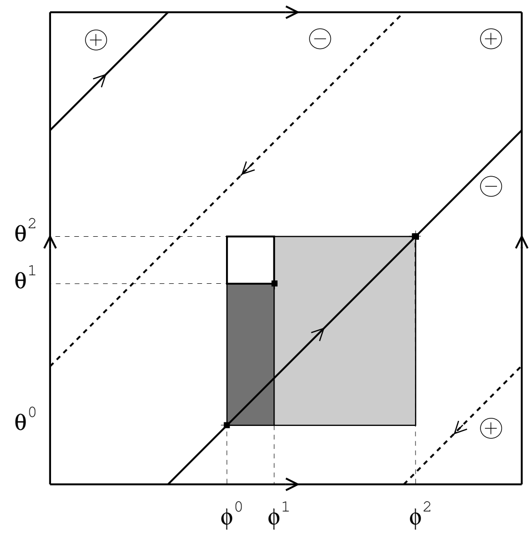

Figure 3.2: $\operatorname{spt}\bar{\gamma}^1_o$ cannot intersect $\bar{\Sigma}^0_P$ at both (ϕ^0, θ^0) and (ϕ^2, θ^2).

It causes no loss of generality to restrict to $d(\phi^0, \phi^2) \le \pi$. Homeomorphism and non-decreasingness of the optimal map $t^+_1 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ yields a point $t^1 = t^+_1(s^1)$ on $]\!]t^0, t^2[\![$ for some $s^1 \in]\!]s^0, s^2[\![$ and consequently a point $(\phi^1, \theta^1) \in \bar{\Sigma}^+ \cap (\,]\!]\phi^0, \phi^2[\![\times]\!]\theta^0, \theta^2[\![\,)$ on $\mathbf{T}^2$ — see Figure-3.2. Since the points (ϕ^k, θ^k) for $k = 0, 1, 2$ belong to the support of the optimal solution on $\mathbf{T}^2$ — which we call $\bar{\gamma}^1_o$ — they must satisfy (1.9) for $n = 2$, namely:

$$\sum_{k=0}^{2} \bar{c}(1, \phi^k, \theta^k) - \bar{c}(1, \phi^{\alpha(k)}, \theta^k) \ge 0. \tag{3.23}$$

We show below that for the cyclic permutation $\alpha(k) = k - 1$, the above inequality fails to hold. Identifying $\phi^{-1} = \phi^2$ one gets

$$\begin{aligned}
&\bar{c}(1, \phi^0, \theta^0) + \bar{c}(1, \phi^1, \theta^1) + \bar{c}\,(1, \phi^2, \theta^2) - \bar{c}\,(1, \phi^2, \theta^0) - \bar{c}\,(1, \phi^0, \theta^1) - \bar{c}\,(1, \phi^1, \theta^2) \\
&= \ [\,\bar{c}\,(1, \phi^0, \theta^0) + \bar{c}\,(1, \phi^1, \theta^1) - \bar{c}\,(1, \phi^0, \theta^1) - \bar{c}\,(1, \phi^1, \theta^0)\,] + [\,\bar{c}\,(1, \phi^2, \theta^2) \\
&\quad - \bar{c}\,(1, \phi^1, \theta^2) - \bar{c}\,(1, \phi^2, \theta^0) + \bar{c}\,(1, \phi^1, \theta^0)\,] \\
&= \int_{\phi^0}^{\phi^1}\int_{\theta^0}^{\theta^1} \frac{\partial^2 \bar{c}}{\partial\phi\partial\theta}(1, \phi, \theta)\, d\phi\, d\theta \ + \int_{\phi^1}^{\phi^2}\int_{\theta^0}^{\theta^2} \frac{\partial^2 \bar{c}}{\partial\phi\partial\theta}(1, \phi, \theta)\, d\phi\, d\theta \\
&= \int_{\phi^0}^{\phi^2}\int_{\theta^0}^{\theta^2} \frac{\partial^2 \bar{c}}{\partial\phi\partial\theta}(1, \phi, \theta)\, d\phi\, d\theta \ - \int_{\phi^0}^{\phi^1}\int_{\theta^1}^{\theta^2} \frac{\partial^2 \bar{c}}{\partial\phi\partial\theta}(1, \phi, \theta)\, d\phi\, d\theta \\
&< \ 0\,.
\end{aligned}$$

We get the first equality by rearranging the terms in the line above and adding and subtracting the term $\bar{c}\,(1, \phi^1, \theta^0)$. The third equality follows from the second by periodicity of functions on torus. The first term in the third equality is zero by (3.22) of Lemma-3.2.6, while the second term is an integration over a strictly positive quantity since the cost is convex type on $]\!]\phi^o, \phi^1[\![\times]\!]\theta^1, \theta^2[\![\subset \bar{\Sigma}^+$. Thus if $\operatorname{spt}\bar{\gamma}_o^1$ intersects $\bar{\Sigma}_P^0$ at more than one point it fails to satisfy c_β-cyclical monotonicity (3.23) for the triples (ϕ^k, θ^k), $k = 0, 1, 2$. The same holds for $\bar{\Sigma}_N^0$. Consequently, by a change of variable, $\operatorname{spt}\gamma_o^1 \cap \Sigma_P^0$ and $\operatorname{spt}\gamma_o^1 \cap \Sigma_N^0$ can at most be singleton subsets of $\mathbb{T}^2$. This concludes the proof of the proposition. □

We conclude the chapter by giving a schematic of the support of optimal solution γ_o^1 on the flat torus $\mathbb{T}^2$:

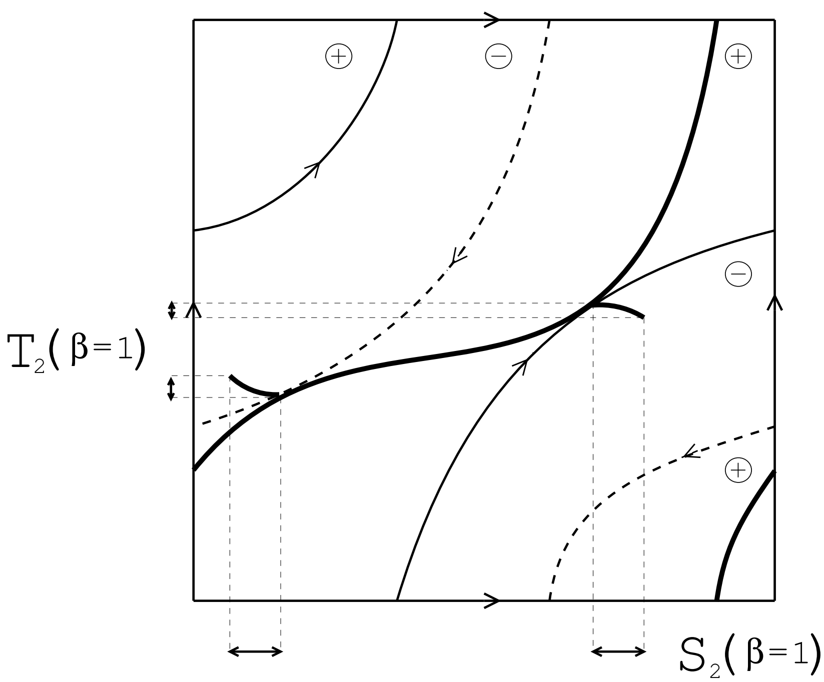

Figure 3.3: $\beta = 1$: the bold solid curves represent a possible support of the optimal solution which can intersect Σ_P^0 at most once and Σ_N^0 at most once. $S_2(\beta = 1) \subset \operatorname{spt}\mu = \mathbb{T}^1$ represent the subset where each point has two potential destinations causing splitting of mass. Notice that the support is locally decreasing in Σ^- and increasing throughout Σ^+.

Chapter 4

Duality: existence and uniqueness of dual solution

4.1 The dual problem

By the Kantorovich duality principle [12] one can write the dual problem to the infinite dimensional linear program (1.5) on the toy model (1.3)-(1.4) as

$$\inf_{(u,v)\in\mathcal{A}_\beta}\Big\{\int_{\mathbb{T}^1} u(s)d\mu(s)+\int_{\mathbb{T}^1} v(t)d\nu(t) =: J_\beta(u,v)\Big\}, \tag{4.1}$$

where $u:\mathbb{T}^1\longrightarrow\mathbb{R}$ and $v:\mathbb{T}^1\longrightarrow\mathbb{R}$ are lower semi-continuous functions on the one dimensional torus while $\mathcal{A}_\beta$ denotes the set of all such pairs (u,v) that satisfy $u(s)+v(t)\geq c(\beta,s,t)$, i.e.

$$\mathcal{A}_\beta := \{(u,v) \mid u(s)+v(t)\geq c(\beta,s,t)\}. \tag{4.2}$$

In this chapter we outline a proofs for the existence of the *dual optimizer* (u^β,v^β) defined by

$$(u^\beta,v^\beta)\in \arg\min_{(u,v)\in\mathcal{A}_\beta} J_\beta(u,v) \tag{4.3}$$

These existence results, though well known, are included to give a background on the characterization of the support of optimal solutions γ_o^β for the primal problem (1.5) in terms of the differentials of the dual solutions. The strategies for some of the proofs are adopted from McCann [18]. To check the lipschitz continuity of the cost function $c(\beta, s, t)$ and the potentials u^β and v^β, we metrize the one-dimensional torus $\mathbb{T}^1$ by the quotient metric:

$$d(s_1, s_2) := \inf_{n\in\mathbb{Z}} |s_1 - s_2 - n|, \tag{4.4}$$

for $s_1, s_2 \in \mathbb{T}^1$. We also introduce some definitions that generalizes the notions of Legendre-Fenchel transforms and subdifferentiability of convex functions to lower semi-continuous functions on $\mathbb{T}^1$.

Definition 4.1.1 (c_β-convexity and c_β-transforms). For each $\beta \in [0,1]$ we call a function $v : \mathbb{T}^1 \longrightarrow \mathbb{R}$ *c_β-convex* if it is the supremum of translates and shifts of the cost function $c_\beta : \mathbb{T}^2 \longrightarrow \mathbb{R}$ (defined by (1.8)) by some lower semi-continuous function $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$; that is for all $t \in \mathbb{T}^1$ if

$$v(t) := \sup_{s\in\mathbb{T}^1} c(\beta, s, t) - u(s), \tag{4.5}$$

which can also be referred to as the *c_β-transform* of $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$ and denoted $u_{c_\beta}(t)$.

Notice that c_β-convexity of $v(t)$ on $\mathbb{R}$ is equivalent to convexity if the cost function is given by $c_\beta(s,t) := st$ on $\mathbb{R}^2$; while c_β-transform of $u(s)$ is an analog of the Legendre-Fenchel transform if $u(s)$ is a convex function on $\mathbb{R}$ with $c_\beta(s,t) := st$ on $\mathbb{R}^2$. Due to the lack of symmetry of the cost function $c(\beta, s, t)$ under the interchange $s \leftrightarrow t$, we identify

$$c^\beta(t, s) := c_\beta(s, t) := c(\beta, s, t) \tag{4.6}$$

and define by analogy:

Definition 4.1.2 (c^β-convexity and c^β-transforms). Following Definition-4.1.1 we define for each $\beta \in [0,1]$ a *c^β-convex* function $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$ when the supremum is taken over $t \in \mathbb{T}^1$ for some lower semi-continuous function $v(t)$ on $\mathbb{T}^1$ according to

$$\begin{aligned} u(s) &:= \sup\nolimits_{t\in\mathbb{T}^1} c(\beta, s, t) - v(t) \\ &= \sup\nolimits_{t\in\mathbb{T}^1} c^\beta(t, s) - v(t) =: v_{c^\beta}(s), \end{aligned} \tag{4.7}$$

and call it the *c^β-transform* of $v(t)$ — denoted $v_{c^\beta}(s)$.

Definition 4.1.3 (c_β-subdifferential). Given $\beta \in [0,1]$ the *c_β-subdifferential* $\partial_{c_\beta} u$ of $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$ consists of the pairs $(s,t) \in \mathbb{T}^2$ for which $u(s') \geq u(s) + c(\beta, s', t) - c(\beta, s, t)$ for all $s' \in \mathbb{T}^1$.

Alternatively, $(s,t) \in \partial_{c_\beta} u$ means $c(\beta, s', t) - u(s')$ attains its maximum at $s' = s$ which then combines with Definition-4.1.1 to give

$$\partial_{c_\beta} u = \{(s,t) \in \mathbb{T}^2 \mid u(s) + u_{c_\beta}(t) = c(\beta, s, t)\}. \tag{4.8}$$

We also define the *c_β-subgradient* of u at $s \in \mathbb{T}^1$ to be the subset $\partial_{c_\beta} u(s) \subset \mathbb{T}^1$ consisting of those $t \in \mathbb{T}^1$ for which $(s,t) \in \partial_{c_\beta} u$, i.e.

$$\partial_{c_\beta} u(s) = \{t \in \mathbb{T}^1 \mid (s,t) \in \partial_{c_\beta} u\}; \tag{4.9}$$

while $\partial_{c_\beta} u(B) := \cup_{s \in B} \ \partial_{c_\beta} u(s)$ for $B \subset \mathbb{T}^1$.

Analogously, the *c^β-subdifferential* $\partial_{c^\beta} v$ of $v(t)$ and the *c^β-subgradient* $\partial_{c^\beta} v(t)$ of $v(t)$ at $t \in \mathbb{T}^1$ are defined according to

$$\partial_{c^\beta} v \ := \ \{(t,s) \in \mathbb{T}^2 \mid v(t) + v_{c^\beta}(s) = c(\beta, s, t)\}, \tag{4.10}$$

$$\partial_{c^\beta} v(t) \ := \ \{s \in \mathbb{T}^1 \mid (t,s) \in \partial_{c^\beta} v\}. \tag{4.11}$$

Remark 4.1.4. Using the identity (4.6) and Lemma-4.1.8 one can check from (4.8) and (4.10) that

$$(s,t) \in \partial_{c_\beta} u \Longleftrightarrow (t,s) \in \partial_{c^\beta} u_{c_\beta}. \tag{4.12}$$

Moreover, comparison with convex functions of $\mathbb{R}$ shows that if $u : \mathbb{R} \longrightarrow \mathbb{R}$ is convex and $c_\beta(s,t) = st$ on $\mathbb{R}^2$ then the c_β-subdifferential coincides with the sub-differential of $u(s)$ defined by

$$\partial.u := \{(s,t) \in \mathbb{T}^2 \mid u(s') \geq u(s) + (s' - s)t \text{ for all } s' \in \mathbb{R}\}. \tag{4.13}$$

Lemma 4.1.5 (lipschitz cost). *The cost function $c : [0,1] \times \mathbb{T}^1 \times \mathbb{T}^1 \longrightarrow \mathbb{R}$ defined by (1.8) for the toy model (1.3) is uniformly lipschitz continuous as a function of s with*

$$|c(\beta, s_1, t) - c(\beta, s_2, t)| \leq v_\Omega \left[M_\Lambda + \|K_\Omega\|_\infty\right] d(s_1, s_2) \tag{4.14}$$

for all $s_1, s_2 \in \mathbb{T}^1$ and for all $(\beta, t) \in [0,1] \times \mathbb{T}^1$.

Proof. Fix any two points s_1, s_2 in $\mathbb{T}^1$. Then for all $(\beta, t) \in [0,1] \times \mathbb{T}^1$ one gets from (1.8)

$$\begin{aligned} & c(\beta, s_2, t) - c(\beta, s_1, t) \\ &= (1-\beta)[\mathbf{x}(s_2) - \mathbf{x}(s_1)] \cdot \mathbf{y}(t) + \beta[\mathbf{n}_\Omega(s_2) - \mathbf{n}_\Omega(s_1)] \cdot \mathbf{n}_\Lambda(t) \\ &= (1-\beta)\mathbf{y}(t) \cdot \int_{s_1}^{s_2} \frac{d\mathbf{x}(s)}{ds}\, ds + \beta\, \mathbf{n}_\Lambda(t) \cdot \int_{s_1}^{s_2} \frac{d\mathbf{n}_\Omega(s)}{ds}\, ds \\ &= v_\Omega \left[(1-\beta)\mathbf{y}(t) \cdot \int_{s_1}^{s_2} \mathbf{T}_\Omega(s)\, ds + \beta\, \mathbf{n}_\Lambda(t) \cdot \int_{s_1}^{s_2} K_\Omega(s) \mathbf{T}_\Omega(s)\, ds \right], \end{aligned}$$

which implies

$$|c(\beta, s_2, t) - c(\beta, s_1, t)| \le v_\Omega \left[(1-\beta)\, M_\Lambda + \beta\, \|K_\Omega\|_\infty \right] |[\![s_1, s_2]\!]|, \tag{4.15}$$

where we used the Cauchy-Schwarz inequality and the facts that

$$\begin{cases} |\mathbf{T}_\Omega(s)| = |\mathbf{n}_\Lambda(t)| = 1 \\ \text{the domains and their curvatures are bounded (see Chapter-2 for notations)} \\ \left| \int_{s_1}^{s_2} ds \right| =: |[\![s_1, s_2]\!]| = \text{length of the positively oriented arc } [\![s_1, s_2]\!] \text{ of } \mathbb{T}^1. \end{cases}$$

Interchanging $s_1 \leftrightarrow s_2$ regenerates (4.15) with the arc length $|[\![s_1, s_2]\!]|$ replaced by $|[\![s_2, s_1]\!]|$. Taking minimum over the two arc lengths and using

$$\begin{cases} d(s_1, s_2) = \min\{|[\![s_1, s_2]\!]|, |[\![s_2, s_1]\!]|\} \\ \text{and } 0 \le \beta \le 1 \end{cases}$$

yields (4.14). □

Lemma 4.1.6 (c^β-convex potentials are lipschitz). *Given $\beta \in [0,1]$ let $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$ denote a c^β-convex potential defined by (4.5) for the cost function $c(\beta, s, t)$ of (1.8) and $v : \mathbb{T}^1 \longrightarrow \mathbb{R}$ a lower semi-continuous function. Then u is uniformly lipschitz on $\mathbb{T}^1$ with*

$$|u(s_1) - u(s_2)| \le v_\Omega \left[M_\Lambda + \|K_\Omega\|_\infty \right] d(s_1, s_2). \tag{4.16}$$

Proof. The proof follows from Lemma-4.1.5 above and from Lemma-2 of McCann [18] that the c-convex functions on a bounded metric space are lipschitz continuous when the cost function is lipschitz. □

Remark 4.1.7 (lipschitz c_β-transforms and bounded cost).
R1. Taking $v(t) = u_{c_\beta}(t) := \sup_{s\in\mathbb{T}^1} c(\beta, s, t) - u(s)$ one gets from Lemma-4.1.6

$$|u_{c_\beta}(t_1) - u_{c_\beta}(t_2)| \leq v_\Lambda \left[M_\Omega + \|K_\Lambda\|_\infty \right] d(t_1, t_2).$$

R2. Letting $M := \max\{v_\Omega \left[M_\Lambda + \|K_\Omega\|_\infty\right], v_\Lambda \left[M_\Omega + \|K_\Lambda\|_\infty\right]\}$ one has

$$\begin{aligned} |u(s_1) - u(s_2)| &\leq M\, d(s_1, s_2), \\ |u_{c_\beta}(t_1) - u_{c_\beta}(t_2)| &\leq M\, d(t_1, t_2), \\ |c(\beta, s_1, t) - c(\beta, s_2, t)| &\leq M\, d(s_1, s_2), \\ |c(\beta, s, t_1) - c(\beta, s, t_2)| &\leq M\, d(t_1, t_2). \end{aligned}$$

R3. Moreover, using $0 \leq \beta \leq 1$:

$$\begin{aligned} |c(\beta, s, t)| &\leq (1-\beta)|\mathbf{x}(s)|\,|\mathbf{y}(t)| + \beta|\mathbf{n}_\Omega(s)|\,|\mathbf{n}_\Lambda(t)| \\ &\leq (1-\beta) M_\Omega\, M_\Lambda + \beta \\ &\leq M_\Omega\, M_\Lambda + 1. \end{aligned}$$

The next lemma states a standard fact about c-convex functions:

Lemma 4.1.8. *A lower semi-continuous function $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$ is c^β-convex if and only if $u = u_{c_\beta c^\beta} := (u_{c_\beta})_{c^\beta}$.*

Proof. The necessary condition follows directly from Definition-4.1.2 according to which $u_{c_\beta c^\beta}(s) = (u_{c_\beta})_{c^\beta}(s)$ is the c^β-transform of the function $u_{c_\beta}(t)$ which is Lipschitz continuous by Lemma-4.1.6 and therefore makes $u_{c_\beta c^\beta}(s)$ a c^β-convex function. For the converse it suffices to show that any lower semi-continuous function $v : \mathbb{T}^1 \longrightarrow \mathbb{R}$ satisfies $v_{c^\beta c_\beta c^\beta} = v_{c^\beta}$. For then setting $u = v_{c^\beta}$ defines an arbitrary c^β-convex function and shows $u = u_{c_\beta c^\beta}$.

