Asymptotics near extinction for nonlinear fast diffusion on a bounded domain

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Outline

1 Introduction to Nonlinear Diffusion

2 History and goals

3 Methods and results
   - The dynamical systems approach
   - Challenges

4 Acknowledgements
Rate (and corrections) at which the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial \tau} = \frac{1}{m} \Delta (\rho^m)$$

in $\Omega \subset \subset \mathbb{R}^n$ open and bounded

$\rho = 0$

on $(0, \infty) \times \partial \Omega \in C^\infty$

$0 \leq \rho = \rho_0 \in L^1(\Omega)$

on $\{\tau = 0\} \times \Omega$

transports heat from $\Omega$ to the sink at its boundary $\partial \Omega$?

Three regimes:
Nonlinear diffusion: basic question

Rate (and corrections) at which the nonlinear diffusion equation

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transports heat from $\Omega$ to the sink at its boundary $\partial \Omega$?

Three regimes:

(PM) Porous medium: $m \in ]1, \infty[

(FD) Sobolev subcritical fast diffusion: $0 < m \in ]\frac{n-2}{n+2}, 1[$

(FD') Sobolev supercritical fast diffusion: $m \in ]-\infty, \frac{n-2}{n+2}[$

Limiting cases: linear heat equation $m = 1$

Sobolev critical diffusion $m = \frac{n-2}{n+2}$
How does this work for the linear heat equation \( m = 1 \)?

Recall: separation of variables yields

\[
\rho(\tau, y) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i \tau} \phi_i(y)
\]

where

\[
0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots ,
\]

and \( c_i = \langle \rho_0, \phi_i \rangle_{L^2} \) where \( \{\phi_i\}_{i=1}^{\infty} \subset H^1_0(\Omega) \) for

\[
H^1_0(\Omega) = \{ \phi \in L^2(\Omega) : \nabla \phi \in L^2(\Omega) \text{ and } \phi = 0 \text{ on } \partial \Omega \}.
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H^1_0(\Omega) = \{ \phi \in L^2(\Omega) \mid D\phi \in L^2(\Omega) \text{ and } \phi = 0 \text{ on } \partial\Omega \}
\]

solve

\[
-\Delta \phi_i = \lambda_i \phi_i \quad \text{on } \Omega
\]

and form an orthonormal basis for \( L^2(\Omega) \)
Do the nonlinear dynamics admit a similar description?

\[
\begin{cases}
\rho(0, y) = \rho_0(y) \\
\frac{\partial \rho}{\partial \tau} = \frac{1}{m} \Delta (\rho^m)
\end{cases}
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\end{aligned}
\]

POROUS MEDIUM REGIME \((m > 1)\)

- fluid in rock; population spreading; temperature dependent conductivity
- rate of diffusion \(\rho^{m-1}\) varies directly with density \(\rho\) of diffusing material
- compactly supported \(\rho_0\) remains compactly supported at \(\tau > 0\)
Motivation: dissipative fluids

\[ \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) (\rho u) = -\nabla P(\rho) - bu \]  

(1)

- if drag negligible \((b \ll 1)\), (1) couples with continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \]  

(2)

to give compressible Euler system
Motivation: dissipative fluids

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to give compressible Euler system

- if drag dominates \((b \gg 1)\), neglect inertial terms in (1); then (2) yields

\[
\frac{\partial \rho}{\partial t} - \frac{1}{b} \nabla \cdot (\rho \nabla P(\rho)) = 0
\] (3)
Motivation: dissipative fluids

