Maximizing angle sums between Euclidean lines

Tongseok Lim & Robert McCann

University of Toronto

Click on ‘Talk 3’ at www.math.toronto.edu/mccann

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Which $k$ lines through the origin in $\mathbb{R}^{n+1}$ maximize the sum of the angles $\theta$ between them?
A family of generalized Fejes Tóth problems

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Find a globally stable configuration of \( k \) charges on projective sphere \( \mathbb{RP}^n \), which repel each other pairwise with a force independent of distance.
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Which $k$ lines through the origin in $\mathbb{R}^{n+1}$ maximize the sum of the angles $\theta$ between them? Equivalently,

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- **Fejes Tóth conjecture (1959):** Distribute the lines / charges as evenly as possible over an orthonormal basis of $\mathbb{R}^{n+1}$

  - obvious for $k \leq n + 1$; also resolved affirmatively for all $k$ with $n = 1$ but remains open otherwise

  - for $n = 1$ and $k > 2$ there are many inequivalent maximimizers as well (accumulating to the uniform distribution)

To make progress, let the force increase with a power $b - 1 > 0$ of the distance, so we minimize $\sum \theta^b$ instead
Identifying $\pm x \in S^n$ yields $RP^n = \frac{2}{\pi} S^n / \{+, -\}$ scaled to diameter 1. Let

$$d_{RP^n}(x, y) = \frac{2}{\pi} \min \{d_{S^n}(x, y), \pi - d_{S^n}(x, y)\}$$

$$\mathcal{P}(RP^n) = \{0 \leq \mu \text{ on } RP^n \mid \int_{RP^n} d\mu = 1\}$$

$$\mathcal{P}_k(RP^n) = \{\mu \in \mathcal{P}(RP^n) \mid \mu = \frac{1}{k} \sum_{i=1}^{k} \delta_{x_i}, \ x_i \in RP^n\}$$
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$$\mathcal{P}(\mathbb{RP}^n) = \{0 \leq \mu \text{ on } \mathbb{RP}^n \mid \int_{\mathbb{RP}^n} d\mu = 1\}$$

$$\mathcal{P}_k^\perp(\mathbb{RP}^n) = \{\mu \in \mathcal{P}(\mathbb{RP}^n) \mid \mu = \frac{1}{k} \sum_{i=1}^{k} \delta_{x_i}, \ x_i \in \mathbb{RP}^n\}$$

Fejes Tóth’s problem becomes the case $b = 1$ of

$$\max_{\mu \in \mathcal{P}_k^\perp(\mathbb{RP}^n)} E_b(\mu)$$

where

$$E_b(\mu) = \frac{1}{2} \iint d_{\mathbb{RP}^n}(x, y)^b d\mu(x) d\mu(y)$$

F.T. Conjecture: standard basis $\{\hat{e}_1, \ldots, \hat{e}_{d+1}\}$ (i.e. maximal projective simplex) $\bar{\mu}_k = \frac{1}{k} \sum_{i=1}^{k} \delta_{\hat{e}_{(k \mod d+1)}}$ achieves maximum for all $b \in [1, \infty]$. 
Theorem (Discrete threshold for simplex maxima)

(a) For $k > n + 1$, there exists $b_{\Delta n}(k) \in [1, \infty)$ such that $\overline{\mu}_k$ maximizes $E_b(\mu)$ on $\mathcal{P}_k^\|= (\mathbb{RP}^n)$ if and only if $b \geq b_{\Delta n}(k)$.

(b) $\overline{\mu}_k$ is the only maximizer up to rotations if and only if $b > b_{\Delta n}(k)$.

RMK: Fejes Tóth conjecture $\iff b_{\Delta n}(k) = 1$ for all $k > n + 1$

Note $2E_b(\overline{\mu}) = \frac{n}{n+1}$ for $\overline{\mu} = \overline{\mu}_k$ when $n + 1$ divides $k$. 
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Note $2E_b(\bar{\mu}) = \frac{n}{n+1}$ for $\bar{\mu} = \bar{\mu}_k$ when $n + 1$ divides $k$.

(c) $\bar{\mu}_k$ maximizes $E_b$ over the full set $\mathcal{P}(\mathbb{RP}^n)$ iff $b \geq b_{\Delta n}$,
(d) $\bar{\mu}$ maximizes uniquely on $\mathcal{P}(\mathbb{RP}^n)$ up to rotations iff $b > b_{\Delta n}$.
Theorem (Discrete threshold for simplex maxima)

(a) For \( k > n + 1 \), there exists \( b_{\Delta n}(k) \in [1, \infty) \) such that \( \bar{\mu}_k \) maximizes \( E_b(\mu) \) on \( \mathcal{P}_k(RP^n) \) if and only if \( b \geq b_{\Delta n}(k) \).
(b) \( \bar{\mu}_k \) is the only maximizer up to rotations if and only if \( b > b_{\Delta n}(k) \).

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Note \( 2E_b(\bar{\mu}) = \frac{n}{n+1} \) for \( \bar{\mu} = \bar{\mu}_k \) when \( n + 1 \) divides \( k \).