1. The definition of c^β-transform (4.7) implies that $v_{c^\beta}(s) + v(t) \geq c(\beta, s, t)$. The c_β-transform of $v_{c^\beta}(s)$ then gives

$$\begin{aligned} v_{c^\beta c_\beta}(t) &= \sup_{s\in\mathbb{T}^1} c(\beta, s, t) - v_{c^\beta}(s) \\ &\leq \sup_{s\in\mathbb{T}^1} v(t) \\ &= v(t) \quad \text{for all} \quad t \in \mathbb{T}^1. \end{aligned} \tag{4.17}$$

2. Interchanging $c^\beta \leftrightarrow c_\beta$ and $s \leftrightarrow t$ one can write from (4.17)

$$v_{c_\beta c^\beta}(s) \leq v(s) \quad \text{for all } s \in \mathbb{T}^1. \tag{4.18}$$

Using this and taking $w = v_{c^\beta}$ yields

$$\begin{aligned} v_{c^\beta c_\beta c^\beta}(s) &= (v_{c^\beta})_{c_\beta c^\beta}(s) \\ &= w_{c_\beta c^\beta}(s) \\ &\leq w(s) = v_{c^\beta}(s) \end{aligned} \tag{4.19}$$

giving $v_{c^\beta c_\beta c^\beta}(s) \leq v_{c^\beta}(s)$ for all $s \in \mathbb{T}^1$.

3. To deduce the reverse inequality one can write from the definition of c^β-transform that

$$\begin{aligned} v_{c^\beta c_\beta c^\beta}(s) &= (v_{c^\beta c_\beta})_{c^\beta}(s) \\ &= \sup_{t\in\mathbb{T}^1} c(\beta, s, t) - v_{c^\beta c_\beta}(t) \\ &\geq \sup_{t\in\mathbb{T}^1} c(\beta, s, t) - v(t) \\ &= v_{c^\beta}(s), \end{aligned} \tag{4.20}$$

where the inequality follows from (4.17) and the last equality from (4.7). This shows $v_{c^\beta c_\beta c^\beta} = v_{c^\beta}$ to complete the lemma. □

4.2 Topology of the space of c^β-convex functions

For each $\beta \in [0, 1]$ we define the set of c^β-convex functions on $\mathbb{T}^1$ by $\mathcal{B}^\beta$:

$$\mathcal{B}^\beta := \left\{u : \mathbb{T}^1 \longrightarrow \mathbb{R} \mid u = u_{c_\beta c^\beta}\right\}. \tag{4.21}$$

From Lemma-4.1.6 and Remark-4.1.7 it follows that the subset of $\mathcal{B}^\beta$ for which every c^β-convex function satisfies $u(s_0) = 0$ for some $s_0 \in \mathbb{T}^1$ are uniformly bounded on $\mathbb{T}^1$:

$$\begin{aligned} |u(s)| &= |u(s) - u(s_o)| \\ &\leq M\, d(s, s_0) \\ &\leq M \quad \text{since } d(s, s_0) \leq 1. \end{aligned} \tag{4.22}$$

We denote this restriction to the uniformly bounded c^β-convex functions by

$$\mathcal{B}_0^\beta := \left\{u : \mathbb{T}^1 \longrightarrow \mathbb{R} \mid u = u_{c_\beta c^\beta} \text{ and } u(s_0) = 0 \text{ for some } s_0 \in \mathbb{T}^1\right\}, \tag{4.23}$$

and define

$$\mathcal{B}_0 := \Big\{(\beta, u) \mid \beta \in [0,1], u \in \mathcal{B}_0^{\beta}\Big\} \tag{4.24}$$

— a subset of the product space $[0,1] \times C(\mathbb{T}^1)$. In the following proposition we prove compactness of the set $\mathcal{B}_0$ under the metric topology on $[0,1]$ and the topology of uniform convergence on $C(\mathbb{T}^1)$. We first prove the lemma:

Lemma 4.2.1. *The c_β-transforms $u^n_{c_n} := u^n_{c_{\beta_n}}$ of the sequence $u^n \in \mathcal{B}_0^{\beta_n}$ converges uniformly to u_{c_β} if $u^n \to u$ uniformly as $\beta_n \to \beta$.*

Proof. For each $n \geq 1$ and $t \in \mathbb{T}^1$ the c_β-transform satisfy

$$u^n_{c_n}(t) = \sup_{s \in \mathbb{T}^1} c(\beta_n, s, t) - u^n(s).$$

Continuity-compactness argument yields an s^n_0 for which the supremum is achieved

$$u^n_{c_n}(t) = c(\beta_n, s^n_0, t) - u^n(s^n_0); \tag{4.25}$$

whereas for all other $s \in \mathbb{T}^1$ one has

$$u^n_{c_n}(t) \geq c(\beta_n, s, t) - u^n(s). \tag{4.26}$$

By (4.22) the sequence $u^n \in \mathcal{B}_0^{\beta_n}$ is uniformly bounded with $|u^n(s)| \leq M$ — for M independent of s and n — which combines with (4.25) and Remark-4.1.7-R2 and -R3 to uniformly bound the c_β-transforms:

$$\begin{aligned} |u^n_{c_n}(t)| &= |c(\beta_n, s^n_0, t) + u^n(s^n_0)| \\ &\leq |c(\beta_n, s^n_0, t)| + |u^n(s^n_0)| \\ &\leq 1 + M_\Omega M_\Lambda + M. \end{aligned} \tag{4.27}$$

Remark-4.1.7-R2 further implies equilipschitzness of the sequence:

$$|u^n_{c_n}(t_1) - u^n_{c_n}(t_2)| \leq M\, d(t_1, t_2).$$

The Arzela-Ascoli argument then provides a subsequence, also denoted $u^n_{c_n}$, that converges uniformly to some lipschitz continuous function $v : \mathbb{T}^1 \longrightarrow \mathbb{R}$, and $s^n_0 \to s_0$ for a further subsequence. In the limit $\beta_n \to \beta$, the uniform convergences $u^n \to u$

and $u^n_{c_n} \to v$, and the continuity of the cost function in β and s force (4.25) and (4.26) to converge to

$$\begin{aligned} v(t) &= c(\beta, s_0, t) - u(s_0) && (4.28)\\ v(t) &\geq c(\beta, s, t) - u(s) \text{ for all } s \in \mathbb{T}^1. && (4.29)\end{aligned}$$

This implies that

$$\begin{aligned} v(t) &= c(\beta, s_0, t) - u(s_0) && \text{by (4.28)}\\ &\leq \sup_{s\in\mathbb{T}^1} c(\beta, s, t) - u(s) &&\\ &\leq v(t) && \text{by (4.29)}\end{aligned}$$

forcing equality and therefore $v(t) = \sup_{s\in\mathbb{T}^1} c(\beta, s, t) - u(s) = u_{c_\beta}(t)$ to complete the proof. □

Proposition 4.2.2 ($\mathcal{B}_0$ is compact). *The set $\mathcal{B}_0$ defined by (4.24) is compact in the product space $[0, 1] \times C(\mathbb{T}^1)$ when $[0, 1]$ is metrized by the Euclidean norm and $C(\mathbb{T}^1)$ by the L^∞ norm.*

Proof. Pick a sequence $(\beta_n, u^n) \in \mathcal{B}_0$. Then by definition $u^n \in \mathcal{B}_0^{\beta_n}$.

Claim ($\mathcal{B}_0^\beta$ **is closed**)*:* If u^n converges uniformly to u as $\beta_n \to \beta$ then $u \in \mathcal{B}_0^\beta$.

Proof of Claim: By c^β-convexity u^n satisfies for each $s \in \mathbb{T}^1$

$$\begin{aligned} u^n(s) &= \sup_{t\in\mathbb{T}^1} c(\beta_n, s, t) - u^n_{c_n}(t),\\ u^n(s) &= u^n_{c_n c^n}(s), \end{aligned} \qquad (4.30)$$

where $u^n_{c_n} := u^n_{c_{\beta_n}}$ is the c_β-transform of u^n. By Lemma-4.2.1 the hypothesis of the claim asserts that $u^n_{c_n} \to u_{c_\beta}$ uniformly as $\beta_n \to \beta$. Interchanging $c^\beta \leftrightarrow c_\beta$ and $s \leftrightarrow t$, the lemma applied to $u^n_{c_n} \to u_{c_\beta}$ implies the uniform convergence $u^n_{c_n c^n} \to u_{c_\beta c^\beta}$. It therefore follows that

$$\begin{aligned} |u(s) - u_{c_\beta c^\beta}(s)| &\leq |u(s) - u^n(s)| + |u^n_{c_n c^n}(s) - u_{c_\beta c^\beta}(s)|\\ &\longrightarrow 0 \qquad \text{as } n \to \infty \end{aligned}$$

giving $u = u_{c_\beta c^\beta}$ — that is u is c^β-convex. Moreover, $u^n(s_0^n) = 0$ and $s_0 = \lim_{n\to\infty} s_0^n$ for a subsequence, also denoted s_0^n. From the uniform convergence $u^n \to u$ and the lipschitiz continuity of u^n one derives

$$\begin{aligned} |u(s_0)| &\leq |u(s_0) - u^n(s_0)| + |u^n(s_0) - u^n(s_0^n)| \\ &\leq \|u - u^n\|_{L^\infty} + M\, d(s_0, s_0^n) \\ &\to 0 \quad \text{ as } n \to \infty \end{aligned} \tag{4.31}$$

which ensures $u \in \mathcal{B}_0^\beta$ to conclude the claim.

To prove compactness consider the sequence $(\beta_n, u^n) \in \mathcal{B}_0$. By compactness of $[0,1]$, β_n has a subsequence that converges to some $\beta \in [0,1]$. By construction the sequence $u^n \in \mathcal{B}_0^{\beta_n}$ — which is equilipschitz by Lemma-4.1.6 and Remark-4.1.7-R2 — is uniformly bounded:

$$|u^n(s)| \leq M \quad \text{ by (4.22)}$$

with M independent of s and n. Then the Arzela-Ascoli argument extracts a convergent subsequence, also denoted u^n, which converges uniformly to a lipschitz function u on $\mathbb{T}^1$ as $n \to \infty$. Passing to convergent sub-sub-sequences, also denoted (β_n, u^n), it follows from the claim that $(\beta_n, u^n) \to (\beta, u) \in \mathcal{B}_0$ as $n \to \infty$. Consequently every sequence in $\mathcal{B}_0$ admits a subsequence that converges in $\mathcal{B}_0$ thus making it compact. □

Remark 4.2.3. For each fixed $\beta \in [0,1]$ the set $\mathcal{B}_0^\beta$ of bounded c^β-convex functions is compact in the sup-norm topology on $C(\mathbb{T}^1)$.

Given μ, ν and $c(\beta, s, t)$ from (1.3) and (1.8), recall from (4.1) that the total cost of the dual problem — for each $\beta \in [0,1]$ and $(u,v) \in \mathcal{A}_\beta$ — is given by

$$J_\beta(u,v) := \int_{\mathbb{T}^1} u(s)\, d\mu(s) + \int_{\mathbb{T}^1} v(t)\, d\nu(t). \tag{4.32}$$

We further recall from the definition of c_β-transforms that every $u \in \mathcal{B}_0^\beta$ satisfies $(u, u_{c_\beta}) \in \mathcal{A}_\beta$. The next proposition shows continuity of the dual cost in $u \in \mathcal{B}_0^\beta$ in the sense that whenever $(\beta_n, u^n) \in \mathcal{B}_0 \to (\beta, u)$, the dual cost $J_{\beta_n}(u^n, u^n_{c_n})$ converges to $J_\beta(u, u_{c_\beta})$.

Proposition 4.2.4 (continuity of dual cost). *The dual cost $J_\beta(u, u_{c_\beta})$ from (4.32) is continuous with respect to (β, u).*

Proof. Pick a sequence $u^n \in \mathcal{B}_0^{\beta_n}$ for $\beta_n \in [0,1]$ and denoting the associated cost function by $c_n := c_{\beta_n}$ let $u^n_{c_n}$ represent the c_n-transforms of u^n. Assume $u^n \to u \in \mathcal{B}_0^\beta$ uniformly as $\beta_n \to \beta$. Then Lemma-4.2.1 shows that the corresponding subsequence of the c_n-transforms satisfies $\|u^n_{c_n} - u_{c_\beta}\|_{L^\infty} \to 0$ as $n \to \infty$. The dominated convergence theorem then yields $\lim_{\beta_n \to \beta} J_{\beta_n}(u^n, u^n_{c_n}) = J_\beta(u, u_{c_\beta})$ to complete the proposition. □

Proposition 4.2.5 (existence of dual solution). *Consider the toy model (1.3)-(1.4). Fix Borel probability measures μ and ν on $\mathbb{T}^1$, mutually continuous with respect to $\mathcal{H}^1\lfloor_{\mathbb{T}^1}$. Then for each $\beta \in [0,1]$ the infimum in (4.1) is attained by the lipschitz potentials $(u^\beta, v^\beta) \in \mathcal{A}_\beta$ satisfying $u^\beta = u^\beta_{c_\beta c^\beta} := (u^\beta_{c_\beta})_{c^\beta}$ and $v^\beta = u^\beta_{c_\beta}$.*

Proof. Fix $\beta \in [0,1]$. Pick a sequence $(\phi^n, \psi^n) \in \mathcal{A}_\beta$ for which $J_\beta(u, v)$ tends to its minimum value on $\mathcal{A}_\beta$. Noting that $(\phi^n_{c_\beta c^\beta}, \phi^n_{c_\beta}) \in \mathcal{A}_\beta$ and

$$
\begin{aligned}
\phi^n_{c_\beta c^\beta} &\leq \phi^n && \text{from (4.18) and} \\
\phi^n_{c_\beta} &\leq \psi^n && \text{from } (\phi^n, \psi^n) \in \mathcal{A}_\beta \text{ and (4.5),}
\end{aligned}
$$

one gets $J_\beta(\phi^n_{c_\beta c^\beta}, \phi^n_{c_\beta}) \leq J_\beta(\phi^n, \psi^n)$. Fix $s_0 \in \mathbb{T}^1$ and set $\phi^n(s_0) = \lambda^n$. Using the fact that $\phi^n_{c_\beta} = \phi^n_{c_\beta c^\beta c_\beta}$ from Lemma-4.1.8, we can now mimic the proof of Proposition-3 of McCann [18] to construct a minimizing sequence $(u^n, u^n_{c_\beta}) := (\phi^n_{c_\beta c^\beta} - \lambda_n,\ \phi^n_{c_\beta} + \lambda_n) \in \mathcal{A}_\beta$. Since by construction u^n is a c^β-convex function that satisfies $u^n(s_0) = 0$ one concludes $u^n \in \mathcal{B}_0^\beta$. Compactness (Remark-4.2.3) yields a convergent subsequence also denoted by u^n and a $u = u_{c_\beta c^\beta} \in \mathcal{B}_0^\beta$ with $\|u^n - u\|_{L^\infty} \to 0$ as $n \to \infty$. Finally taking $\beta_n \equiv \beta$, the continuity of the dual cost from Proposition-4.2.4 yields $\lim_{n\to\infty} J_\beta(u^n, u^n_{c_\beta}) = J_\beta(u, u_{c_\beta}) =$ the minimum dual cost. □

4.3 Characterization of the optimal solutions by the dual optimizers

In this section we give a characterization of the optimal solutions γ_o^β for the transport problem (1.5) in terms of a dual optimizer from Proposition-4.2.5 and show that any such dual optimizer is unique up to some additive constant. We recall from Smith and Knott [26] that optimality forces each $\gamma_o^\beta \in \Gamma(\mu,\nu)$ to have c_β-cyclically monotone support — the union of all such supports itself being a c_β-cyclically monotone set (Corollary-2.4 of Gangbo and McCann [9]) implying that for each optimal solution $\operatorname{spt}\gamma_o^\beta$ is contained in the same c_β-cyclically monotone set. Theorem-2.7 in [9] due to Rüschendorf [25] gives a c^β-convex function whose c_β-subdifferential contains $\operatorname{spt}\gamma_o^\beta$ — a necessary and sufficient condition for c_β-cyclical monotonicity. For notational convenience we define:

$$\Gamma_\beta := \{\gamma_o^\beta \in \arg\min_{\Gamma(\mu,\nu)} \mathcal{C}_\beta(\gamma)\} \tag{4.33}$$

$$Z_\beta := \cup_{\gamma_o^\beta \in \Gamma_\beta} \operatorname{spt}\gamma_o^\beta, \tag{4.34}$$

and show that $Z_\beta \subset \partial_{c_\beta} u^\beta$ for the dual optimizer u^β from Proposition-4.2.5. We further recall that compactness makes the metric space $(\mathbb{T}^1, d)$ a Polish space and enables one to deduce from Theorem-1.3 in Villani [29] that the total optimal costs for the primal (1.5) and dual (4.1) problems are equal, i.e.

$$\sup_{\gamma\in\Gamma(\mu,\nu)} \mathcal{C}_\beta(\gamma) = \inf_{(u,v)\in\mathcal{A}_\beta} J_\beta(u,v), \tag{4.35}$$

for given μ, ν and $c(\beta,s,t)$ from (1.3) and (1.8). An equivalent statement is that for each fixed $\beta \in [0,1]$, $(u^\beta, v^\beta) \in \mathcal{A}_\beta$ and $\gamma_o^\beta \in \Gamma(\mu,\nu)$ solve

$$\int_{T^2} [u^\beta(s) + v^\beta(t) - c(\beta,s,t)]\, d\gamma_o^\beta(s,t) = 0 \tag{4.36}$$

if and only if γ_o^β and (u^β, v^β) are the primal and dual optimizers from equations (1.6) and (4.3) respectively.