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\]  
- polytropic equation of state \(P(\rho) = \frac{b}{m-1} \rho^m\) gives nonlinear diffusion (3)
Subcritical fast diffusion regime \((\frac{n-2}{n+2} < m < 1)\)

\[
\frac{\partial \rho}{\partial \tau} = \frac{1}{m} \Delta (\rho^m) = \nabla \cdot (\rho^{m-1} \nabla \rho) \quad \text{in} \quad \Omega \subset\subset \mathbb{R}^n
\]

\[
\rho = 0 \quad \text{on} \quad (\tau, y) \in (0, \infty) \times \partial \Omega \in C^\infty
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0 \leq \rho = \rho_0 \in L^1(\Omega) \quad \text{on} \quad \{\tau = 0\} \times \Omega
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- rate of diffusion varies \textit{inversely} with density of diffusing material
- \(m = 1/2\) used to model plasma ions diffusing across a \(\vec{B}\) field
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- solution vanishes at (finite) time \( T = T(\rho_0) < \infty \)
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- Rescale (Berryman-Holland '78-'80): if \( p = 1/m \) then

\[
v(t, x)^p = \frac{\rho(\tau, x)}{\frac{1-m}{m} (T - \tau)^{1-m/1-m}} \quad \text{and} \quad t = -\frac{1-m}{m} \log |1 - \frac{\tau}{T}|
\]

satisfy

\[
\frac{\partial}{\partial t} \left( \frac{v^p}{p} \right) = \Delta v + v^p
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Subcritical fast diffusion regime \( \left( \frac{n-2}{n+2} < m < 1 \right) \)

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\]

satisfy

\[
\frac{\partial}{\partial t} \left( \frac{v^p}{p} \right) = \Delta v + v^p = \frac{\delta E}{\delta v} \quad \text{in } (t, x) \in (0, \infty) \times \Omega
\]
where
\[
E(v) := \int_\Omega \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{p+1} v^{p+1} \right] \, dx
\]
is a Lyapunov function for the rescaled dynamics.

- **Berger '77**: $\frac{\delta E}{\delta v} = 0$ has positive solutions in $H^1_0(\Omega)$ ‘ground states’
- **Berryman-Holland '80**: $v(t) \to$ ground state as $t \to \infty$ along a subsequence; conjectured limit unique & higher-order asymptotics
- **Brezis-Nirenberg '83**: ground states non-unique on certain domains
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- **Feireisl-Simondon '00:** \( \lim_{t \to \infty} v(t) = V \), but depends on \( v(0) \)
- **Bonforte-Grillo-Vazquez '12:** \( \left\| \frac{v(t)}{V} - 1 \right\|_\infty \to 0 \) (rate if \( m \sim 1 \))
- **Jin-Xiong '20+:** \( \left\| \frac{v(t)}{V} - 1 \right\|_\infty \leq \frac{C}{t^{\alpha}} \) for some \( C, \alpha > 0 \) if \( m \in \left[ \frac{n-2}{n+2}, 1 \right] \)
- **Bonforte-Figalli '21:** \( \left\| \frac{v(t)}{V} - 1 \right\|_\infty \leq C e^{-\lambda t} \), where the spectral gap \( \lambda > 0 \) for an open \( C^{2,\alpha} \) dense set of domains \( \Omega \), including the ball
- **Akagi '21+** energetic (rather than entropic) proof
relative error

\[ h(t) := \frac{v(t)}{V} - 1 \]

satisfies

\[ \frac{\partial h}{\partial t} + L_V h = N(h) = M_V(h) \]

where

\[ L_V h = -\frac{1}{V} \Delta (hV) - ph \]

\[ = -V^{-p-1} \nabla \cdot (V^2 \nabla h) - (p - 1) h \]

\[ \geq (1 - p) h \]

\[ N(h) = (1 + h)^p - 1 - ph - ((1 + h)^p - 1) p \frac{\partial h}{\partial t} \]

\[ M_V(h) = \frac{1}{(1 + h)^{p-1}} \left[ (1 + h)^p - 1 - ph + ((1 + h)^p - 1) L_V h \right] \]
\[ \|f\|_{L^q} := \left( \int_{\Omega} |f(x)|^q V(x)^r \, dx \right)^{1/q} \]

\[ \langle f, g \rangle_r := \langle f, g \rangle_{L^2} = \int_{\Omega} fg V(x)^r \, dx \]

implies \( L_V \) is self-adjoint on \( L^2_{p+1} \) and has a complete basis of eigenvectors, which are critical points for the restriction of the weighted Dirichlet energy.
\[ \|f\|_{L^q_r} := \left( \int_{\Omega} |f(x)|^q V(x)^r \, dx \right)^{1/q} \]
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\[ Q_V(\phi) := \|D\phi\|_{L^2_r}^2 = \int_{\Omega} |D\phi|^2 V^2 \, dx \]

to the \( L^2_{p+1} \) unit-sphere, with the boundary trace of \( \phi V \) vanishing.