Theorem (Continuum threshold for simplex maxima)

If \( n + 1 \) divides \( k \) then there exists \( b_{\Delta n} \in [1, \infty) \) such that

(c) \( \bar{\mu} = \mu_k \) maximizes \( E_b \) over the full set \( \mathcal{P}(RP^n) \) iff \( b \geq b_{\Delta n} \),
(d) \( \bar{\mu} \) maximizes uniquely on \( \mathcal{P}(RP^n) \) up to rotations iff \( b > b_{\Delta n} \)
(e) \( b_{\Delta n} < 2 \)

RMK: (e) improves \( b_{\Delta n} \leq 2 \) (Bilyk, Glazyrin, Matzke, Park, Vlasiuk)

\( b = 2 \) is the threshold for mild repulsion in attractive-repulsive models
Easy proof (apart from ‘only if’):

• $0 \leq d_{\text{RP}^n}(x, y) \leq 1$ with equality iff $x = y$ or $x \perp y$

• $d_{\text{RP}^n}(x, y)^b \geq d_{\text{RP}^n}(x, y)^{b+\epsilon}$ with the same conditions for equality

• $(\text{spt} \hat{\mu}_k) = \{\hat{e}_1, \ldots, \hat{e}_{n+1}\}^2$ lies in the equality set

• if $\hat{\mu}_k$ maximizes $E_b$ for some $b$, it also maximized for all $b + \epsilon > b$

• and every measure not supported in the equality set does strictly worse
Easy proof (apart from ‘only if’):

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- if $\hat{\mu}_k$ maximizes $E_b$ for some $b$, it also maximized for all $b + \epsilon > b$
- and every measure not supported in the equality set does strictly worse
- nonunique $n = b = 1 = \frac{k}{3}$ maximizer implies $b_{\Delta^n}(k) \geq 1$ if $k > n + 1$
- we argued $b_{\Delta^n}(k) < \infty$ in an earlier work; can also be viewed as a consequence of Turan’s theorem in graph theory (Bilyk et al if $k = \infty$)
- ‘only if’ asserts $\arg\max E_{\infty} \subset \arg\max E_{b_{\Delta^n}(k)}$ strictly
Where is the Optimal Transport?

$L^p$-Kantorovich-Rubinstein-Wasserstein distance $d_p$ on $\mu, \nu \in \mathcal{P}(\mathbb{RP}^n)$

$$d_p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[d_{\mathbb{RP}^n}(X, Y)^p]^{1/p} \quad p \in [1, \infty]$$

$$= \min_{\gamma \in \Gamma(\mu, \nu)} \left( \int \int d_{\mathbb{RP}^n}(x, y)^p d\gamma(x, y) \right)^{1/p} \quad p \in [1, \infty)$$

- $p < \infty$: metrizes weak-⋆ convergence of measures
- $d_\infty = \lim_{p \to \infty} d_p$ metrizes a much finer topology
- in this finer topology, $E_b(\mu)$ can have more local maxima
- c.f. McCann (PhD 1994, HJM 2006) stable binary stars
Weighted maximal projective simplices

For $0 \leq m_1 \leq m_2 \leq \cdots \leq m_{n+1}$ with $\sum_{i=1}^{n+1} m_i = 1$ set
\[ \vec{m} = (m_1, \ldots, m_{n+1}) \]

\[ \mu \vec{m} = \sum_{i=1}^{n+1} m_i \delta_{\hat{e}_i} \]

**Theorem (Weighted simplices are strict $d_\infty$-locally maxima $\forall b > 1$)**

Given $b_0 \in (1, \infty)$, $m_0 > 0$ and $n \in \mathbb{N}$ there exists $r = r_n(b_0, m_0)$ such that if $b \geq b_0$, $m_1 \geq m_0$ and $\mu \in \mathcal{P}(\mathbb{RP}^n)$ satisfy $d_\infty(\mu, \mu \vec{m}) < r$ then $E_b(\mu) \geq E_b(\mu \vec{m})$ and the inequality is strict unless $\mu$ is a rotate of $\mu \vec{m}$
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- crucial for proving $\arg\max E_\infty \subset \arg\max E_{b\Delta n(k)}$ strictly (i.e. ‘only if’)
- e.g. $\mu_\epsilon \in \arg\max_{\mathcal{P}(\mathbb{R}P^n)} E_{b\Delta n(k)} - \epsilon$ $d_2$-accumulates to $\mu_0 \in \arg\max_{\mathcal{P}(\mathbb{R}P^n)} E_{b\Delta n(k)}$
- on $\mathcal{P}_k^\infty(\mathbb{R}P^n)$, $d_\infty$ gives the same topology as $d_2$
Further evidence for Fejes Tóth’s conjecture?

**Corollary (Discontinuous bifurcation unless \( b_{\Delta n} = 1 \))**

*Fix \( n \in \mathbb{N} \). No curve \((\mu_b)_{b>0}\) of optimizers*

\[
\mu_b \in \arg\max_{\mathcal{P}(\mathbb{R}P^n)} E_b
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*can be \( d_\infty \)-continuous at \( b = b_{\Delta n} \) except possibly if \( b_{\Delta n} = 1 \).*
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can be \(d_\infty\)-continuous at \(b = b_{\Delta n}\) except possibly if \(b_{\Delta n} = 1\).

**Corollary (On connectedness of the threshold set of optimizers)**

Fix \(n \in \mathbb{N}\). If the set of threshold optimizers

\[
\arg\max_{\mathcal{P}(\mathbb{R}P^n)} E_{b_{\Delta n}}
\]

forms a \(d_\infty\)-connected subset of \(\mathcal{P}(\mathbb{R}P^n)\), then \(b_{\Delta n} = 1\).
Thank you!