In the following lemma we claim that the support of a doubly stochastic measure on $\mathbb{T}^2$ — with $\mathcal{H}^1\lfloor_{\mathbb{T}^1}$ for marginals — projects onto $\mathbb{T}^1$ under the projections $\pi_1(s,t) = s$ or $\pi_2(s,t) = t$.

Lemma 4.3.1 (support of a doubly stochastic measure on $\mathbb{T}^2$). *Let μ and ν be Borel probability measures on $\mathbb{T}^1$ mutually continuous with respect to $\mathcal{H}^1\lfloor_{\mathbb{T}^1}$ and let $\gamma \in \Gamma(\mu,\nu)$ be a doubly stochastic measure on the flat torus $\mathbb{T}^2$. Then for each $s \in \mathbb{T}^1$ there exists a point $t \in \mathbb{T}^1$ for which $(s,t) \in \operatorname{spt}\gamma$.*

Proof. Assume the statement is false. Then there exists an $s_0 \in \mathbb{T}^1$ for which $(s_0, t) \notin \operatorname{spt}\gamma$ for all $t \in \mathbb{T}^1$. Then there exists an open set $U \subset \mathbb{T}^2$ (for example $U = \mathbb{T}^2 \backslash \operatorname{spt}\gamma$) containing the slice $\{s_0\} \times \mathbb{T}^1$ so that $\gamma[U] = 0$. By a standard fact in topology (referred to as the *tube lemma* in Munkres [21]), U contains some tube $A \times \mathbb{T}^1$ about s_0 — where $A \subset \mathbb{T}^1$ is an open arc containing s_0. By mutual continuity of μ with respect to $\mathcal{H}^1\lfloor_{\mathbb{T}^1}$ it then follows that

$$0 < \mu[A] = \gamma[A \times \mathbb{T}^1] \leq \gamma[U] = 0,$$

which is a contradiction — hence the lemma. □

Proposition 4.3.2 (uniqueness of dual optimizer). *Consider the toy model (1.3) and its constant speed parametrizations (1.4). For each $0 \leq \beta \leq 1$ if $(u^\beta = u^\beta_{c_\beta c^\beta}, u^\beta_{c_\beta})$ represent the dual optimizers from Proposition-4.2.5, then the c_β-sub-differential $\partial_{c_\beta} u^\beta$ contains Z_β. Moreover apart from an additive constant, u^β is uniquely determined a.e. on $\mathbb{T}^1$*

Proof. Fix a $\beta \in [0, 1]$. For each $t_0 \in \mathbb{T}^1$ define the function $F_{t_0} : \mathbb{T}^1 \longrightarrow \mathbb{R}$ by $F_{t_0}(s) := u^\beta(s) + u^\beta_{c_\beta}(t_0) - c(\beta, s, t_0)$. Then $F_{t_0}(s)$ is non-negative on $\mathbb{T}^1$ since $(u^\beta, u^\beta_{c_\beta}) \in \mathcal{A}_\beta$. By (4.36), $u^\beta(s) + u^\beta_{c_\beta}(t) - c(\beta, s, t) = 0$ at γ^β_o-a.e. $(s, t) \in \mathbb{T}^2$ — that is for all $(s, t) \in Z_\beta$. In other words $F_{t_0}(s)$ is minimized for all $s \in \mathbb{T}^1$ for which (s, t_0) belongs to Z_β. Lemma-4.3.1 asserts the existence of an $s_0 \in \mathbb{T}^1$ for which $(s_0, t_0) \in Z_\beta$. Then $F_{t_0}(s)$ vanishes at $s = s_0$ forcing $(s_0, t_0) \in \partial_{c_\beta} u^\beta$ by (4.8). This is true for all $(s_0, t_0) \in Z_\beta$ implying $Z_\beta \subset \partial_{c_\beta} u^\beta$.

uniqueness: suppose $u^\beta, \hat{u}^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ are c^β-convex functions with $(u^\beta = u^\beta_{c_\beta c^\beta}, u^\beta_{c_\beta})$ and $(\hat{u}^\beta = \hat{u}^\beta_{c_\beta c^\beta}, \hat{u}^\beta_{c_\beta})$ maximizing $J_\beta(u, v)$ among all pairs of functions in $\mathcal{A}_\beta$. Then $\partial_{c_\beta} u^\beta$ and $\partial_{c_\beta} \hat{u}^\beta$ both contain Z_β. By lipschitz continuity (Lemma-4.1.6) $u^\beta, \hat{u}^\beta$ are differentiable μ-a.e. — Rademacher. Let D^β, $\widehat{D}^\beta$ denote their domains of differentiability in $\mathbb{T}^1$. Then $\mu[\mathbb{T}^1 \backslash D^\beta] = 0 = \mu[\mathbb{T}^1 \backslash \widehat{D}^\beta]$ and their intersection $D^\beta \cap \widehat{D}^\beta$ carries the full μ-measure. Then for all $(s, t) \in [(D^\beta \cap \widehat{D}^\beta) \times \mathbb{T}^1] \cap Z_\beta$ one has $\frac{du^\beta}{ds} = \frac{\partial c}{\partial s}(\beta, s, t) = \frac{d\hat{u}^\beta}{ds}$. But $u^\beta, \hat{u}^\beta$ are absolutely continuous with respect to the quotient metric (4.4) by their lipschitz continuity. Then the difference $u^\beta - \hat{u}^\beta$ is also absolutely continuous with $\frac{d}{ds}(u^\beta - \hat{u}^\beta) = 0$ μ-a.e. on $\mathbb{T}^1$. This implies $u^\beta(s) - \hat{u}^\beta(s) = \text{constant}$ μ-a.e. $s \in \mathbb{T}^1$. This completes the proof of the proposition. $\square$

The next proposition gives a convergence property for the primal and the dual solutions:

Proposition 4.3.3. *Let a sequence $\gamma_o^n \in \Gamma_{\beta_n}$ of optimal solutions converge weak-$*$ to γ. If $\operatorname{spt}\gamma_o^n$ lies in the c_β-subdifferential $\partial_{c_n} u^n$ of the c^β-convex functions $u^n \in \mathcal{B}_0^{\beta_n}$ for each n and if u^n converges uniformly to u, then $\operatorname{spt}\gamma \subset \partial_{c_{\beta_0}} u$ where $\beta_0 = \lim_{n\to\infty}\beta_n$.*

Proof. From Lemma-9 of McCann [20] the weak-$*$ limit γ has the same marginals as γ_o^n and that $\operatorname{spt}\gamma$ is c_β-cyclically monotone. By compactness of $\mathcal{B}_0$ from Proposition-4.2.2 the uniform limit of the potentials satisfies $u = u_{c_{\beta_o} c^{\beta_0}} \in \mathcal{B}_0^{\beta_0}$; and from Lemma-4.2.1 the c_n-transforms satisfy $u^n_{c_n} \to u_{c_{\beta_0}}$ — here $c_n := c_{\beta_n}$. By optimality, for each n one gets from (4.36):

$$\int_{\mathbb{T}^2} [u^n(s) + u^n_{c_n}(t) - c(\beta_n, s, t)]\, d\gamma_o^n(s,t) = 0. \tag{4.37}$$

Denoting the cost function by c_β and suppressing s and t for convenience, one can write using (4.37):

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2}[u(s) + u_{c_{\beta_0}}(t) - c(\beta_0, s, t)]\, d\gamma(s,t)\right| \\
=\ &\left|\int_{\mathbb{T}^2}(u^n + u^n_{c_n} - c_n)\, d\gamma_o^n - \int_{\mathbb{T}^2}(u + u_{c_{\beta_0}} - c_{\beta_0})\, d\gamma\right| \\
=\ &\Big|\int_{\mathbb{T}^2}(u^n - u)\, d\gamma_o^n + \int_{\mathbb{T}^2}(u^n_{c_n} - u_{c_{\beta_0}})\, d\gamma_o^n + \int_{\mathbb{T}^2}(c_n - c_{\beta_0})\, d\gamma_o^n \\
&+ \int_{\mathbb{T}^2}(u + u_{c_{\beta_0}} - c_{\beta_0})\, d\gamma_o^n - \int_{\mathbb{T}^2}(u + u_{c_{\beta_0}} - c_{\beta_0})\, d\gamma\Big| \qquad (4.38)\\
\leq\ &\|u^n - u\|_{L^\infty} + \|u^n_{c_n} - u_{c_{\beta_0}}\|_{L^\infty} + \|c_n - c_{\beta_0}\|_{L^\infty} \\
&+ \left|\int_{\mathbb{T}^2}[u + u_{c_{\beta_0}} - c_{\beta_0}]\, d\gamma_o^n - \int_{\mathbb{T}^2}[u + u_{c_{\beta_0}} - c_{\beta_0}]\, d\gamma\right|
\end{aligned}$$

where we used $\int_{\mathbb{T}^2} d\gamma_o^n = 1$ in the last step. Letting the limit go to infinity in (4.38), the uniform convergences of $u^n, u^n_{c_n}$ and c_n and the weak-$*$ convergence of γ_o^n yield

$$\int_{\mathbb{T}^2} [u(s) + u_{c_{\beta_0}}(t) - c(\beta_0, s, t)]\, d\gamma(s,t) = 0\,. \tag{4.39}$$

The remarks on equation (4.36) then enables one to conclude:

$$(u, u_{c_{\beta_0}}) \in \arg\min_{\mathcal{A}_{\beta_0}} J_{\beta_0}(u, v) \quad \text{and} \quad \gamma \in \Gamma_{\beta_0}.$$

The claim then follows from Proposition-4.3.2 by which $\partial_{c_{\beta_0}} u$ contains the support of all optimal solutions in Γ_{β_0} — in particular $\operatorname{spt} \gamma \subset \partial_{c_{\beta_0}} u$. □

Chapter 5

Persistence of uniqueness under perturbation of the cost

5.1 Dual potentials and optimal transport maps

Based on the geometry of the optimal measures for $\beta = 0, 1$, we develop in this chapter a perturbative argument to achieve uniqueness for the general case with β ranging over values close to zero or one where the the cost function (1.8) penalizes both translation and rotation. All the arguments pertain to the toy model with the ultimate goal to determine γ_o^β uniquely in terms of the prescribed measures μ, ν and the optimal transport maps in Definition-5.1.4. Most of the analysis is restricted to $1 - \epsilon < \beta \leq 1$ for $\epsilon > 0$ — the conclusions for $0 \leq \beta < \epsilon$ can be retrieved by replacing β by $1 - \beta$. We first state without proof a lemma from Gangbo and McCann [10] that characterizes a pair of distinct points on the boundary of a strictly convex domain in $\mathbb{R}^2$ in terms of their outward unit normals.

Lemma 5.1.1. *Take distinct points* $\mathbf{y}_1, \mathbf{y}_2 \in \partial\Lambda$ *on the boundary of a strictly convex domain* $\Lambda \subset \mathbb{R}^2$*. Denote by* $N_\Lambda(\mathbf{y}_k)$ *the set of outward unit normals to* $\partial\Lambda$ *at* $\mathbf{y}_k$ *for* $k = 1, 2$ *— see Definition-3.1.1. Then every outward unit normal* $\mathbf{q}_1 \in N_\Lambda(\mathbf{y}_1)$ *to* $\partial\Lambda$ *at* $\mathbf{y}_1$ *satisfies* $\mathbf{q}_1 \cdot (\mathbf{y}_1 - \mathbf{y}_2) > 0$*. Similarly, each* $\mathbf{q}_2 \in N_\Lambda(\mathbf{y}_2)$ *satisfies* $\mathbf{q}_2 \cdot (\mathbf{y}_1 - \mathbf{y}_2) < 0$*.*

By lipschitz continuity and Rademacher's theorem the optimal dual potentials

$u^\beta \in \mathcal{B}_0^\beta$ are differentiable $\mathcal{H}^1\lfloor_{\mathbb{T}^1}$ and hence μ-a.e. on $\mathbb{T}^1$. For each fixed $\beta \in [0,1]$ we denote

$$D^\beta := \text{ the domain of differentiability of } u^\beta(s) \text{ on } \mathbb{T}^1, \tag{5.1}$$

and demonstrate in the next proposition that each $s \in D^\beta$ supplies to at most two potential destinations on the support of ν.

Proposition 5.1.2 (at most two images a.e.). *Consider the toy model (1.3)-(1.4). Let $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ represent the c^β-convex potential from Proposition-4.3.2 for which* $\operatorname{spt}\gamma_o^\beta \subset \partial_{c_\beta} u^\beta$ *for each optimal solution γ_o^β from (1.6). Then given $\beta \in [0,1]$ for each $s \in D^\beta$ from (5.1), the c_β-subgradient $\partial_{c_\beta} u^\beta(s)$ contains at most two points of* $\operatorname{spt}\nu$ *with $\partial_{c_\beta} u^\beta(s) \subseteq \{t_1, t_2\}$ satisfying $\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t_1) \geq 0$ and $\mathbf{n}_\Omega(\mathbf{s}_0) \cdot \mathbf{n}_\Lambda(t_2) \leq 0$ with strict inequalities unless $t_1 = t_2$.*

Proof. Fix $\beta \in [0,1]$, $s_0 \in D^\beta$ and $t \in \partial_{c_\beta} u^\beta(s_0)$. Then by hypothesis the function $u^\beta(s) + u^\beta_{c_\beta}(t) - c(\beta, s, t) \geq 0$ and is minimized by (4.8) and (4.9) at $s = s_0$ — in which case one has

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=s_0} u^\beta(s) &= \frac{\partial}{\partial s}\bigg|_{s=s_0} c(\beta, s, t) \\ &= v_\Omega \, \mathbf{T}_\Omega(s_0) \cdot [(1-\beta)\mathbf{y}(t) + \beta \, K_\Omega(s_0) \, \mathbf{n}_\Lambda(t)]. \end{aligned}$$

The set $\{(1-\beta)\mathbf{y}(t) + \beta \, K_\Omega(s_0) \, \mathbf{n}_\Lambda(t) \mid t \in \mathbb{T}^1\}$ represent the boundary of a uniformly blown up copy $(1-\beta)\Lambda + r\mathbf{B}_1(\mathbf{0})$ of the domain $(1-\beta)\Lambda$; here $r = r(\beta, s_0) := \beta \, K_\Omega(s_0)$ and $\mathbf{B}_1(\mathbf{0})$ the closed unit ball in $\mathbb{R}^2$. Denoting these points by $\mathbf{b}_\Lambda \circ \mathbf{y}(t)$ for $\mathbf{y}(t) \in \partial\Lambda$, the above equation can be rewritten as

$$\mathbf{T}_\Omega(s_0) \cdot \mathbf{b}_\Lambda \circ \mathbf{y}(t) = \mathrm{C}_\beta(\mathrm{s}_0) \tag{5.2}$$

for some constant $\mathrm{C}_\beta(s_0)$ depending on β, s_0 and u^β. The solutions are those $t \in \mathbb{T}^1$ for which the line $\mathrm{L} := \{\mathbf{z} \in \mathbb{R}^2 \mid \mathbf{z} \cdot \mathbf{T}_\Omega(s_0) = \mathrm{C}_\beta(s_0)\}$, perpendicular to $\mathbf{T}_\Omega(s_0)$, intersects the blown up boundary $\mathbf{b}_\Lambda(\partial\Lambda)$. Lemma-4.3.1 and Proposition-4.3.2 guarantee at least one solution of (5.2) by non-emptiness of $\partial_{c_\beta} u^\beta(s_0)$, while strict convexity of $\partial\Lambda$ and hence of $\mathbf{b}_\Lambda(\partial\Lambda)$ ensures at most two solutions $t_1 \neq t_2 \in \mathbb{T}^1$ for which $\mathbf{b}_\Lambda \circ \mathbf{y}(t_1)$ and $\mathbf{b}_\Lambda \circ \mathbf{y}(t_2)$ belong to $\mathrm{L} \cap \mathbf{b}_\Lambda(\partial\Lambda)$. Interchange $t_1 \leftrightarrow t_2$ if necessary to make $\mathbf{b}_\Lambda \circ \mathbf{y}(t_1) - \mathbf{b}_\Lambda \circ \mathbf{y}(t_2)$ parallel to $\mathbf{n}_\Omega(s_0)$. Then Lemma-5.1.1 asserts that

$$\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t_1) \geq 0 \quad \text{and} \quad \mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t_2) \leq 0,$$

where we identified the outward unit normals $\tilde{\mathbf{n}}(\mathbf{b}_\Lambda \circ \mathbf{y}(t)) = \tilde{\mathbf{n}}_\Lambda(\mathbf{y}(t)) = \mathbf{n}_\Lambda(t)$ for all $t \in \mathbb{T}^1$. When L is tangent to $\mathbf{b}_\Lambda(\partial\Lambda)$, a unique solution exists with $t_1 = t_2 = t$ and $\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t) = 0$ — the point $\mathbf{x}(s_0) \in \partial\Omega$ is then mapped onto a unique point $\mathbf{y}(t) \in \partial\Lambda$ where the outward unit normal $\mathbf{n}_\Lambda(t)$ is at $90°$ with the initial orientation $\mathbf{n}_\Omega(s_0)$. This completes the proof of the proposition. □

Remark 5.1.3 (c_β-subdifferentials and images). Geometrically what it means for a point $(s_0, t_0) \in \mathbb{T}^2$ on the flat torus to belong to the c_β-subdifferential $\partial_{c_\beta} u^\beta$ of the potential $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ is that $t_0 \in \mathbb{T}^1$ gives a translate of the shifted cost $c(\beta, s, t_0) - v^\beta(t_0)$ which is dominated by $u^\beta(s)$, with equality at $s = s_0$ where it is tangent to $u^\beta(s)$. By construction $u^\beta(s)$ is finite on $\mathbb{T}^1$ and therefore c_β-subdifferentiable at each $s_0 \in \mathbb{T}^1$; while Lemma-4.3.1 and Proposition-4.3.2 guarantee at least one $t_0 \in \mathbb{T}^1$ for which $(s_0, t_0) \in \partial_{c_\beta} u^\beta$. By Proposition-5.1.2, for all $s_0 \in D^\beta$ this $t_0 \in \partial_{c_\beta} u^\beta(s_0)$ is characterized uniquely by the equation $\frac{du^\beta}{ds}(s_0) = \frac{\partial c}{\partial s}(\beta, s_0, t_0)$ and the sign of the dot product $\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t_0)$. If however $u^\beta(s)$ fails to be differentiable at $s = s_0$, then the c_β-subdifferentiability, Lemma-4.3.1 and Proposition-4.3.2 ensure that $u^\beta(s)$ still supports a shifted translate of the cost $c(\beta, s, t)$ touching it from below for some $t_0 \in \mathbb{T}^1$ that satisfies $c(\beta, s_0, t_0) = u^\beta(s_0) + u^\beta_{c_\beta}(t_0)$. An argument analogous to that in the proof of Lemma-C.7 of Gangbo and McCann [9] shows that: if $(s_0, t_0) \in \partial_{c_\beta} u^\beta$ then the subgradient of $u^\beta(s)$ at $s = s_0$ contains the subgradient of $c(\beta, s, t_0)$ at $s = s_0$, i.e. $\partial_{s}.c(\beta, s_0, t_0) \subset \partial.u^\beta(s_0)$ — where the subscript s indicates that the subgradient of the cost function is with respect to the variable s (see Chapter-2 for notations and definitions). C^2-differentiability of the cost function on $\mathbb{T}^2$ forces $\partial_{s}.c(\beta, s, t_0) = \{\frac{\partial c}{\partial s}(\beta, s, t_0)\} \neq \emptyset$ for each $s \in \mathbb{T}^1$ and consequently ensuring the subdifferentiability of $u^\beta(s)$ everywhere on $\mathbb{T}^1$. By Lemma-B.1.2, $u^\beta(s)$ is uniformly semi-convex on $\mathbb{T}^1$ and therefore has left and right derivatives where it fails to be differentiable. Denoting by $u^{\beta'}_{-}(s)$ and $u^{\beta'}_{+}(s)$ the left and right derivatives of $u^\beta(s)$ on $\mathbb{T}^1$ and by $[x, y]$ the convex hull $[x, y] := \{\alpha x + (1 - \alpha) y \mid 0 \leq \alpha \leq 1\}$ of the points x, y on the real line $\mathbb{R}$, we note that for each $s \in \mathbb{T}^1$ the subgradient of $u^\beta(s)$ is given by $\partial.u^\beta(s) = [{u^\beta}'_+(s), {u^\beta}'_-(s)]$ with ${u^\beta}'_+(s) = {u^\beta}'_-(s) = \frac{du^\beta}{ds}(s)$ on D^β that is μ-a.e. $s \in \mathbb{T}^1$.