Proof: \( \tilde{L} := V \circ (L_V + pl) \circ V^{-1} \geq I \) has compact inverse on \( L^2_{p-1} \) ... \( \square \)
Denote the eigenvalues by

\[ \lambda_{-I} < \lambda_{-I+1} \leq \cdots \leq \lambda_{-3} \leq \lambda_{-2} \leq \cdots \leq \lambda_{K-1} \leq 0 < \lambda_K \leq \lambda_{K+1} \leq \cdots \]

where \( I \geq 1 \) counts the number of unstable modes, \( K \geq 0 \) the zero modes (if any), and \( \lambda_K > 0 \) is the first positive eigenvalue or ‘spectral gap’

Note \( LV + (p - 1) \geq 0 \) and \( LV1 = (1 - p)1 \) imply \( \lambda_{-I} = 1 - p \) and simple; this corresponds to time translation symmetry in the original variables.

Bonforte-Figalli ‘21’s exponential convergence rate \( \lambda = \lambda_K \) follows from the fact that the unstable modes are suppressed (Feireisl-Simondon ‘00), while \( C^{2,\alpha} \)-generic domains admit no zero modes (Saut-Teman ‘79).

Can we (a) close the gap between Bonforte-Figalli ‘21’s exponential and Jin-Xiong ‘20’s algebraic rate of convergence, and/or (b) access higher asymptotics conjectured (for \( n = 1 \)) by Berryman-Holland ‘80? (c) When do zero modes spoil exponential convergence?
First dichotomy

**Theorem (Choi-M.-Seis)**

Fix $\Omega \subset \subset \mathbb{R}^n$ bounded with $\partial \Omega \in C^\infty$ and $0 < m \in \left]\frac{n-2}{n+2}, 1\right[$.

If $0 \leq v \in L^\infty([0, \infty[ \times \Omega)$ solves dynamics and $h(t) := \frac{v(t)}{V} - 1 \to 0$ uniformly, there exist $\epsilon, C(p, V)$ and $\lambda \geq \lambda_K$ such that $\|h\|_{L^\infty(\mathbb{R}_+ \times \Omega)} \leq \epsilon$ implies either

$$C\|h(t)\|_{L^\infty} \geq \|h(t)\|_{L^2} \geq \frac{1}{Ct} \quad \forall t \gg 1 \quad (4)$$

or

$$\frac{1}{C}\|h(t)\|_{L^2} \leq \|h(t)\|_{L^\infty} \leq C e^{-\lambda t}\|h(0)\|_{L^2} \quad \forall t \geq 1. \quad (5)$$
Fix $\Omega \subset \subset \mathbb{R}^n$ bounded with $\partial \Omega \in C^\infty$ and $0 < m \in ]\frac{n-2}{n+2}, 1[$. If $0 \leq v \in L^\infty([0, \infty[ \times \Omega)$ solves dynamics and $h(t) := \frac{v(t)}{V} - 1 \to 0$ uniformly, there exist $\epsilon, C(p, V)$ and $\lambda \geq \lambda_K$ such that $\|h\|_{L^\infty(\mathbb{R}_+ \times \Omega)} \leq \epsilon$ implies either

$$C \|h(t)\|_{L^\infty} \geq \|h(t)\|_{L^2_{p+1}} \geq \frac{1}{Ct} \quad \forall t \gg 1 \quad (4)$$

or

$$\frac{1}{C} \|h(t)\|_{L^2_{p+1}} \leq \|h(t)\|_{L^\infty} \leq Ce^{-\lambda t} \|h(0)\|_{L^2_{p+1}} \quad \forall t \geq 1. \quad (5)$$

If (5) holds and $2\lambda \in ]\lambda_J, \lambda_{J+1}[$, then there exist $c_i = c_i(h(0)) \in \mathbb{R}$ and $\tilde{C} = \tilde{C}(V, p, \lambda, C)$ such that the eigenfunctions $L_V \phi_i = \lambda_i \phi_i \in L^2_{p+1}(\Omega)$ yield

$$\left\|h(t) - \sum_{i=K}^{J} c_i e^{-\lambda_i t} \phi_i\right\|_{L^2_{p+1}} \leq \tilde{C} \|h(0)\|_{L^2_{p+1}} te^{-2\lambda t}$$
Let $S \subset H^1_0(\Omega)$ denote the set of fixed points $V \geq 0$ of the rescaled dynamics. Then $S \subset C^{3,\alpha}(\Omega)$ and $V, W \in S$ implies $V/W \in L^\infty$. Topologize $S$ using the relatively uniform ‘balls’

$$B_r(V) := \{ W \in S \mid \| W/V - 1 \|_\infty < r \}$$

as a base. Call $V \in S$ an ordinary limit iff $S$ forms a manifold of dimension $\text{dim}(S) = K := \text{dim}(\text{Ker}L_V)$ near $V$, which the error relative to $V$ embeds differentiably into $L^2_{p+1}(\Omega)$.

**Theorem (Second dichotomy)**

*Under the hypotheses of the preceding theorem, convergence is exponentially fast if $V$ is an ordinary limit.*
Remark: All tangent vectors to the embedding of $S$ at $V$ lie in $\text{Ker} L_V$. Conversely, if $V$ is an ordinary limit, then each $u \in \text{Ker} L_V$ is tangent to the embedding of $S$. In the latter case the kernel is said to be integrable, a notion exploited by Allard-Almgren ’81 and Simon ’85 for related purposes in the context of minimal surfaces and geometric evolution equations.

We expect ordinary limits to be in some sense generic in $S$. 
Some history for $\Omega = \mathbb{R}^n$ with $m_q = 1 - \frac{2}{n+q}$

For compactly supported non-negative initial data with $\|\rho_0\|_1 = \|\tilde{\rho}_0\|_1$

Friedman-Kamin '80 $m > m_0 \Rightarrow \|\rho(\tau, \cdot) - \tilde{\rho}(\tau, \cdot)\|_{L^1} = o(1)$ as $\tau \to \infty$
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Carrillo-Toscani '99, Dolbeault-del Pino '00, Otto '01

$m = m_q > m_n$ implies $\|\rho(\tau, \cdot) - \tilde{\rho}(\tau, \cdot)\|_{L^1} = O(\tau^{-\frac{1}{2}(1 + \frac{n}{q})})$ as $\tau \to \infty$
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Some history for $\Omega = \mathbb{R}^n$ with $m_q = 1 - \frac{2}{n+q}$

For compactly supported non-negative initial data with $\|\rho_0\|_1 = \|\tilde{\rho}_0\|_1$

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$m = m_q > m_n$ implies $\|\rho(\tau, \cdot) - \tilde{\rho}(\tau, \cdot)\|_{L^1} = O(\tau^{-\frac{1}{2}(1+n/q)})$ as $\tau \to \infty$

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Carrillo-Vazquez (’03, $m > m_0 = 1 - \frac{2}{n}$) radial symmetry implies $\|\rho(\tau, \cdot) - \tilde{\rho}(\tau, \cdot)\|_{L^1} = O(\tau^{-1})$ as $\tau \to \infty$

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Seis ’14 ($m > 1$) linearized spectrum for the porous medium equation
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Seis '14 ($m > 1$) linearized spectrum for the porous medium equation

Angenent '88 invariant manifolds $m \geq 1 = n$; Koch '99, Seis '15+ $n \geq 1$
A (finite dimensional) dynamical systems approach

\[ x'(t) = -F(x(t)) \in \mathbb{R}^n \quad \text{with} \quad x(0) = x_0 \]