The above string of arguments motivates the following definition for images under the optimal transportation (1.5):

Definition 5.1.4 (optimal maps). For the optimal transport problem (1.5) on the toy model (1.3)-(1.4), we define for each $\beta \in [0,1]$ the mappings $t^\pm_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ — called the *optimal transport maps* — as follows: for each $s \in \operatorname{spt}\mu = \mathbb{T}^1$, the images $t^\pm_\beta(s) \in \operatorname{spt}\nu = \mathbb{T}^1$ under these maps satisfy

$$\begin{array}{rcl} \dfrac{\partial c}{\partial s}(\beta, s, t^\pm_\beta(s)) & \in & [{u^\beta}'_+(s), {u^\beta}'_-(s)] \\ u^\beta(s) + u^\beta_{c_\beta}(t^\pm_\beta(s)) & = & c(\beta, s, t^\pm_\beta(s)) \\ \mathbf{n}_\Omega(s)\cdot\mathbf{n}_\Lambda(t^+_\beta(s)) & \geq & 0 \\ \mathbf{n}_\Omega(s)\cdot\mathbf{n}_\Lambda(t^-_\beta(s)) & \leq & 0 \end{array} \tag{5.3}$$

with equalities in the dot products if and only if the point $\mathbf{x}(s) \in \partial\Omega$ gets mapped to a unique point $\mathbf{y}(t^+_\beta(s)) = \mathbf{y}(t^-_\beta(s)) \in \partial\Lambda$ whose outward unit normal on $\partial\Lambda$ is orthogonal to $\mathbf{n}_\Omega(s)$. Here $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ and is its c_β-transform $u^\beta_{c_\beta}$ are the unique dual optimizer from Proposition-4.3.2.

5.2 Perturbation of β

The purpose of this section is to develop the necessary formulations to achieve uniqueness of optimal solutions when β is perturbed from the value one to include the effects of both rotation and translation in the cost function.

Lemma 5.2.1. *Consider the toy model (1.3)-(1.4). Given $(\beta_0, s_0, t_0) \in [0,1] \times \mathbb{T}^1 \times \mathbb{T}^1$ and the cost function $c : [0,1]\times\mathbb{T}^1\times\mathbb{T}^1 \longrightarrow \mathbb{R}$ defined by (1.8), if $(s_0, t_0) \in \Sigma^+ \cup \Sigma^-$ then there exists a unique $t_1 \in \mathbb{T}^1 \backslash \{t_0\}$ so that*

$$\frac{\partial c}{\partial s}(\beta_0, s_0, t_0) = \frac{\partial c}{\partial s}(\beta_0, s_0, t_1)\,. \tag{5.4}$$

Furthermore $(s_0, t_0) \in \Sigma^+$ if and only if $(s_0, t_1) \in \Sigma^-$. On the other hand if $(s_0, t_0) \in \Sigma^0$ then no $t_1 \in \mathbb{T}^1 \backslash \{t_0\}$ satisfies (5.4).

Proof. Fix $s_0 \in \mathbb{T}^1$, $t_0 \in \mathbb{T}^1$ and $\beta_0 \in [0,1]$. For any $t \in \mathbb{T}^1$ that solves equation (5.4) one has $\frac{\partial c}{\partial s}(\beta_0, s_0, t_0) - \frac{\partial c}{\partial s}(\beta_0, s_0, t) = 0$ — following the proof of the above Proposition-5.1.2 this can be rewritten as

$$\mathbf{T}_\Omega(s_0) \cdot [\mathbf{b}_\Lambda \circ \mathbf{y}(t_0) - \mathbf{b}_\Lambda \circ \mathbf{y}(t)] = 0\,, \tag{5.5}$$

where

$$\mathbf{b}_\Lambda \circ \mathbf{y}(t) := (1-\beta)\mathbf{y}(t) + \beta\, K_\Omega(s_0)\, \mathbf{n}_\Lambda(t). \tag{5.6}$$

Denote by L_0 the line in $\mathbb{R}^2$ that is perpendicular to $\mathbf{T}_\Omega(s_0)$ and passes through the point $\mathbf{b}_\Lambda \circ \mathbf{y}(t_0)$ of the uniformly blown-up boundary $\mathbf{b}_\Lambda(\partial\Lambda)$. The solutions of equation (5.5) are those $t \in \mathbb{T}^1$ for which L_0 intersects $\mathbf{b}_\Lambda(\partial\Lambda)$ at $\mathbf{b}_\Lambda \circ \mathbf{y}(t)$. Strict convexity of $\partial\Lambda$ and hence of $\mathbf{b}_\Lambda(\partial\Lambda)$ ensures the existence of exactly one such t, denoted t_1 — with $t_1 = t_0$ when L_0 is tangent to $\mathbf{b}_\Lambda(\partial\Lambda)$ at $\mathbf{b}_\Lambda \circ \mathbf{y}(t_0)$. For $(s_0, t_0) \in \Sigma^+ \cup \Sigma^-$, depending on whether $\mathbf{b}_\Lambda \circ \mathbf{y}(t_0) - \mathbf{b}_\Lambda \circ \mathbf{y}(t_1)$ is parallel or anti-parallel to $\mathbf{n}_\Omega(s_0)$, the dot product $\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t_0)$ is strictly positive or strictly negative (by Lemma-5.1.1) thus forcing $(s_0, t_0) \in \Sigma^+$ if and only if $(s_0, t_1) \in \Sigma^-$ — see (3.12) for definitions of $\Sigma^\pm$. If $(s_0, t_0) \in \Sigma^0$ then $\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t_0) = 0$ which forces L_0 to be tangent to $\mathbf{b}_\Lambda(\partial\Lambda)$ and t_1 to degenerate into t_0 — whence follows the lemma. ∎

The following definitions are based on the observation of Lemma-5.2.1 and the c_β-monotonicity (1.10) or rather its variant (1.11):

Definition 5.2.2. *Define the maps* $f : [0,1] \times \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1 \longrightarrow \mathbb{R}$ *and* $F : [0,1] \times \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1 \longrightarrow \mathbb{R}$ *by*

$$f(\beta, s, t', t'') := \int_{t'}^{t''} \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t)\, dt \tag{5.7}$$

$$F(\beta, \hat{s}, s_0, t', t'') := \int_{s_0}^{\hat{s}} \int_{t'}^{t''} \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t)\, ds\, dt. \tag{5.8}$$

Remark 5.2.3 (f, F and $\partial_{c_\beta} u^\beta$). Given $\beta \in [0,1]$, $s \in \mathbb{T}^1$ and $t' \neq t''$, observe that $(s_0, t_0, t_1) = (s, t', t'')$ satisfies (5.4) if and only if $f(\beta, s, t', t'') = 0$. For distinct points $(s_1, t_1), (s_2, t_2) \in \partial_{c_\beta} u^\beta \subset \mathbb{T}^2$, the c_β-monotonicity of $\partial_{c_\beta} u^\beta$ is equivalent to non-negativity of $F(\beta, s_2, s_1, t_1, t_2)$ — compare (5.8) with (1.10) and (1.11). In other words the inequality

$$F(\beta, \hat{s}, s_0, t', t'') \geq 0 \tag{5.9}$$

gives a reformulation of the c_β-monotonicity (1.10) for the points (s_0, t') and $(\hat{s}, t'')$ formed by pairing the second argument with the fifth and the third argument with the fourth. This gives a consistency check (through Propositions-5.2.6 and -5.2.8 under suitable constraints) for a pair of points on $\mathbb{T}^2$ to belong to spt γ_o^β.

Remark 5.2.4 (notation convention). In the following analysis whenever we restrict the functions $f(\beta, s, t', t'')$ and $F(\beta, \hat{s}, s_0, t', t'')$ to points (t', t'') for which Lemma-5.2.1 holds, as a convention we will use double prime superscript on t when it belongs to $\overline{\Sigma^-}(s)$ and single prime when $t \in \overline{\Sigma^+}(s)$. Here the bar over the sets $\Sigma^\pm(s)$ denotes the closures: $\overline{\Sigma^\pm}(s) = \Sigma^\pm(s) \cup \Sigma^0(s)$.

We recall from Proposition-3.2.7 that for $\beta = 1$ there can exist at most one point where spt γ_o^1 intersects Σ_P^0 and one point where spt γ_o^1 intersects Σ_N^0. In the remainder of this section our aim is to interpret disjointness of spt γ_o^1 from Σ^0 as a non-degeneracy condition for critical points of the function $F(1, \hat{s}, s_0, t', t'')$. We then find an $\epsilon > 0$ for which spt γ_o^β continues to have empty intersection with Σ^0 for all values of β in $1 - \epsilon < \beta < 1$ — Propositions-5.2.7. The stability of non-degenerate critical points under small perturbations then enable us to give a perturbative argument for the persistence of uniqueness of optimal solutions γ_o^β for each β in the range $1 - \epsilon < \beta < 1$ — Proposition-5.2.8 and Theorem-5.4.7. With this aim we define:

Definition 5.2.5 ($S_0(\beta)$). For each fixed $\beta \in [0, 1]$, let $(u^\beta = u^\beta_{c_\beta c^\beta}, u^\beta_{c_\beta}) \in \mathcal{A}_\beta$ be the unique dual optimizer of Proposition-4.3.2. Define by $S_0(\beta) \subset \mathbb{T}^1$ the subset consisting of all $s \in \mathbb{T}^1$ for which the c_β-subgradient $\partial_{c_\beta} u^\beta(s)$ intersects $\Sigma^0(s)$ at some $t \in \mathbb{T}^1$ so that $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t) = 0$, i.e.

$$S_0(\beta) := \{s \in \mathbb{T}^1 \mid \partial_{c_\beta} u^\beta(s) \cap \Sigma^0(s) \neq \emptyset\}. \tag{5.10}$$

Proposition 5.2.6. *For $0 \leq \beta \leq 1$ and $s_0 \in \mathbb{T}^1$, fix $t', t'' \in \mathbb{T}^1$ so that $f(\beta, s_0, t', t'')$ $= 0$ with either t' or t'' in $\partial_{c_\beta} u^\beta(s_0)$. Assume $S_0(\beta = 1) = \emptyset$. Then the function $\hat{s} \longrightarrow F(1, \hat{s}, s_0, t', t'')$, defined by (5.8), is $C^2(\mathbb{T}^1)$ smooth, non-negative and has no critical points except for a global minimum at $\hat{s} = s_0$ and a global maximum at some $s_1 \neq s_0$. Both critical points are non-degenerate, meaning $\frac{\partial^2 F}{\partial \hat{s}^2}(1, s_k, s_0, t', t'') \neq 0$ for $k = 0, 1$.*

Proof. Set $\beta = 1$. Let $u \in \mathcal{B}^1_0$ denote the c^β-convex potential for which the c_β-subdifferential $\partial_{c_1} u$ contains the support of the corresponding optimal solution γ^1_o. The hypothesis $S_0(\beta = 1) = \emptyset$ precludes $\partial_{c_1} u$ from intersecting Σ^0 — thus forcing $t' \neq t''$ for all $s \in \mathbb{T}^1$ whenever $f(1, s, t', t'') = 0$ with either t' or t'' in $\partial_{c_1} u(s)$. Proposition-3.1.4 asserts that

$$\{(s, t^+_1(s))\}_{s \in \mathbb{T}^1} \subset \operatorname{spt} \gamma^1_o \subset \partial_{c_1} u \tag{5.11}$$

with the optimal map $t^+_1 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ (Definition-5.1.4) a homeomorphism, so that one can identify $t' = t^+_1(s)$ for all $s \in \mathbb{T}^1$ whenever s, t', t'' satisfy the hypotheses.

Fix $s_0 \in \mathbb{T}^1$. Fix $t'_{01} := t^+_1(s_0)$ and $t''_{01} := t''(s_0, t'_0)$ in accordance with the hypotheses. Then the C^2-smoothness of $F(\beta, \hat{s}, s_0, t'_{01}, t''_{01})$ on $\hat{s}$ follows directly from equation (5.8) by the C^2-differentiability of the cost function $c(1, s, t) = \mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t)$ and the continuous dependence of t'_{01} hence of t''_{01} on s_0 through $f(1, s_0, t'_{01}, t''_{01}) = 0$. Using $f(1, s, t'_{01}, t''_{01}) = \int_{t'_{01}}^{t''_{01}} \frac{\partial^2 c}{\partial s \partial t}(1, s, t)\, dt$ rewrite (5.8) as

$$F(1, \hat{s}, s_0, t'_{01}, t''_{01}) = \int_{s_0}^{\hat{s}} f(1, s, t'_{01}, t''_{01})\, ds\,. \tag{5.12}$$

Differentiating (5.12) with respect to $\hat{s}$ one gets

$$\begin{aligned} \frac{\partial F}{\partial \hat{s}}(1, \hat{s}, s_0, t'_{01}, t''_{01}) &= f(1, \hat{s}, t'_{01}, t''_{01}) \\ &= \frac{\partial c}{\partial s}(1, \hat{s}, t''_{01}) - \frac{\partial c}{\partial s}(1, \hat{s}, t'_{01}) \\ &= -v_\Omega\, K_\Omega(\hat{s})\, \mathbf{T}_\Omega(\hat{s}) \cdot [\, \mathbf{n}_\Lambda(t'_{01}) - \mathbf{n}_\Lambda(t''_{01})]. \end{aligned} \tag{5.13}$$

By construction $f(1, \hat{s}, t'_{01}, t''_{01}) = 0$ at $\hat{s} = s_0$ forcing the vector $\mathbf{n}_\Lambda(t'_{01}) - \mathbf{n}_\Lambda(t''_{01})$ to be perpendicular to $\mathbf{T}_\Omega(s_0)$. This vector is in fact parallel to $\mathbf{n}_\Omega(s_0)$ by Lemma-5.1.1 through the identity $t'_{01} = t^+_1(s_0)$ satisfying $\mathbf{n}_\Omega(s_0) \cdot \mathbf{n}_\Lambda(t^+_1(s_0)) \geq 0$. The hypothesis $S_0(\beta = 1) = \emptyset$ and the strict convexity of Ω imply there can be exactly two distinct points $\hat{s} \in \mathbb{T}^1$ where $f(1, \hat{s}, t'_{01}, t''_{01})$ vanishes. These points — called s_0 and s_1 — constitute the critical points of $F(1, \hat{s}, s_0, t'_{01}, t''_{01})$ by (5.13) and are characterized respectively by the normals $\mathbf{n}_\Omega(s_0)$ and $\mathbf{n}_\Omega(s_1)$ parallel and anti-parallel to $\mathbf{n}_\Lambda(t'_{01}) - \mathbf{n}_\Lambda(t''_{01})$. Using $\dot{\mathbf{T}}_\Omega(s) = -v_\Omega\, K_\Omega(s)\, \mathbf{n}_\Omega(s)$, a second derivative of (5.13) with respect to $\hat{s}$ gives

$$\begin{aligned} \frac{\partial^2 F}{\partial \hat{s}^2}(1, \hat{s}, s_0, t'_{01}, t''_{01}) &= \frac{\dot{K}_\Omega(\hat{s})}{K_\Omega(\hat{s})}\, \frac{\partial F}{\partial \hat{s}}(1, \hat{s}, s_0, t'_{01}, t''_{01}) \\ &+ v^2_\Omega\, K_\Omega(\hat{s})^2\, \mathbf{n}_\Omega(\hat{s}) \cdot [\, \mathbf{n}_\Lambda(t'_{01}) - \mathbf{n}_\Lambda(t''_0)]\,. \end{aligned} \tag{5.14}$$

Whence one can conclude

$$\begin{aligned} \frac{\partial^2 F}{\partial \hat{s}^2}(1, s_0, s_0, t'_{01}, t''_{01}) &= \frac{\partial f}{\partial \hat{s}}(1, s_0, t'_{01}, t''_{01}) > 0 \\ \frac{\partial^2 F}{\partial \hat{s}^2}(1, s_1, s_0, t'_{01}, t''_{01}) &= \frac{\partial f}{\partial \hat{s}}(1, s_1, t'_{01}, t''_{01}) < 0\,, \end{aligned} \tag{5.15}$$

so that $\hat{s} = s_0$ and $\hat{s} = s_1$ are respectively the global (being the only critical points) minimizer and maximizer of $F(1, \hat{s}, s_0, t'_{01}, t''_{01})$ on $\mathbb{T}^1$.

For the non-negativity condition we note that by continuity (5.15) further implies $f(1, s, t'_{01}, t''_{01}) > 0$ on the open arc $]\!]s_0, s_1[\![$ of $\mathbb{T}^1$ while $f(1, s, t'_{01}, t''_{01}) \leq 0$ on the complement $[\![s_1, s_0]\!]$, with equality at $s = s_0$ and $s = s_1$. This together with the fact that $\int_{\mathbb{T}^1} f(1, s, t'_{01}, t''_{01})\, ds = 0$ (by periodicity) applied to (5.12) yields $F(1, \hat{s}, s_0, t'_{01}, t''_{01}) \geq 0$ for all $\hat{s} \in \mathbb{T}^1$ and vanishing at $\hat{s} = s_0$, the unique minimizer.