LINEARIZE around fixed point \( F(x_\infty) = 0 \) to get:

\[ (x(t) - x_\infty)' = -DF(x_\infty)(x(t) - x_\infty)' + O(x(t) - x_\infty)^2 \]
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If \( \sigma(DF(x_\infty)) = \{0 < \lambda_1 \leq \cdots \leq \lambda_n\} \) with eigenvectors \( \hat{\phi}_i \), as \( t \to \infty \) expect

\[ x(t) - x_\infty = \sum_{i=1}^{n} c_i \hat{\phi}_i e^{-\lambda_i t} \]
A (finite dimensional) dynamical systems approach

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If \( \sigma(DF(x_\infty)) = \{0 < \lambda_1 \leq \cdots \leq \lambda_n\} \) with eigenvectors \( \hat{\phi}_i \), as \( t \to \infty \) expect

\[ x(t) - x_\infty = \sum_{i=1}^{n} c_i \hat{\phi}_i e^{-\lambda_i t} + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \hat{\phi}_i \hat{\phi}_j e^{-(\lambda_i + \lambda_j)t} + \sum_{i} \sum_{j} \sum_{k} \cdots \]

- to simplify, only strive for asymptotics to order \( O(e^{-2\lambda_1 t}) \)
- differentiability of \( F(x_0) \) or \( x(t) = X(t, x_0) \) wrt \( x_0 \in \mathbb{R}^n \) was crucial
New challenges

- coping with unstable and zero modes
New challenges

- coping with unstable and zero modes
- estimating the nonlinearity quadratically \( \| N(\nu(t)) \|_{L^{p+1}_2} \leq \| \nu(t-1) \|_{L^{p+1}_2}^2 \)

using \( |N(\nu(t))| \leq |\nu(t)| \| \frac{\partial}{\partial t} \nu(t) \|_{L^\infty} \) to reduce to one of the model cases

\( \dot{a}(t) = -Ca^2 \) so that \( a(t) = \frac{1}{Ct + a(0)^{-1}} \)

or
New challenges

- coping with unstable and zero modes
- estimating the nonlinearity quadratically
  \[ \| N(v(t)) \|_{L_p}^2 \leq \| v(t-1) \|_{L_p}^{2p+1} \]

using

\[ |N(v(t))| \leq |v(t)| \| \frac{\partial}{\partial t} v(t) \|_{L^\infty} \]

so to reduce to one of the model cases

\[ \dot{a}(t) = -Ca^2 \quad \text{so that} \quad a(t) = \frac{1}{Ct + a(0)^{-1}} \]

or

\[ \dot{x}_i = -\lambda_i x_i + N_i(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \]

where

\[ N(x) \leq k|x|^2 \quad \text{if} \quad |x| \leq \epsilon^2, \quad \text{and} \quad \lambda_i \geq \lambda > 0. \]

In this case Gronwall implies

\[ f(t) = |x(t)|^2 \]

satisfies

\[ f(T) \leq f(0) \exp[-2(\lambda - k\epsilon)T], \]

and then, if

\[ f(t) \leq C^2 \exp[-2\Lambda t] \quad \text{for some} \quad \Lambda > 0, \quad C \quad \text{and all} \quad t \geq 0, \]

\[ |x(T)| \leq |x(0)| \exp[-\lambda T + \frac{C}{2\Lambda}] \]
Lemma (K. Choi-Haslhofer-Hershkovits ’18+)

Let $X(s)$, $Y(s)$, and $Z(s)$ be non-negative AC functions on $[0, \infty)$ satisfying

\[
\frac{dX}{ds} - X \geq -\epsilon(Y + Z),
\]

\[
\left| \frac{dY}{ds} \right| \leq \epsilon(X + Y + Z),
\]

\[
\frac{dZ}{ds} + Z \leq \epsilon(X + Y)
\]

for each $\epsilon \in (0, \frac{1}{100})$ and a.e. $s \in [s_0(\epsilon), \infty)$. 