□

Proposition 5.2.7. *If $S_0(\beta = 1) = \emptyset$ then there exists an $\epsilon > 0$ so that $S_0(\beta) = \emptyset$ for all $1 \geq \beta > 1 - \epsilon$.*

Proof. To derive a contradiction assume that $S_0(\beta = 1) = \emptyset$ but for all $\epsilon > 0$ there exists a $\beta \in\,]\,1 - \epsilon, 1\,]$ for which $S_0(\beta)$ is non-empty. Then by assumption, for each $n \geq 1$, there exists $1 > \beta_n > 1 - \frac{1}{n}$ for which $S_0(\beta_n) \neq \emptyset$. Let $s_n := s(\beta_n) \in S_0(\beta_n)$. Denote by c_n the cost function associated with β_n and by $u^n := u^{\beta_n} = u^{\beta_n}_{c_n c^n}$, $v^n := u^n_{c_n}$ the corresponding dual optimizers with $u^n \in \mathcal{B}_0^{\beta_n}$ of (4.23). Non-emptiness of $S_0(\beta_n)$ implies there exists a $t_n := t(\beta_n, s_n) \in \partial_{c_n} u^n(s_n)$ for which

$$\begin{aligned} u^n(s_n) + v^n(t_n) &= c(\beta_n, s_n, t_n) \\ \frac{\partial^2 c}{\partial s \partial t}(\beta_n, s_n, t_n) &= 0. \end{aligned} \tag{5.16}$$

Since $(\beta_n, u^n) \in \mathcal{B}_0$ from (4.24) and $\mathcal{B}_0$ is compact by Proposition-4.2.2, a subsequence — also denoted (β_n, u^n) — converges to $(1, u) \in \mathcal{B}_0$ as $n \to \infty$. Then $u \in \mathcal{B}_0^1$ by (4.24). Lemma-4.2.1 forces a corresponding subsequence of the c_n-transforms, also denoted by v^n, to converge to $v = u_{c_1}$ as $n \to \infty$. Moreover, compactness of $\mathbb{T}^2$ implies the sequence $(s_n, t_n) \in \partial_{c_n} u^n \subset \mathbb{T}^2$ has a convergent subsequence that converges to some $(s_\infty, t_\infty) \in \mathbb{T}^2$. Hence for a sub-subsequence of (u^n, v^n, s_n, t_n), also denoted by n, equation (5.16) continues to hold. Taking the limit $n \to \infty$ yields:

$$\begin{aligned} u(s_\infty) + u_{c_1}(t_\infty) &= c(1, s_\infty, t_\infty) \\ \frac{\partial^2 c}{\partial s \partial t}(1, s_\infty, t_\infty) &= 0\,. \end{aligned} \tag{5.17}$$

We used continuity in all the arguments β, s and t of the cost function $c(\beta, s, t)$ and its mixed partial $\frac{\partial^2}{\partial s \partial t} c(\beta, s, t)$ and the Lipschitz continuity (hence uniform continuity) of the potentials u^n and v^n to get (5.17). But (5.17) combines with (4.8), (3.12), (5.10) and the hypothesis to assert that $s_\infty \in S_0(\beta = 1) = \emptyset$ — a contradiction — thus proving the proposition. $\square$

The next proposition extends a result proved for $\beta = 1$ in Proposition-5.2.6 to β which are merely close to 1. It employs a perturbation argument which relies on the geometrical condition $S_0(\beta = 1) = \emptyset$ to preclude degeneracies.

Proposition 5.2.8. *If $S_0(\beta = 1) = \emptyset$, there exists an $\epsilon > 0$ such that for fixed $\beta > 1 - \epsilon$ and $f(\beta, s_0, t', t'') = 0$ with either t' or t'' in $\partial_{c_\beta} u^\beta(s_0)$, the function $\hat{s} \longrightarrow F(\beta, \hat{s}, s_0, t', t'')$ is $C^2(\mathbb{T}^1)$ smooth and has no critical points except for a global minimum at $\hat{s} = s_0$ and a global maximum at some $\hat{s}_\beta \neq s_0$. Moreover $F(\beta, \hat{s}, s_0, t', t'') \geq 0$ with equality if and only if $\hat{s} = s_0$. Here t' and t'' are labeled in accordance with Remark-5.2.4 and $u^\beta \in \mathcal{B}_0^\beta$ is the optimal dual potential from Proposition-4.3.2.*

Proof. The hypothesis $S_0(\beta = 1) = \emptyset$ combines with Proposition-5.2.7 to provide an $\epsilon > 0$ for which the set $S_0(\beta)$ continues to be empty for all $1 - \epsilon < \beta \leq 1$. For each such β one therefore has $t'_{0\beta} := t'(\beta, s_0)$ distinct from $t''_{0\beta} := t''(\beta, s_0)$ whenever $f(\beta, s_0, t'_{0\beta}, t''_{0\beta}) = 0$ and either $t'_{0\beta}$ or $t''_{0\beta}$ belongs to $\partial_{c_\beta} u^\beta(s_0)$. Consider this $\epsilon > 0$. The strategy for the proof is to show the uniform convergence of $F(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta})$ and its first and second partials with respect to $\hat{s}$ as $\beta \to 1$ to the corresponding quantities of Proposition-5.2.6 and extend the conclusions there to β close 1.

2. *Claim:* When either $t'_{0\beta}$ or $t''_{0\beta}$ belongs to $\partial_{c_\beta} u^\beta(s_0)$ with $f(\beta, s_0, t'_{0\beta}, t''_{0\beta}) = 0$, a subsequence of $t'_{0\beta}$ converges to $t_1^+(s_0)$ as $\beta \to 1$.

Proof of Claim: Compactness of $\mathbb{T}^2$ allows one to extract a subsequence, also denoted $(t'_{0\beta}, t''_{0\beta})$, that converges to some $(t'_{01}, t''_{01}) \in \mathbb{T}^2$. By hypotheses the convergent subsequence satisfies

$$
\begin{aligned}
{u^\beta}'_-(s_0) \le \frac{\partial c}{\partial s}(\beta, s_\beta, t'_{0\beta}) &\le {u^\beta}'_+(s_0) && (5.18)\\
u^\beta(s_0) + u^\beta_{c_\beta}(t'_{0\beta}) &\ge c(\beta, s_0, t'_{0\beta}) && (5.19)\\
\frac{\partial^2 c}{\partial s^2}(\beta, s_0, t'_{0\beta}) &> 0, && (5.20)
\end{aligned}
$$

where the primes on the potentials represent the s-derivatives. The point $t''_{0\beta}$ satisfies (5.18), (5.19) and a strict reverse inequality in (5.20). Moreover, (5.19) is an equality whenever the point belongs to $\partial_{c_\beta} u^\beta(s_0)$. Recall from Lemma-4.2.1 that as $\beta \to 1$ the uniform convergence $u^\beta \to u$ implies $u^\beta_{c_\beta} \to u_{c_1}$ uniformly. By Proposition-4.3.3 the weak-$*$ limit of the corresponding optimal solutions γ^β_o, with $\operatorname{spt}\gamma^\beta_o \subset \partial_{c_\beta} u^\beta$, then satisfies $\operatorname{spt}\gamma^1_o \subset \partial_{c_1} u$ with $u \in \mathcal{B}^1_0$ the unique dual potential of Proposition-4.3.2 and γ^1_o the unique primal optimizer from Theorem-3.1.3. The potentials u^β and the limit u are uniformly semi-convex by Lemma-B.1.2, while u is continuously differentiable on $\mathbb{T}^1$ — see the remark following Proposition-3.4 of Gangbo and McCann [10]. Passing to sub-subsequences, also denoted by β, the uniform convergence ${u^\beta}'_\pm \to u'$ from Lemma-B.2.1 and the C^2- differentiability of the cost function yield

$$
\begin{aligned}
\frac{\partial c}{\partial s}(1, s_0, t'_{01}) &= \frac{du}{ds}(s_0)\\
u(s) + u_{c_1}(t'_{01}) &\ge c(1, s_0, t'_{01}) && (5.21)\\
\frac{\partial^2 c}{\partial s^2}(1, s_0, t'_{01}) &\ge 0
\end{aligned}
$$

as $\beta \to 1$. Similar relations hold for t''_{01} with the last inequality reversed. The equation $\frac{\partial c}{\partial s}(1, s_0, t) = \frac{du}{ds}(s_0)$ has at most two solutions $t \in \mathbb{T}^1$ with at least one of them in $\partial_{c_1} u(s_0)$ by hypotheses — thus making either t'_{01} or t''_{01} belong to $\partial_{c_1} u(s_0)$. This combines with $\operatorname{graph}(t^+_1) \subset \operatorname{spt}\gamma^1_o \subset \partial_{c_1} u$ from Proposition-3.1.4 to assert $t'_{01} = t^+_1(s_0)$ to prove the claim.

3. Setting $f(\beta, s_0, t'_{0\beta}, t''_{0\beta}) = 0$, the C^2-differentiability of the cost function and continuity of all its derivatives in β force the corresponding limits $(t^+_1(s_0), t''_{01})$ to satisfy $f(1, s_0, t_1^+(s_0), t''_{01}) = 0$. Using (5.8) one gets

$$\begin{aligned} \frac{\partial F}{\partial \hat{s}}(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta}) &= f(\beta, \hat{s}, t'_{0\beta}, t''_{0\beta}) & (5.22)\\ \frac{\partial^2 F}{\partial \hat{s}^2}(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta}) &= \frac{\partial^2 c}{\partial \hat{s}^2}(\beta, \hat{s}, t''_{0\beta}) - \frac{\partial^2 c}{\partial \hat{s}^2}(\beta, \hat{s}, t'_{0\beta}), & (5.23)\end{aligned}$$

to derive the uniform convergences:

$$\begin{aligned} F(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta}) &\longrightarrow F(1, \hat{s}, s_0, t_1^+(s_0), t''_{01}) & (5.24)\\ \frac{\partial F}{\partial \hat{s}}(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta}) &\longrightarrow \frac{\partial F}{\partial \hat{s}}(1, \hat{s}, s_0, t_1^+(s_0), t''_{01}) & (5.25)\\ \frac{\partial^2 F}{\partial \hat{s}^2}(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta}) &\longrightarrow \frac{\partial^2 F}{\partial \hat{s}^2}(1, \hat{s}, s_0, t_1^+(s_0), t''_{01}) & (5.26)\end{aligned}$$

as $\beta \to 1$ and to conclude the C^2-differentiability of $\hat{s} \longrightarrow F(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta})$.

4. From Proposition-5.2.6, $\hat{s} \longrightarrow F(1, \hat{s}, s_0, t'_{01}, t''_{01})$ is non-negative with exactly two non-degenerate critical points — a global minimum at $\hat{s} = s_0$ and a global maximum at $\hat{s} = s_1$:

$$\begin{aligned} \frac{\partial F}{\partial \hat{s}}(1, s_0, s_0, t'_{01}, t''_{01}) &= 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial \hat{s}^2}(1, s_0, s_0, t'_{01}, t''_{01}) > 0\\ \frac{\partial F}{\partial \hat{s}}(1, s_1, s_0, t'_{01}, t''_{01}) &= 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial \hat{s}^2}(1, s_1, s_0, t'_{01}, t''_{01}) < 0 \end{aligned} \qquad (5.27)$$

— see (5.15), with the function strictly increasing on the arc $]\!]s_0, s_1[\![\subset \mathbb{T}^1$ and strictly decreasing on $]\!]s_1, s_0[\![$.

5. In the following analysis we suppress the dependence on s_0, $t'_{0\beta}$ and $t''_{0\beta}$ for convenience to write

$$\begin{aligned} F_\beta(\hat{s}) &:= F(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta})\\ F_1(\hat{s}) &:= F(1, \hat{s}, s_0, t'_{01}, t''_{01}). \end{aligned} \qquad (5.28)$$

From **4** since $\frac{\partial^2 F_1}{\partial \hat{s}^2}(s_0) > 0$ and $\frac{\partial^2 F_1}{\partial \hat{s}^2}(s_1) < 0$, there must exist at least two points of inflections $s_2 \in]\!]s_0, s_1[\![$ and $s_3 \in]\!]s_1, s_0[\![$ with $\frac{\partial^2 F_1}{\partial \hat{s}^2}(s_k) = 0$, $k = 2, 3$, and $\frac{\partial F_1}{\partial \hat{s}}(s_2) > 0$ and $\frac{\partial F_1}{\partial \hat{s}}(s_3) < 0$. Let A_0 and A_1 be closed arcs about the points s_0 and s_1 respectively and small enough so that they do not contain s_2 or s_3 and

$$\begin{aligned} \tfrac{\partial^2 F_1}{\partial \hat{s}^2}(\hat{s}) &> 0 \quad \text{on } A_0\\ \tfrac{\partial^2 F_1}{\partial \hat{s}^2}(\hat{s}) &< 0 \quad \text{on } A_1 \end{aligned} \qquad (5.29)$$

hold by (5.27) and C^2-differentiability of $F_1(\hat{s})$. Then the complement of $A_0 \cup A_1$ on $\mathbb{T}^1$ is the disjoint union of two open arcs called A_2 and A_3 with $s_2 \in A_2$ and $s_3 \in A_3$. This therefore gives by **4** above:

$$\begin{array}{lll} \frac{\partial F_1}{\partial \hat{s}}(\hat{s}) & > 0 & \text{on } A_2 \\ \frac{\partial F_1}{\partial \hat{s}}(\hat{s}) & < 0 & \text{on } A_3, \end{array} \tag{5.30}$$

consequently by the uniform convergences from (5.26) and (5.25) we get for all β close to 1:

$$\begin{array}{lll} \frac{\partial^2 F_\beta}{\partial \hat{s}^2}(\hat{s}) & > 0 & \text{on } A_0 \\ \frac{\partial^2 F_\beta}{\partial \hat{s}^2}(\hat{s}) & < 0 & \text{on } A_1 \end{array} \tag{5.31}$$

and

$$\begin{array}{lll} \frac{\partial F_\beta}{\partial \hat{s}}(\hat{s}) & > 0 & \text{on } A_2 \\ \frac{\partial F_\beta}{\partial \hat{s}}(\hat{s}) & < 0 & \text{on } A_3. \end{array} \tag{5.32}$$

By (5.32) there exist at least two points $\tilde{s}_0 \in A_0$ and $\tilde{s}_1 \in A_1$ where $\frac{\partial F_\beta}{\partial \hat{s}}(\hat{s})$ vanishes. Since by (5.31), $F_\beta(\hat{s})$ is strictly convex on A_0 and strictly concave on A_1 there are exactly two such points $\tilde{s}_0$ and $\tilde{s}_1$. Thus the points $\hat{s} = \tilde{s}_0$ and $\hat{s} = \tilde{s}_1$ constitute the only critical points of the function $F_\beta(\hat{s})$, and they are non-degenerate by (5.31) — with $F_\beta(\tilde{s}_0)$ a global minimum and $F_\beta(\tilde{s}_1)$ a global maximum.

6. non-negativity: Given $\beta > 1 - \epsilon$ and $s_0 \in \mathbb{T}^1$, let $t'_{0\beta} \neq t''_{0\beta} \in \mathbb{T}^1$ satisfy the hypotheses. Then $\frac{\partial F}{\partial \hat{s}}(\beta, s_0, s_0, t'_{0\beta}, t''_{0\beta}) = f(\beta, s_0, t'_{0\beta}, t''_{0\beta}) = 0$ — which in the limit $\beta \to 1$ converges to $\frac{\partial F}{\partial \hat{s}}(1, s_0, s_0, t_1^+(s_0), t''_{01}) = 0$ by (5.25). By (5.15) from Proposition-5.2.6 one has $\frac{\partial^2 F}{\partial \hat{s}^2}(1, s_0, s_0, t_1^+(s_0), t''_{01}) > 0$. The uniform convergence (5.26) then implies

$$\frac{\partial^2 F}{\partial \hat{s}^2}(\beta, s_0, s_0, t'_{0\beta}, t''_{0\beta}) > 0, \tag{5.33}$$

for all β near 1. This makes $\hat{s} = s_0$ a local minimizer for $\hat{s} \longrightarrow F(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta})$ — which is the only minimizer by **5** giving $\tilde{s}_0 = s_0$. By construction the function vanishes at $\hat{s} = s_0$ thus attaining a minimum of zero on $\mathbb{T}^2$, consequently $F(\beta, \hat{s}, s_0, t'_{0\beta}, t''_{0\beta}) \geq 0$ for all $\hat{s} \in \mathbb{T}^1$ to complete the proposition. □

5.3 Regularity of potentials and smoothness of transport maps

The next proposition is a regularity result which extends the a.e. statement of Proposition-5.1.2 to every point $s_0 \in \mathbb{T}^1$. For $\beta = 0$ this was shown by Gangbo and McCann [10].

Proposition 5.3.1 (at most two images everywhere). *Let $s, t \in \mathbb{T}^1$ denote the constant speed parameters for the toy model (1.3)-(1.4) and let $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ be the c^β-convex potential of Proposition-4.3.2 whose c_β-subdifferential contains the supports Z_β (4.34) of all optimal solutions γ_o^β (1.6). If $S_0(\beta = 1) = \emptyset$, then for each $s_0 \in \mathbb{T}^1$ and each $1 - \epsilon < \beta \leq 1$ (for the $\epsilon > 0$ of Proposition-5.2.7) exactly one of the following statements holds:*
(i) $\partial_{c_\beta} u^\beta(s_0) = \{t_0\}$ *with* $(s_0, t_0) \in \Sigma^+$
(ii) $\partial_{c_\beta} u^\beta(s_0) = \{t_0, t_1\}$ *with* $(s_0, t_0) \in \Sigma^+$ *and* $(s_0, t_1) \in \Sigma^-$.

Proof. Fix a $\beta \in]1 - \epsilon, 1]$. Then by Proposition-5.2.7, $\partial_{c_\beta} u^\beta(s) \cap \Sigma^0(s) = \emptyset$ for all $s \in \mathbb{T}^1$. The key factor in the proof is the c_β-monotonicity (1.10) of Z_β and of the c_β-subdifferential $\partial_{c_\beta} u^\beta \supset Z_\beta$ of the potential that contains it.

1. *Claim-1:* Given $s_0 \in \mathbb{T}^1$, if the c_β-subgradient $\partial_{c_\beta} u^\beta(s_0)$ at s_0 has non-empty intersection with the subset $\Sigma^-(s_0) \subset \mathbb{T}^1$, then for each $t'' \in \partial_{c_\beta} u^\beta(s_0) \cap \Sigma^-(s_0)$ the corresponding $t' \neq t''$ in $\Sigma^+(s_0)$, satisfying $f(\beta, s_0, t', t'') = 0$, must also belong to $\partial_{c_\beta} u^\beta(s_0)$.

Proof of Claim-1: To produce a contradiction assume there exists an $s_0 \in \mathbb{T}^1$ where the c_β-subgradient $\partial_{c_\beta} u^\beta(s_0)$ of the potential intersects $\Sigma^-(s_0)$ at some t_0 while its counterpart t_2 in $\Sigma^+(s_0)$, satisfying $f(\beta, s_0, t_2, t_0) = 0$, fails to be in $\partial_{c_\beta} u^\beta(s_0)$. By Lemma-4.3.1 and Proposition-4.3.2 one therefore has $(s_0, t_2) \notin Z_\beta$; while the same lemma and proposition ensure the existence of an s_2 in $\mathbb{T}^1 \backslash \{s_0\}$ for which $(s_2, t_2) \in Z_\beta \subset \partial_{c_\beta} u^\beta \subset \Sigma^+ \cup \Sigma^-$ — Figure-5.1. Conforming to the notations introduced in Remark-5.2.4, the points t_0, t_2 can be represented as $t_0 = t''$ and $t_2 = t'$ with $f(\beta, s_0, t', t'') = 0$. Now for the pairs $(s_0, t''), (s_2, t') \in \partial_{c_\beta} u^\beta$, c_β-monotonicity implies:

$$\begin{aligned}
0 &\leq c(\beta, s_0, t'') + c(\beta, s_2, t') - c(\beta, s_0, t') - c(\beta, s_2, t'') \\
&= -\int_{s_0}^{s_2}\int_{t'}^{t''} \frac{\partial^2 c}{\partial s \partial t}(\beta, s, t) ds dt \\
&= -F(\beta, s_2, s_0, t', t'') < 0.
\end{aligned}$$

Where the strict inequality follows from $F(\beta, s_2, s_0, t', t'') \geq 0$ vanishing at $s_2 = s_0$ only (by Proposition-5.2.8) — which is not the case by the above assumption — the contradiction then confirms the claim.

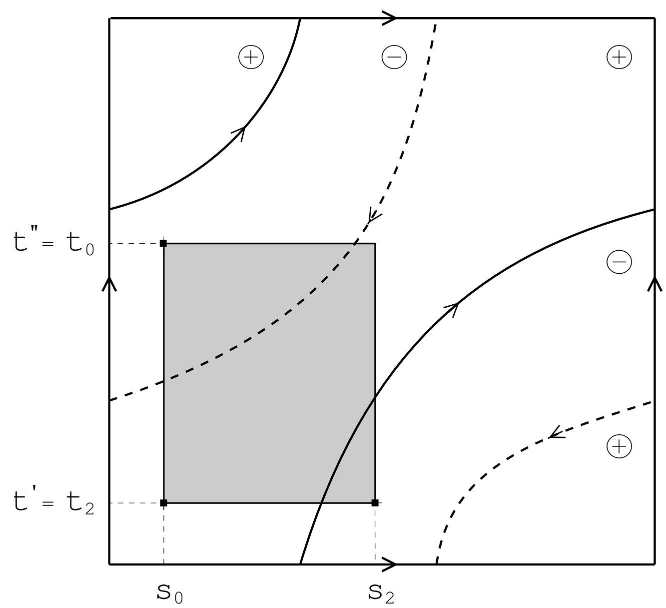

Figure 5.1: c_β-monotonicity: having $(s_0, t_0) \in \Sigma^- \cap \partial_{c_\beta} u^\beta$ forces $t_2 \in \Sigma^+(s_0)$, satisfying $\frac{\partial c}{\partial s}(\beta, s_0, t_2) = \frac{\partial c}{\partial s}(\beta, s_0, t_0)$, to be in $\partial_{c_\beta} u^\beta(s_0)$.

2. *Claim-2:* For each $s \in \mathbb{T}^1$ the subset $\{s\} \times \Sigma^+(s)$ of $\mathbb{T}^2$ has non-empty intersection with Z_β.

Proof of Claim-2: We recall from Lemma-4.3.1 and Proposition-4.3.2 that Z_β is a non-empty subset of $\partial_{c_\beta} u^\beta$. Then Claim-1 applied to Z_β confirms the statement.

(i) The proof of (i) is a direct consequence of Claim-1. If there exists an $s_0 \in \mathbb{T}^1$ at which the c_β-subgradient of the potential is the singleton set $\{t_0\} \subset \mathbb{T}^1$, then Claim-1 precludes t_0 from belonging to $\Sigma^-(s_0)$ — confirming the statement in (i).

3. We remark that the conclusions of Claims-1 and -2 are equally true under the interchange $(\mathbb{T}^1, \mu, u^\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta})$. These observations are used in steps **6** and **5-c** of the proof of (ii) respectively. We now proceed to prove (ii).

(ii) If $s_0 \in \mathbb{T}^1$ is a point of differentiability of $u^\beta(s)$ — i.e. if $s_0 \in D^\beta$ — then the c_β-subgradient there satisfies $t_0 \in \partial_{c_\beta} u^\beta(s_0) \subseteq \{t_0, t_1\}$ and the conclusion of (ii) follows readily from Proposition-5.1.2. We therefore focus on any point $s_0 \in \mathbb{T}^1$ where differentiability of $u^\beta(s)$ allegedly fails.

4. By Remark-5.1.3, the subgradient of $u^\beta(s)$ at each $s_0 \in \mathbb{T}^1 \backslash D^\beta$ is given by the convex subset $\partial.u^\beta(s_0) = [u^{\beta'}_+(s_0), u^{\beta'}_-(s_0)] \subset \mathbb{R}$ so that all $t \in \mathbb{T}^1$ that satisfy $\frac{\partial c}{\partial s}(\beta, s_0, t) \in \partial.u^\beta(s_0)$ belong to the c_β-subgradient $\partial_{c_\beta} u^\beta(s_0)$ at s_0 provided $u^\beta(s_0) + u^\beta_{c_\beta}(t) = c(\beta, s_0, t)$.

5. *Claim-3:* $\partial_{c_\beta} u^\beta(s) \cap \Sigma^+(s)$ is a singleton set for each $s \in \mathbb{T}^1$.

Proof of Claim-3: Claim-1 combines with Lemma-4.3.1 and Proposition-4.3.2 to assert that the subset $\partial_{c_\beta} u^\beta(s) \cap \Sigma^+(s) \subset \mathbb{T}^1$ is non-empty for each $s \in \mathbb{T}^1$ and is in fact a singleton set for each $s \in D^\beta$ by Proposition-5.1.2. We claim that this continues to be true for each $s \in \mathbb{T}^1 \backslash D^\beta$. Assume the contrary — then there exists an $s_0 \in \mathbb{T}^1 \backslash D^\beta$ with at least two distinct points $t_0 \neq t_2$ in $\partial_{c_\beta} u^\beta(s_0) \cap \Sigma^+(s_0)$. Interchange $t_0 \leftrightarrow t_2$ if necessary to get $\rrbracket t_0, t_2 \llbracket \subset \Sigma^+(s_0)$. We further claim that any point $t \in \rrbracket t_0, t_2 \llbracket$ does not belong to $\partial_{c_\beta} u^\beta(s) \cap \Sigma^+(s)$ for all $s \in \mathbb{T}^1 \backslash \{s_0\}$. If it does then one would have

5-a. either $(s_0, t_0), (s, t) \in \partial_{c_\beta} u^\beta \cap \Sigma^+$ with $\rrbracket s, s_0 \llbracket \times \rrbracket t_0, t \llbracket \subset \Sigma^+$

5-b. or $(s_0, t_2), (s, t) \in \partial_{c_\beta} u^\beta \cap \Sigma^+$ with $\rrbracket s_0, s \llbracket \times \rrbracket t, t_2 \llbracket \subset \Sigma^+$, so that in either case the upper-left and lower-right corners of the rectangles in Σ^+ will be in $\partial_{c_\beta} u^\beta$ — Figure-5.2. By Lemma-1.2.4 this precludes $\partial_{c_\beta} u^\beta$ from being c_β-monotone — and hence c_β-cyclically monotone.

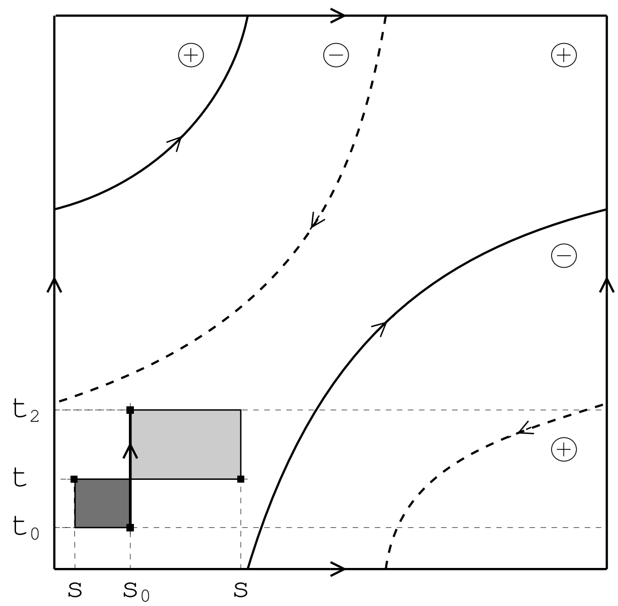

Figure 5.2: c_β-monotonicity forbids multiple images $t_0 \neq t_2$ of s_0 on $\Sigma^+(s_0)$.

5-c. Denote by $\Sigma^{\pm *} := \{(t,s) \mid (s,t) \in \Sigma^\pm\}$ and $Z^*_\beta = \{(t,s) \mid (s,t) \in Z_\beta\}$ the reflections of the sets $\Sigma^\pm$ and Z_β under $(s,t) \longrightarrow (t,s)$. Then Claim-2, together with **5-a**, **-b** and the symmetry $(\mathbb{T}^1, \mu, u^\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta})$, forces $\{s_0\} \times \rrbracket t_0, t_1 \llbracket \subset Z_\beta$, otherwise there will be a $t \in \rrbracket t_0, t_1 \llbracket$ that satisfies $(\Sigma^{+*}(t) \times \{t\}) \cap Z^*_\beta = \emptyset$ — contrary to the claim — see the remark in **3**. By **5-a** and **-b** no other point in a δ-neighborhood of $s_0 \in \operatorname{spt} \mu = \mathbb{T}^1$ supplies the arc $\rrbracket t_0, t_1 \llbracket$ — this therefore assigns to $\{s_0\}$ a positive μ-mass equal to $\nu(\rrbracket t_0, t_1 \llbracket)$:

$$\begin{aligned} 0 &< \nu(\rrbracket t_0, t_1 \llbracket) \\ &= \gamma(\{s_0\} \times \rrbracket t_0, t_1 \llbracket) \\ &\leq \mu(\{s_0\}). \end{aligned}$$

This contradicts the fact that μ is mutually continuous with respect to $\mathcal{H}^1 \lfloor_{\mathbb{T}^1}$. We therefore conclude that $\partial_{c_\beta} u^\beta(s) \cap \Sigma^+(s)$ for each $s \in \mathbb{T}^1$ contains exactly one point proving the claim.

6. With $s_0 \in \mathbb{T}^1 \backslash D^\beta$, denote by t_0 the single point in $\partial_{c_\beta} u^\beta(s_0) \cap \Sigma^+(s_0)$ (from Claim-3). Corresponding to this t_0 one can find, using Lemma-5.2.1, a unique $t_1 \in \Sigma^-(s_0)$ for which $f(\beta, s_0, t_0, t_1) = 0$. We claim that t_1 is the only element in the set $\partial_{c_\beta} u^\beta(s_0) \cap \Sigma^-(s_0)$. Since any point t_3 in $\partial_{c_\beta} u^\beta(s_0) \cap \Sigma^-(s_0)$ other than t_1 will satisfy $\frac{\partial}{\partial s} c(\beta, s_0, t_3) \in \partial . u^\beta(s_0)$ forcing, by Claim-1, the counterpart $t_4 \in \Sigma^+(s_0)$ with $f(\beta, s_0, t_4, t_3) = 0$ to be in $\partial_{c_\beta} u^\beta(s_0)$ — this contradicts Claim-3 proved in **5**. This concludes the proof of the proposition. □

The above proposition motivates the following definitions:

Definition 5.3.2 (trichotomy). Using the c^β-convex potential $u^\beta \in \mathcal{B}_0^\beta$ and the conclusion of Proposition-5.3.1 one can define (for each $\beta \in]1-\epsilon, 1]$) a disjoint decomposition $\mathbb{T}^1 = S_0(\beta) \cap S_1(\beta) \cup S_2(\beta)$ of the s-parameter space with

(o) $S_0(\beta) := \{s \in \mathbb{T}^1 \mid \partial_{c_\beta} u^\beta(s) = \{t_1\}$ with $(s, t_1) \in \Sigma^0\}$ — see Definition-5.2.5

(i) $S_1(\beta) := \{s \in \mathbb{T}^1 \mid \partial_{c_\beta} u^\beta(s) = \{t_1\}$ with $(s, t_1) \in \Sigma^+\}$

(ii) $S_2(\beta) := \{s \in \mathbb{T}^1 \mid \partial_{c_\beta} u^\beta(s) = \{t_1, t_2\}$ with $(s, t_1) \in \Sigma^+$ and $(s, t_2) \in \Sigma^-\}$

Definition 5.3.3 (symmetry and inverse maps). The symmetry under the interchange $(\mathbb{T}^1, \mu, u^\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta})$ allows a similar decomposition of the t-parameter space as $\mathbb{T}^1 = T_0(\beta) \cup T_1(\beta) \cup T_2(\beta)$ with

(o) $T_0(\beta) := \{t \in \mathbb{T}^1 \mid \partial_{c^\beta} u^\beta_{c_\beta}(t) = \{s_1\}$ with $(s_1, t) \in \Sigma^0\}$

(i) $T_1(\beta) := \{t \in \mathbb{T}^1 \mid \partial_{c^\beta} u^\beta_{c_\beta}(t) = \{s_1\}$ with $(s_1, t) \in \Sigma^+\}$

(ii) $T_2(\beta) := \{t \in \mathbb{T}^1 \mid \partial_{c^\beta} u^\beta_{c_\beta}(t) = \{s_1, s_2\}$ with $(s_1, t) \in \Sigma^+$ and $(s_2, t) \in \Sigma^-\}$

and the existence of the *inverse optimal transport maps* $s^\pm_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$, for the corresponding inverse optimal transport problem, defined similarly to (5.3) using the symmetry.

We now see these maps are well-defined everywhere and not merely almost everywhere. According to Proposition-5.3.1 and the above Definitions-5.3.2-5.3.3, given $1-\epsilon < \beta < 1$ for which $S_0(\beta) = \emptyset$, the c_β-subgradient of the dual optimizer u^β at each $s \in \mathbb{T}^1$ satisfies $t_1 \in \partial_{c_\beta} u^\beta(s) \subseteq \{t_1, t_2\}$ for some $t_1 \neq t_2 \in \mathbb{T}^1$ with $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t_1) > 0$ and $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t_2) < 0$ and $\partial_{c_\beta} u^\beta(s) = \{t_1, t_2\}$ whenever $s \in S_2(\beta)$. Since the optimal solutions γ^β_o from (1.6) satisfy $\operatorname{spt} \gamma^\beta_o \subset \partial_{c_\beta} u^\beta$, a comparison with (5.3) then shows that under the optimal transport problem (1.5), each $s \in S_1(\beta)$ is transported to a unique destination $t^+_\beta(s) \in \operatorname{spt}\nu = \mathbb{T}^1$ whereas for each $s \in S_2(\beta)$ there are two possible destinations $t^+_\beta(s) \neq t^-_\beta(s)$ on $\operatorname{spt}\nu$. The subsets $T_1(\beta)$ and $T_2(\beta)$ can be interpreted similarly under the inverse transport problem. Accordingly we redefine the optimal and the inverse optimal transport maps — for mere technical convenience later in the uniqueness proof — as follows:

Definition 5.3.4 (optimal and inverse optimal transport maps - redefined). For each $1-\epsilon < \beta \leq 1$ with $S_0(\beta) = \emptyset$, we define the *optimal transport maps* $t^\pm_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ by $t^\pm_\beta(s) \in \partial_{c_\beta} u^\beta(s)$ and

1. $t^+_\beta(s) = t^-_\beta(s)$ identified for all $s \in S_1(\beta)$ with $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t^\pm_\beta(s)) > 0$, and

2. t^+_β is distinct from t^-_β on the subset $S_2(\beta)$ with $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t^+_\beta(s)) > 0$ and $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t^-_\beta(s)) < 0$.

By the symmetry under $(\mathbb{T}^1, \mu, u^\beta, t^\pm_\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta}, s^\pm_\beta)$ the *inverse optimal transport maps* $s^\pm_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ are redefined similarly with $S_k(\beta)$ replaced by $T_k(\beta)$ for $k = 1, 2$.

Proposition 5.3.5 (optimal maps are homeomorphisms). *Consider the toy model (1.3)-(1.4). When $S_0(\beta = 1) = \emptyset$ and $1-\epsilon < \beta < 1$ for $\epsilon > 0$ of Proposition-5.2.7, the optimal and the inverse optimal transport maps are homeomorphisms and they satisfy:*
(i) $t^+_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ *is continuous with continuous inverse* $s^+_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$.
(ii) $t^-_\beta : S_2(\beta) \longrightarrow T_2(\beta)$ *is a homeomorphism with* $(t^-_\beta \lfloor_{S_2(\beta)})^{-1} = s^-_\beta \lfloor_{T_2(\beta)}$.

Proof. Fix a β in $]1-\epsilon, 1]$. Then $S_0(\beta) = \emptyset$ by Proposition-5.2.7. Let $(u^\beta = u^\beta_{c_\beta c^\beta}, u^\beta_{c_\beta})$ denote the optimal solutions to the dual problem (4.1).

(i) A. **continuity:** Pick a sequence $s_n \in \mathbb{T}^1$ and set $t_n = t^+_\beta(s_n)$. By (5.3) and Proposition-5.3.1 one has $(s_n, t_n) \in \partial_{c_\beta} u^\beta$ with $\mathbf{n}_\Omega(s_n) \cdot \mathbf{n}_\Lambda(t_n) > 0$ for each $n \geq 1$. By compactness, $\mathbb{T}^2$ admits a subsequence also denoted by (s_n, t_n) that converges to some $(s,t) \in \mathbb{T}^2$. Since $\partial_{c_\beta} u^\beta$ is closed by the continuity of $c(\beta, s, t)$ and $u^\beta(s)$, one has $(s,t) \in \partial_{c_\beta} u^\beta$. Strict convexity and differentiability (C^4) of the domain boundaries $\partial\Omega$, $\partial\Lambda$ make the maps $\mathbf{n}_\Omega : \mathbb{T}^1 \longrightarrow \mathbf{S^1}$ and $\mathbf{n}_\Lambda : \mathbb{T}^1 \longrightarrow \mathbf{S^1}$ continuous. Consequently $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t) \geq 0$; the inequality is in fact strict because $S_0(\beta) = \emptyset$. Thus $t = t^+_\beta(s)$ which implies that $t^+_\beta(s_n) \to t = t^+_\beta(s)$ whenever $s_n \to s$ — confirming the continuity of t^+_β on $\mathbb{T}^1$. Exploiting the symmetry $(\mathbb{T}^1, \mu, u^\beta, t^\pm_\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta}, s^\pm_\beta)$ a similar conclusion can be drawn for the inverse optimal maps $s^+_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$.

B. **inverse:** It suffices to show $s^+_\beta(t^+_\beta(s)) = s$ for each $s \in \mathbb{T}^1$ — for this will prove t^+_β is one-to-one with continuous inverse and that s^+_β is onto. Symmetry then gives $t^+_\beta(s^+_\beta(t)) = t$ establishing $t^+_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ is a bijection with $(t^+_\beta)^{-1} = s^+_\beta$. To prove $s^+_\beta(t^+_\beta(s)) = s$: fix $s \in \mathbb{T}^1$ and set $t = t^+_\beta(s)$. This implies $(s,t) \in \partial_{c_\beta} u^\beta$ with $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t) > 0$ — strict inequality since $S_0(\beta) = \emptyset$. Since the dual optimizer satisfies $u^\beta = u^\beta_{c_\beta c^\beta}$ it follows from (4.12) that $s \in \partial_{c^\beta} u^\beta_{c_\beta}(t)$. The inequality $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t) > 0$ then forces $s = s^+_\beta(t)$; consequently $s = s^+_\beta(t^+_\beta(s))$.