If $\lim_{s \to \infty} (X(s) + Y(s) + Z(s)) = 0$ then $X \leq 2\epsilon(Y + Z)$ for $s \geq s_0(\epsilon)$ and either $X(s) + Z(s) = o(Y(s))$ as $s \to \infty$ (6) or $X(s) + Y(s) \leq 100\epsilon Z(s)$ for $s \geq s_0(\epsilon)$. (7)
Lemma (K. Choi-Haslhofer-Hershkovits ’18+)

Let $X(s)$, $Y(s)$, and $Z(s)$ be non-negative AC functions on $[0, \infty)$ satisfying

$$\frac{dX}{ds} - X \geq -\epsilon(Y + Z),$$

$$|\frac{dY}{ds}| \leq \epsilon(X + Y + Z),$$

$$\frac{dZ}{ds} + Z \leq \epsilon(X + Y)$$

for each $\epsilon \in (0, \frac{1}{100})$ and a.e. $s \in [s_0(\epsilon), \infty)$. If $\lim_{s \to \infty} (X + Y + Z)(s) = 0$ then $X \leq 2\epsilon(Y + Z)$ for $s \geq s_0(\epsilon)$ and either

$$X(s) + Z(s) = o(Y(s)) \text{ as } s \to \infty \quad (6)$$

or

$$X(s) + Y(s) \leq 100\epsilon Z(s) \text{ for } s \geq s_0(\epsilon). \quad (7)$$
Quadratic Hilbert-space estimate for the nonlinearity:

First dichotomy: apply the lemma to $X(t)$, $Y(t)$ and $Z(t)$ defined as the Hilbert norm $L_{p+1}^2 := L^2(V^{p+1})$ of the orthogonal projection of $h(t)$ onto the unstable, neutral, and stable modes respectively. To absorb the nonlinearity into the $\epsilon$ corrections, apply the following theorem with $t = 1$. 
Quadratic Hilbert-space estimate for the nonlinearity:

First dichotomy: apply the lemma to $X(t)$, $Y(t)$ and $Z(t)$ defined as the Hilbert norm $L^2_{p+1} := L^2(V^{p+1})$ of the orthogonal projection of $h(t)$ onto the unstable, neutral, and stable modes respectively. To absorb the nonlinearity into the $\epsilon$ corrections, apply the following theorem with $t = 1$.

**Theorem (Spatially uniform control of time derivatives)**

Let $k \in \{0, 1, 2, \ldots\}$ and $t > 0$ fixed. Then if $\|h\|_{L^\infty} \leq \epsilon$ with $\epsilon$ sufficiently small, there exists a constant $C = C(t, k, m, V)$ such that

$$\|\partial_t^k h(t)\|_{L^\infty} \leq C \|h_0\|_{L^2_{p+1}}.$$  

Proof: degenerate parabolic smoothing, with delicate control near the boundary of $\Omega$ where $V(x) \sim d_{\partial \Omega}(x)$. 

c.f. Jin-Xiong ’19+
Among other ingredients, second dichotomy relies on

**Lemma (K. Choi-Sun ’20+)**

Suppose \(X(s), Y(s),\) and \(Z(s)\) are non-negative absolutely continuous functions on some interval \([-L, L]\) such that \(0 < X + Y + Z < \eta\) for some \(\eta > 0\). Suppose that there exist two constants \(\sigma > 0\) and \(\Lambda > 0\) such that

\[
\frac{dX}{ds} - \Lambda X \geq -\sigma (Y + Z),
\]

\[
\left| \frac{dY}{ds} \right| \leq \sigma (X + Y + Z),
\]

\[
\frac{dZ}{ds} + \Lambda Z \leq \sigma (X + Y),
\]

for any \(s \in [-L, L]\). Then there exists \(\sigma_0 = \sigma_0(\Lambda)\) such that if \(0 < \sigma < \sigma_0\) it holds

\[
X + Z \leq \frac{8\sigma}{\Lambda} Y + 4\eta e^{-\frac{\Lambda L}{4}}\text{ for any } s \in \left[-\frac{L}{2}, \frac{L}{2}\right].
\]
Thank you!