(ii) C. **continuity:** Noting that $S_0(\beta) = \emptyset$, a similar argument as in A applied to the map $t^-_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ shows that for any sequence of points $s_n \in S_2(\beta)$ setting $t_n = t^-_\beta(s_n)$ yields a convergent subsequence, also denoted by n, for which $(s_n, t_n) \to (s,t) \in \partial_{c_\beta} u^\beta$ with $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t) < 0$ so that $s \in S_2(\beta)$ with $t = t^-_\beta(s)$. This enables one to conclude compactness of $S_2(\beta)$ in addition to continuity of $t^-_\beta \lfloor_{S_2(\beta)}$. Moreover, on the subset $S_1(\beta) \subset \mathbb{T}^1$, t^-_β is continuous by the identity $t^-_\beta = t^+_\beta$ and A above — making $t^-_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ a piecewise continuous function on $\mathbb{T}^1$. By symmetry $s^-_\beta \lfloor_{T_2(\beta)}$ is also continuous.

D. **inverse:** Mimicking the proof in B: if $t = t^-_\beta(s)$ for some $s \in S_2(\beta)$ then $(s,t) \in \partial_{c_\beta} u^\beta$ with $\mathbf{n}_\Omega(s) \cdot \mathbf{n}_\Lambda(t) < 0$. The sign of the dot product of the normals together with $u^\beta = u^\beta_{c_\beta c^\beta}$ and (4.12) then implies $(t,s) \in \partial_{c^\beta} u^\beta_{c_\beta}$ with $s = s^-_\beta(t)$ for which t belongs to $T_2(\beta)$ by Definition-5.3.3. Thus for each $s \in S_2(\beta)$, one has $s^-_\beta(t^-_\beta(s)) = s$ with $t^-_\beta(s) \in T_2(\beta)$. While $t^-_\beta(s^-_\beta(t)) = t$ on $T_2(\beta)$ with $s^-_\beta(t) \in S_2(\beta)$ can be argued using the symmetry under the interchange $(\mathbb{T}^1, \mu, u^\beta, t^\pm_\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta}, s^\pm_\beta)$ — this proves the claim in (ii) to conclude the proof of the proposition.

□

5.4 Uniqueness of optimal correlation

This section develops the characteristic geometry of a c_β-cyclically monotone set in $\mathbb{T}^2$ that makes the optimal solution unique among all possible correlations on $\mathbb{T}^2$ under the hypotheses of the toy model. The final theorem states this uniqueness result by defining γ_o^β uniquely in terms of the given quantities (1.3).

Lemma 5.4.1 (cover of $\operatorname{spt}\gamma_o^\beta$). *Let γ_o^β denote an optimizer (1.6) for the optimal transport problem (1.5) for the toy model (1.3)-(1.4). Assume $S_0(\beta = 1) = \emptyset$. Then for the $\epsilon > 0$ of Proposition-5.2.7 and each $1 - \epsilon < \beta < 1$,*

$$\{(s, t_\beta^+(s))\}_{s\in\mathbb{T}^1} \subset \operatorname{spt}\gamma_o^\beta \subset \{(s, t_\beta^+(s))\}_{s\in\mathbb{T}^1} \cup \{(s, t_\beta^-(s))\}_{s\in S_2(\beta)}. \tag{5.34}$$

Proof. Fix $1 - \epsilon < \beta < 1$. Let $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ denote the dual optimizer from Proposition-4.3.2. From Proposition-5.3.5 the maps t_β^+ and $t_\beta^-\lfloor_{S_2(\beta)}$ are continuous. Under the redefinition of the optimal maps, Proposition-5.3.1 then asserts that the c_β-subdifferential of u^β is the union of the graphs of these maps, i.e. $\partial_{c_\beta}u^\beta = \operatorname{graph}(t_\beta^+) \cup \operatorname{graph}(t_\beta^-\lfloor_{S_2(\beta)})$. The second inclusion in (5.34) then follows from Proposition-4.3.2 which claims that $\operatorname{spt}\gamma_o^\beta \subset \partial_{c_\beta}u^\beta$. By Lemma-4.3.1, for each $s \in \operatorname{spt}\mu = \mathbb{T}^1$ there exists a $t \in \operatorname{spt}\nu = \mathbb{T}^1$ for which (s,t) belongs to $\operatorname{spt}\gamma_o^\beta$. This combines with the non-empty intersection of the subset $\{\{s\} \times \Sigma^+(s)\}_{s\in\mathbb{T}^1} \subset \mathbb{T}^2$ with $\operatorname{spt}\gamma_o^\beta$ from Claim-2 of Proposition-5.3.1 and the continuity of t_β^+ to conclude that *graph*(t_β^+) is contained in $\operatorname{spt}\gamma_o^\beta$ — giving the first inclusion of (5.34). ☐

Remark 5.4.2. Since Lemma-1.2.4 applies to any C^2-differentiable function independent of $\beta \in [0,1]$ and since $t_\beta^+ : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ is a homeomorphism, a similar argument as in Proposition-3.2.5-(i) shows that the graph of t_β^+ is an increasing subset of $\mathbb{T}^2$ in the sense of Definition-1.2.1, while for t_β^- it is locally decreasing by (ii) of the same proposition.

Definition 5.4.3 (hinge: convex type and concave type). A set $Z \subset \mathbb{T}^2$ is said to contain a *hinge* if Z contains points (s', t) and (s, t') with $s \neq s'$ and $t \neq t'$ such that $(s,t) \in Z$. The hinge is *convex type* if $(s,t) \in Z \cap \Sigma^+$ and *concave type* if $(s,t) \in Z \cap \Sigma^-$.

See Figure-5.3 for an illustration of hinges.

Lemma 5.4.4 (no convex type hinge). *Consider the toy model (1.3)-(1.4). Assume* $S_0(\beta) = \emptyset$ *for each* $1 - \epsilon < \beta < 1$ *and* $\epsilon > 0$. *If* γ_o^β *and* $u^\beta = u^\beta_{c_\beta c^\beta}$ *are the optimal solutions for the primal (1.6) and the dual (4.1) transport problems, then each* $s \in S_2(\beta)$ *satisfies* $\operatorname{spt}\mu \cap \partial_{c^\beta} u^\beta_{c_\beta}(t^+_\beta(s)) = \{s\}$.

Proof. Fix $1 - \epsilon < \beta < 1$ and $s \in S_2(\beta)$. If $s_0 \in \operatorname{spt}\mu \cap \partial_{c^\beta} u^\beta_{c_\beta}(t^+_\beta(s))$ for some $s_0 \in \operatorname{spt}\mu = \mathbb{T}^1$ then one has $(s_0, t^+_\beta(s)), (s, t^-_\beta(s)) \in \partial_{c_\beta} u^\beta$. Then the reformulation (5.9) of c_β-monotonicity yields $F(\beta, s, s_o, t^+_\beta(s), t^-_\beta(s)) \geq 0$ or equivalently $F(\beta, s_o, s, t^+_\beta(s), t^-_\beta(s)) \leq 0$ from (5.8) by periodicity. Non-negativity of the function $s_0 \longrightarrow F(\beta, s_o, s, t^+_\beta(s), t^-_\beta(s))$ from Proposition-5.2.8 then forces

$$F(\beta, s_o, s, t^+_\beta(s), t^-_\beta(s)) = 0$$

and consequently $s_0 = s$ — since $s_0 = s$ is the unique minimizer of the function. This concludes the proof of the lemma. □

Remark 5.4.5 (underlying geometry and optimal transport scheme). Fix $1 - \epsilon < \beta \leq 1$, where ϵ is small enough so that Proposition-5.2.7 implies $S_0(\beta) = \emptyset$. By Lemma-5.4.1, each $s \in \operatorname{spt}\mu$ has at most two destinations $t^\pm_\beta(s)$ on $\operatorname{spt}\nu$ with $(s, t^+_\beta(s))$ always in $\operatorname{spt}\gamma_o^\beta$. We therefore call the image $t^+_\beta(s)$ under the map $t^+_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ the *primary destination* of s. Since the map t^+_β is a homeomorphism, each point on $\operatorname{spt}\nu$ can be the primary destination of exactly one $s \in \operatorname{spt}\mu$. However, if the density at s satisfies

$$\frac{d\mu}{ds}(s) > \frac{dt^+_\beta}{ds}(s)\,\frac{d\nu}{dt}(t^+_\beta(s))\,, \tag{5.35}$$

meaning s has an excess mass after saturating its primary destination, then the surplus is transported to what we call a *secondary destination*, denoted $t^-_\beta(s)$, by the map $t^-_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$. A comparison with Proposition-5.3.1 shows that all such s, where the mass is split into two destinations $t^+_\beta(s) \neq t^-_\beta(s)$ belong to the subset $S_2(\beta) \subset \operatorname{spt}\mu$. Lemma-5.4.4 then precludes the primary image $t^+_\beta(s)$ of $s \in S_2(\beta)$ from receiving mass from any point on $\operatorname{spt}\mu$ other than s itself — making s the *sole supplier* of $t^+_\beta(s)$. Thus the non-existence of convex type hinges is an extension to β near zero or one of the characteristic geometry of optimal solutions in Gangbo and McCann [10]. It therefore enables us to adopt the strategy for the uniqueness proof in [10] predicated on the notions of c_β monotonicity and sole supplier.

- **strategy for uniqueness proof:** Making necessary changes in notation the strategy in [10] can be paraphrased as follows: whatever μ-mass of $S^2(\beta)$ is destined for spt ν under the map $t^+_\beta : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ is first transported backward to spt μ through $t \longrightarrow s^+_\beta(t) = s^-_\beta(t)$ to obtain a measure $\mu_1 \leq \mu$ on $\mathbb{T}^1 = \text{spt}\,\mu$, the difference $\mu_2 := \mu - \mu_1$ is then pushed forward to spt $\nu = \mathbb{T}^1$ through the map $s \longrightarrow t^-_\beta(s) = t^+_\beta(s)$ — thus enabling one to define γ uniquely in terms of these maps $t^\pm_\beta$, $s^\pm_\beta$ and the marginals μ, ν.

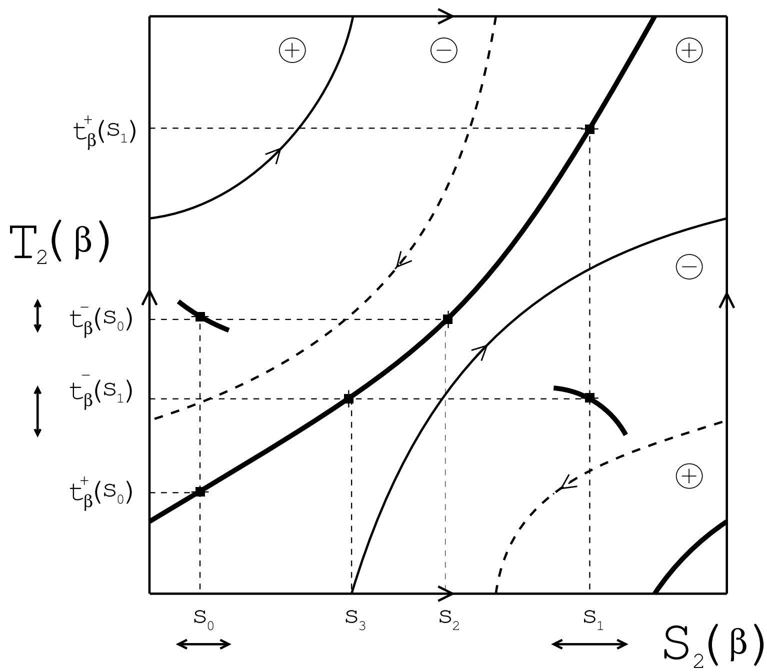

Figure 5.3: The bold solid curves represent the schematics for spt γ^β_o. The points $(s_0, t^-_\beta(s_0))$ and $(s_1, t^-_\beta(s_1))$ on Σ^- are concave type hinges — c_β-monotonicity forbids any such hinges on Σ^+.

We now state a lemma from Gangbo and McCann [10] which plays a crucial role in the uniqueness proof for γ^β_o:

Lemma 5.4.6 (Measures on Graphs are Push-Forwards). *Let* $(\mathbf{X}, \mathrm{d})$ *and* $(\mathbf{Y}, \rho)$ *be metric spaces with a Borel measure* μ *on* $\mathbf{X}$ *and Borel map* $\mathbf{t} : S \longrightarrow \mathbf{Y}$ *defined on a (Borel) subset* $S \subset \mathbf{X}$ *of full measure* $\mu[\mathbf{X} \setminus S] = 0$. *If a non-negative*

Borel measure γ on the product space $\mathbf{X} \times \mathbf{Y}$ has left marginal μ and satisfies

$$\int_{\mathbf{X}\times\mathbf{Y}} \rho(\mathbf{t}(\mathbf{x}), \mathbf{y})\, d\gamma(\mathbf{x}, \mathbf{y}) = 0,$$

then $\gamma = (\mathbf{id} \times \mathbf{t})_{\#}\mu$.

Theorem 5.4.7 (uniqueness of γ for $1 - \epsilon < \beta \leq 1$). *Consider the optimal transport problem (1.5) on the toy model (1.3)-(1.4) under the constraint $S_0(\beta = 1) = \emptyset$. Take $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ and $\epsilon > 0$ as in Proposition-5.3.1. Then for each $1 - \epsilon < \beta < 1$ the supremum in (1.5) is uniquely attained. The optimizer $\gamma_o^\beta \in \Gamma(\mu, \nu)$ can be defined uniquely in terms of the prescribed measures μ and ν, and the cost function c_β given by (1.8).*

Proof. Fix a β in $1 - \epsilon < \beta \leq 1$. Then by Proposition-5.2.7 $S_0(\beta) = \emptyset$ so that $\operatorname{spt}\mu = \mathbb{T}^1 = S_1(\beta) \cup S_2(\beta)$ from Definition-5.3.2; whereas by symmetry $\operatorname{spt}\nu$ can be decomposed as $\operatorname{spt}\nu = \mathbb{T}^1 = T_1(\beta) \cup T_2(\beta)$. Let

$$\nu_1 := \nu\lfloor_{T_1(\beta)}$$

denote the restriction of ν to the subset $T_1(\beta) \subset \operatorname{spt}\nu$ where the inverse maps have unique images $s_\beta^+(t) = s_\beta^-(t)$ on $\operatorname{spt}\mu$. Let γ_o^β denote an optimal solution for (1.3)-(1.5) and set

$$\gamma_{o1}^\beta := \gamma_o^\beta\lfloor_{\mathbb{T}^1\times T_1(\beta)}.$$

1. Define by $\gamma_{o1}^{\beta*} := R_{\#}\gamma_{o1}^\beta$ the reflection of γ_{o1}^β under $R(s,t) := (t,s)$. Then Proposition-4.2.5 together with the symmetry $(\mathbb{T}^1, \mu, u^\beta, t_\beta^\pm) \leftrightarrow (\mathbb{T}^1, \nu, u_{c_\beta}^\beta, s_\beta^\pm)$ implies $\operatorname{spt}\gamma_{o1}^{\beta*} \subset \partial_{c^\beta} u_{c_\beta}^\beta$. It therefore follows that the set

$$(T_1(\beta) \times \mathbb{T}^1) \cap \partial_{c_\beta} {u^\beta}_{c_\beta} = \{(t, s_\beta^+(t)) \mid t \in T_1(\beta)\}$$

carries the full mass of $\gamma_{o1}^{\beta*}$. Then

$$\int_{\mathbb{T}^1\times\mathbb{T}^1} d(s_\beta^+(t), s)\, d\gamma_{o1}^{\beta*}(t,s) = \int_{(T_1(\beta)\times\mathbb{T}^1)\cap\partial_{c_\beta} u_{c_\beta}^\beta} d(s_\beta^+(t), s_\beta^+(t))\, d\gamma_{o1}^{\beta*}(t,s) = 0.$$

Noting that $d : \mathbb{T}^1 \times \mathbb{T}^1 \longrightarrow \mathbb{R}$ defines a metric on the one-dimensional torus $\mathbb{T}^1$ and $\gamma_{o1}^{\beta*}$ has ν_1 as its left marginal we can use Lemma-5.4.6 to conclude that $\gamma_{o1}^{\beta*} = (id \times s_\beta^+)_{\#}\,\nu_1$ which under reflection yields $\gamma_{o1}^\beta = (s_\beta^+ \times id)_{\#}\,\nu_1$ that has

$\mu_1 := s^+_{\beta\,\#}\nu_1$ for left marginal.

2. Subtracting γ^β_{o1} from γ^β_o we define $\gamma^\beta_{o2} := \gamma^\beta_o - \gamma^\beta_{o1}$ which has $\mu_2 := \mu - \mu_1$ for left marginal. We claim that

Claim: If $(s,t) \in (\mathbb{T}^1 \times T_2(\beta)) \cap \partial_{c_\beta} u^\beta$ then $t = t^-_\beta(s)$.

Proof of Claim: Let $(s,t) \in (\mathbb{T}^1 \times T_2(\beta)) \cap \partial_{c_\beta} u^\beta$. If s belongs to $S_1(\beta)$ then $t = t^+_\beta(s) = t^-_\beta(s)$. Or else $s \in S_2(\beta)$ in which case either (i) $t = t^+_\beta(s)$ or (ii) $t = t^-_\beta(s)$. We show below that c_β-monotonicity of $\partial_{c_\beta} u^\beta$ implies

$$[S_2(\beta) \times T_2(\beta)] \cap \Sigma^+ = \emptyset$$

which then precludes (i) from occurring — making (ii) the only possibility. Assume $t = t^+_\beta(s)$ for some $(s,t) \in S_2(\beta) \times T_2(\beta)$. Then the symmetry $(\mathbb{T}^1, \mu, u^\beta, t^\pm_\beta) \leftrightarrow (\mathbb{T}^1, \nu, u^\beta_{c_\beta}, s^\pm_\beta)$ gives $s = s^+_\beta(t)$. Definitions-5.3.2 and 5.3.3 of the sets $S_2(\beta)$ and $T_2(\beta)$ then imply that there exist a $t_1 = t^-_\beta(s) \in \partial_{c_\beta} u^\beta(s)$ and an $s_1 = s^-_\beta(t) \in \partial_{c^\beta} u^\beta_{c_\beta}(t)$. The fact that $s^+_\beta \neq s^-_\beta$ on $T_2(\beta)$ then yields $s = s^+_\beta(t) \neq s^-_\beta(t) = s_1$. It therefore follows from above that $s \neq s_1 \in \partial_{c^\beta} u^\beta_{c_\beta}(t^+_\beta(s))$ which contradicts Lemma-5.4.4 — thus forcing $t = t^-_\beta(s)$ whenever $(s,t) \in \mathbb{T}^1 \times T_2(\beta)$ to complete the claim.

3. By optimality $\operatorname{spt}\gamma^\beta_{o2} \subset \partial_{c_\beta} u^\beta$ (Proposition-4.2.5), while the definition of γ^β_{o2} implies that $\gamma^\beta_{o2} = \gamma^\beta\lfloor_{\mathbb{T}^1 \times T_2(\beta)}$ so that

$$\begin{aligned} \int_{\mathbb{T}^1 \times \mathbb{T}^1} d(t^-_\beta(s), t)\, d\gamma^\beta_{o2}(s,t) &= \int_{\mathbb{T}^1 \times T_2(\beta)} d(t^-_\beta(s), t)\, d\gamma^\beta_{o2}(s,t) \\ &= \int_{(\mathbb{T}^1 \times T_2(\beta)) \cap \partial_{c_\beta} u^\beta_{c_\beta}} d(t^-_\beta(s), t^-_\beta(s))\, d\gamma^\beta_{o2}(s,t) \\ &= 0. \end{aligned}$$

Using Lemma-5.4.6 again we conclude that $\gamma^\beta_{o2} = (id \times t^-_\beta)_\# \mu_2$.

From **1** and **3** one can conclude that the optimal solution γ^β_o for $1 - \epsilon < \beta < 1$

can be written as $\gamma_o^\beta = \gamma_{o1}^\beta + \gamma_{o2}^\beta$ with:

$$\begin{aligned} \gamma_{o1}^\beta &= (s_\beta^+ \times id)_\# \, \nu\lfloor_{T_1(\beta)} \\ \gamma_{o2}^\beta &= (id \times t_\beta^-)_\# \left(\mu - s_\beta^+{}_\# \nu\lfloor_{T_1(\beta)}\right) \end{aligned} \tag{5.36}$$

determined uniquely in terms of the prescribed measures μ and ν and the optimal maps $t_\beta^\pm, s_\beta^\pm : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$. Definition-5.3.4 shows these maps depend on μ, ν and c_β only through the unique dual optimizer u^β of Proposition-4.3.2. This completes the proof of the theorem. $\square$

Appendix A

The Monge-Kantorovich optimal transportation problem

A historical development of the optimal transport problem due to Monge (1781) and Kantorovich (1942) has been chronicled in Gangbo and McCann [9], McCann [19], Rachev and Ruschendorf [24], Villani [29] with references to applications in many fields of mathematical sciences — e.g. physics, economics, probability, material science, atmospheric science, geometry, inequalities, partial differential equations — while [29] contains an exhaustive and comprehensive picture of the development of this topic into a powerful analytical technique. The original formulation of the Monge problem is in terms of volume preserving maps $\mathbf{u} : \Omega \longrightarrow \Lambda$ between two subsets $\Omega, \Lambda \subset \mathbb{R}^3$ with optimality measured against a cost function $c(\mathbf{x}, \mathbf{y}) := |\mathbf{x} - \mathbf{y}|$ defined as the Euclidean distance. While Kantorovich's formulation transforms the optimization problem into a linear problem which is solved for a joint measure γ that satisfies:

$$\inf_{\gamma \in \Gamma(\rho_1, \rho_2)} \int_{\Omega \times \Lambda} c(\mathbf{x}, \mathbf{y}) \, d\gamma(\mathbf{x}, \mathbf{y}), \tag{A.1}$$

where Ω and Λ can be any locally compact, σ-compact Hausdorff spaces with ρ_1 and ρ_2 probability measures on these domains and $\Gamma(\rho_1, \rho_2)$ represents the convex set of all joint measures on $\Omega \times \Lambda$ with ρ_1 and ρ_2 for marginals. Only optimal measures that are concentrated on graphs of measure preserving maps $\mathbf{u} : \Omega \longrightarrow \Lambda$ are allowed to compete in the more restrictive generalization of Monge's problem:

$$\inf_{\mathbf{u}_{\#}\rho_1=\rho_2} \int_\Omega c(\mathbf{x}, \mathbf{u}(\mathbf{x}))\, d\rho_1(\mathbf{x}), \tag{A.2}$$

— see Brenier [2], Gangbo and McCann [9], McCann [19], Evans [7]. Regularity of these maps for the convex cost $c(\mathbf{x}-\mathbf{y}) := |\mathbf{x}-\mathbf{y}|^2$ were studied by Caffarelli [4], [5] for convex domains $\Omega, \Lambda \subset \mathbb{R}^d$ with absolutely continuous probability measures $d\rho_1(\mathbf{x}) := f(\mathbf{x})d\mathbf{x}$ and $d\rho_2(\mathbf{y}) := g(\mathbf{y})d\mathbf{y}$ for which $f(\mathbf{x})$ and $g(\mathbf{y})$ are bounded away from zero and infinity. In [4] he showed interior regularity if the target domain Λ was convex. In [5] he showed this regularity extends to the boundary if both domains are convex (and smooth).

The dual problem to (A.1) by Kantorovich's duality principle [12] is

$$\sup_{(\phi,\psi)} \left\{ \int_\Omega \phi(\mathbf{x})\, d\rho_1(\mathbf{x}) + \int_\Lambda \psi(\mathbf{y})\, d\rho_2(\mathbf{y}) \mid \phi(\mathbf{x}) + \psi(\mathbf{y}) \leq c(\mathbf{x}, \mathbf{y}) \right\}. \tag{A.3}$$

Ma, Trudinger and Wang [17] proved C^3-smoothness of the dual potentials $\phi : \Omega \longrightarrow \mathbb{R}$ and $\psi : \Lambda \longrightarrow \mathbb{R}$ on certain bounded domains $\Omega, \Lambda \subset \mathbb{R}^d$ for C^4-differentiable cost function satisfying the non-degeneracy condition (1.16) with additional hypothesis on higher derivatives of the cost, and densities satisfying $f \in C^2(\Omega)$, $g \in C^2(\Lambda)$.

For a Polish space $\mathbf{X}$ with the metric d, let $P_p(\mathbf{X})$ denote the space of Borel probability measures on $\mathbf{X}$ with finite p-th moments. Then the *Wasserstein-p distance* defined by

$$W_p(\mu, \nu) := \left\{ \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbf{X}\times\mathbf{X}} d(\mathbf{x}, \mathbf{y})^p d\gamma(\mathbf{x}, \mathbf{y}) \right\}^{1/p} \tag{A.4}$$

metrizes $P_p(\mathbf{X})$ in terms of weak convergence in $L^p(\mathbf{X})$. Some non-linear partial differential equations, e.g. heat equation, porous medium equation, Fokker-Planck equation, can be formulated as gradient flow equations with respect to Wasserstein-2 distance to study stability and rates of convergence — see Jordan, Kinderlehrer and Otto [11] and Otto [22] for reference. Wasserstein gradient flows for $p \neq 2$ were studied by Agueh [1].

Appendix B

Semi-convexity of the c_β-convex potentials

The purpose of this appendix is to establish semi-convexity of the c_β-convex potential of the dual problem (4.1) to ensure existence of its left and right derivatives where it fails to be differentiable.

B.1 Uniform semi-convexity

Definition B.1.1 (uniformly semi-convex). A function $\phi : \mathbb{T}^1 \longrightarrow \mathbb{R}$ is said to be *locally semi-convex* at $s_0 \in \mathbb{T}^1$ if there is an open interval $U_0 \subsetneq \mathbb{T}^1$ around s_0 and a constant $0 < \lambda_0 < \infty$ so that $\phi(s) + \lambda_0 s^2$ is a convex function on U_0. We call $\phi : \mathbb{T}^1 \longrightarrow \mathbb{R}$ *uniformly semi-convex* if $\phi(s)$ is locally semi-convex at each $s \in \mathbb{T}^1$ and the constant $0 < \lambda < \infty$, that makes $\phi(s) + \lambda s^2$ locally convex, can be chosen to be independent of s or any other parameter that ϕ might depend on. The constant λ is called the *modulus of semi-convexity*.

Lemma B.1.2 (uniform semi-convexity of c_β-convex potentials). *The dual optimizer $u^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$ of Proposition-4.3.2 is uniformly semi-convex on $\mathbb{T}^1$.*

Proof. We first show the uniform semi-convexity of the cost function — then use the definition of c_β-transform to prove the lemma. Fix $s_0 \in \mathbb{T}^1$. Differentiate $c(\beta, s, t)$

twice with respect to s to get:

$$\begin{aligned}\frac{\partial^2 c}{\partial s^2}(\beta, s_0, t) = & - (1-\beta)v_\Omega^2\, K_\Omega(s_0)\,\mathbf{n}_\Omega(s_0)\cdot\mathbf{y}(t) - \beta v_\Omega^2\, K_\Omega(s_0)^2\,\mathbf{n}_\Omega(s_0)\cdot\mathbf{n}_\Lambda(t) \\ & + \beta v_\Omega\, \dot{K}_\Omega(s_0)\,\mathbf{T}_\Omega(s_0)\cdot\mathbf{n}_\Lambda(t) \\ \\ = & - (1-\beta)v_\Omega^2\, K_\Omega(s_0)\,\mathbf{n}_\Omega(s_0)\cdot\mathbf{y}(t) - v_\Omega^2\,\beta\, K_\Omega(s_0)^2\,\mathbf{n}_\Omega(s_0)\cdot\mathbf{n}_\Lambda(t) \\ & - \beta\,[\ddot{\mathbf{T}}_\Omega(s_0)\cdot\mathbf{n}_\Omega(s_0)]\,[\mathbf{T}_\Omega(s_0)\cdot\mathbf{n}_\Lambda(t)]\end{aligned}$$

we get the second equality using $K_\Omega(s_0) = -\,v_\Omega^{-1}\dot{\mathbf{T}}_\Omega(s_0)\cdot\mathbf{n}_\Omega(s_0)$ and $\mathbf{T}_\Omega(s_0)\cdot\dot{\mathbf{T}}_\Omega(s_0) = 0$. Noting that the normals and the tangents are of unit length, $0 \le \beta \le 1$, Ω and Λ are bounded planar domains and that the curves parametrizing their boundaries are C^4 smooth, one gets using Cauchy-Schwarz

$$\begin{aligned}\left|\frac{\partial^2 c}{\partial s^2}(\beta, s_0, t)\right| \;\le\;& (1-\beta)|K_\Omega(s_0)|\,|\mathbf{y}(t)| + \beta|K_\Omega(s_0)|^2 + \beta\,|\ddot{\mathbf{T}}_\Omega(s_0)| \\ \le\;& \|K_\Omega\|_\infty\,(M_\Lambda + \|K_\Omega\|_\infty) + Q_\Omega \\ \le\;& \|K_\Omega\|_\infty\, M + Q_\Omega \\ \le\;& \lambda < 2\lambda\end{aligned}$$

where,

$$0 < \lambda := max\,\{\,\|K_\Omega\|_\infty\, M + Q_\Omega,\ \|K_\Lambda\|_\infty\, M + Q_\Lambda\} \tag{B.1}$$

and

$$\begin{aligned} Q_\Omega \;&:=\; \sup_{s\in\mathbb{T}^1}|\ddot{\mathbf{T}}_\Omega(s)|, \\ Q_\Lambda \;&:=\; \sup_{t\in\mathbb{T}^1}|\ddot{\mathbf{T}}_\Lambda(t)| \end{aligned} \tag{B.2}$$

while the other symbols are defined in chapter-2 for notations. It therefore follows that $\frac{\partial^2}{\partial s^2}[\,c(\beta, s_0, t) + \lambda\, s_0^2\,] > 0$. For each fixed $\beta \in [0,1]$ and $t \in \mathbb{T}^1$ the function $c(\beta, s, t) + \lambda\, s^2$ is C^2 on $\mathbb{T}^1$ — thus by continuity there is an open interval $U_0 \subsetneq \mathbb{T}^1$ around s_0 with $\frac{\partial^2}{\partial s^2}[\,c(\beta, s, t) + \lambda\, s^2\,] \ge 0$ for all $s \in U_0$ — this makes the function $c(\beta, s, t) + \lambda\, s^2$ convex on the subset $U_0 \subsetneq \mathbb{T}^1$. Since $s_0 \in \mathbb{T}^1$ is arbitrary and $0 < \lambda < \infty$ is independent of β, s and t, one can conclude that $c(\beta, s, t)$ is a uniformly semi-convex function of s.

Let $0 < \lambda < \infty$ be given by (B.1). Given $s \in \mathbb{T}^1$, choose an open neighborhood $U_s \subset \mathbb{T}^1$ of s so that $c(\beta, s, t) + \lambda s^2$ is convex on U_s. For some lower semi-continuous function $v^\beta : \mathbb{T}^1 \longrightarrow \mathbb{R}$, the definition (4.5) of c_β-convexity gives

$$u^\beta(s) + \lambda\, s^2 = \sup_{t\in\mathbb{T}^1} c(\beta, s, t) + \lambda\, s^2 - v^\beta(t)\,.$$

Being the supremum of a family of convex functions of $s \in U_s$, the function $u^\beta(s) + \lambda\, s^2$ is itself convex on U_s. This is true for all $s \in \mathbb{T}^1$. While non-dependence of $0 < \lambda < \infty$ on β, s and t makes $u^\beta(s)$ uniformly semi-convex on $\mathbb{T}^1$. $\square$

B.2 Convergence of derivatives

Lemma B.2.1 (convergence of $u^{\beta\prime}_{\pm}$). *Let $u^n : \mathbb{T}^1 \longrightarrow \mathbb{R}$ be a sequence of uniformly semi-convex functions sharing the same modulus λ of semi-convexity. If the sequence converges uniformly to some C^1-function $u : \mathbb{T}^1 \longrightarrow \mathbb{R}$. Then the left and right derivatives $u^{n\prime}_{-}$ and $u^{n\prime}_{+}$ of u^n converge uniformly to $\frac{du}{ds}$.*

Proof. By hypothesis there is a constant $0 < \lambda < \infty$ and for each $s \in \mathbb{T}^1$ there is an open interval $U_s \subsetneq \mathbb{T}^1$ on which the functions $\phi^n_\lambda(s) := u^n(s) + \lambda\, s^2$ and $\phi_\lambda(s) := u(s) + \lambda\, s^2$ are convex. We denote by $\phi^{n\prime}_{\lambda\pm}(s) := u^{n\prime}_{\pm}(s) + 2\,\lambda\, s$ the right and left derivatives of $\phi^n_\lambda(s)$ at $s \in U_s$.

Claim: $\phi^{n\prime}_{\lambda\pm}(s)$ converge pointwise to $\frac{d\phi_\lambda}{ds}(s)$ for all $s \in \mathbb{T}^1$.

Proof of Claim: Pick some $s_0 \in \mathbb{T}^1$ and choose a δ-neighborhood U_0 as above. Denote by $l^n_0 := \phi^{n\prime}_{\lambda-}(s_0)$ the left derivative of $\phi^n_\lambda(s)$ at s_0. Then by convexity of $\phi^n_\lambda(s)$ on U_0 one gets

$$\phi^n_\lambda(s) \;\geq\; \phi^n_\lambda(s_0) \;+\; l^n_0(s - s_0) \quad \forall\, s \in U_0 \tag{B.3}$$

$$|l^n_0| \;\leq\; Lip\lfloor_{U_0}(\phi^n_\lambda) \;\leq\; Lip\lfloor_{U_0}(\phi_\lambda) + \epsilon \;\leq\; Lip\lfloor_{\mathbb{T}^1}(\phi_\lambda) + \epsilon \tag{B.4}$$

for each $\epsilon > 0$ provided n is large enough. Because l^n_0 is a bounded sequence of real numbers, by Bolzano-Weierstrass one can extract a convergent subsequence $l^{n(k)}_0 \to l$ converging to some $l \in \mathbb{R}$. Then passing to the convergent subsequence and taking the limit $k \to \infty$ in (B.3) one gets $\phi_\lambda(s) \geq \phi_\lambda(s_0) + l\,(s - s_0)$ — which implies $l \in \partial.\phi_\lambda(s_0)$. Differentiability of $\phi_\lambda(s)$ at $s_0 \in \mathbb{T}^1$ forces $\partial.\phi_\lambda(s_0) = \left\{\frac{d\phi_\lambda}{ds}(s_0)\right\}$ and consequently $l = \frac{d\phi_\lambda}{ds}(s_0)$. Thus ${\phi_{\lambda-}^{n(k)}}'(s_0) \to \frac{d\phi_\lambda}{ds}(s_0)$ as $k \to \infty$. Uniqueness of the limit then implies pointwise convergence for the entire sequence ${\phi_{\lambda-}^{n}}'$. Since ${\phi_{\lambda+}^{n}}'(s_0) \geq {\phi_{\lambda-}^{n}}'(s_0)$ also satisfies the above inequalities, by interchanging $- \leftrightarrow +$ one gets ${\phi_{\lambda\pm}^{n}}'(s_0) \to \frac{d\phi_\lambda}{ds}(s_0)$. By uniform semi-convexity this pointwise convergence holds for all $s \in \mathbb{T}^1$. This completes the proof of the claim.

Being continuous on a compact domain the derivative $\frac{d\phi_\lambda}{ds} : \mathbb{T}^1 \longrightarrow \mathbb{R}$ is uniformly continuous and by the convexity of $\phi_\lambda^n(s)$ and $\phi_\lambda(s)$ the derivatives ${\phi_{\lambda\pm}^{n}}'(s)$ and $\frac{d\phi_\lambda}{ds}(s)$ are non-decreasing on U_s. Given $\epsilon > 0$, choose $\delta > 0$ for which $\left|\frac{d\phi_\lambda}{ds}(s_1) - \frac{d\phi_\lambda}{ds}(s_2)\right| < \epsilon$ whenever $d(s_1, s_2) < \delta$. Define a finite cover of $\mathbb{T}^1$ by open arcs $U_i := \rrbracket s_{i-1}, s_{i+1}\llbracket$, $i = 1, ..., N$, with s_i's so numbered that $s_0, ..., s_N$ are positively oriented on $\mathbb{T}^1$ with $s_0 = s_{N+1}$ and $d(s_{i-1}, s_{i+1}) < \delta$. Choose $\delta > 0$ small enough to make $\phi_\lambda^n(s)$ and $\phi_\lambda(s)$ convex on each open arc U_i of the cover. Pick some $s \in \mathbb{T}^1$. Then s belongs to some U_i, $1 \leq i \leq N$, in the cover with s_{i-1}, s, s_{i+1} positively oriented. Monotonicity of ${\phi_{\lambda\pm}^{n}}'(s)$ and $\phi_\lambda(s)$ on U_i gives

$$
\begin{aligned}
{\phi_{\lambda\pm}^{n}}'(s) - \frac{d\phi_\lambda}{ds}(s) &\leq {\phi_{\lambda\pm}^{n}}'(s_{i+1}) - \frac{d\phi_\lambda}{ds}(s_{i-1}) \\
&\leq \left|{\phi_{\lambda\pm}^{n}}'(s_{i+1}) - \frac{d\phi_\lambda}{ds}(s_{i+1})\right| + \left|\frac{d\phi_\lambda}{ds}(s_{i+1}) - \frac{d\phi_\lambda}{ds}(s_{i-1})\right| \\
&< 2\,\epsilon.
\end{aligned}
$$

The last inequality follows from the pointwise convergence of ${\phi_{\lambda\pm}^{n}}'(s)$ and the uniform continuity of $\frac{d\phi_\lambda}{ds}(s)$. With ${\phi_{\lambda\pm}^{n}}'$ and $\frac{d\phi_\lambda}{ds}$ interchanged one finally gets

$$
\left|{\phi_{\lambda\pm}^{n}}'(s) - \frac{d\phi_\lambda}{ds}(s)\right| < 2\,\epsilon.
$$

This is true for any $s \in \mathbb{T}^1$. Replacing ${u^{n}_{\pm}}'(s) = {\phi_{\lambda\pm}^{n}}'(s) - 2\lambda\, s$ and $\frac{d}{ds}u(s) = \frac{d\phi_\lambda}{ds}(s) - 2\lambda\, s$, and taking supremum over all $s \in \mathbb{T}^1$ one gets $\left\|{u^{n}_{\pm}}' - \frac{du}{ds}\right\|_\infty \to 0$ as $n \to \infty$. That concludes the proof of the lemma. $\square$

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