A nonsmooth approach to Einstein’s theory of gravity

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Einstein’s gravity is formulated on smooth Lorentzian manifolds, but often predicts such manifolds are geodesically incomplete. Due to e.g. black hole (Penrose) or big bang (Hawking) type singularities a nonsmooth theory is highly desirable.

Example (Inspiring positive signature developments)

In metric(-measure) geometry with positive signature, there are theories of:

- sectional curvature bounds based on triangle comparison (Aleksandrov . . .)
- pointed Gromov-Hausdorff limits of manifolds under lower Ricci and upper dimensional bounds (Fukaya, Gromov, Cheeger-Colding, . . .)
- Ricci lower bounds via displacement convexity of entropy (Bakry-Emery, Lott-Sturm-Villani, Ambrosio-Gigli-Savare, . . .)
• Einstein’s gravity is formulated on smooth Lorentzian manifolds, but often predicts such manifolds are geodesically incomplete.

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Example (Inspiring positive signature developments)

In metric(-measure) geometry with positive signature, there are theories of

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Can something similar be done in Lorentzian geometry?

• tidal forces (Kunzinger-Sämann ’18)

• convergence of spaces (Müller 22+, Minguzzi-Suhr 22+)

• Einstein eq (M 20, Mondino-Suhr 23, Cavalletti-Mondino 20+, Braun 23)
Elliptic v hyperbolic geometry (c.f. BBCGMORS octet)

**ELLIPITC:** $\mathbb{R}^n$ equipped with Euclidean norm $\|v\|_E := (\sum v_i^2)^{1/2}$

- $\|v + w\|_E \leq \|v\|_E + \|w\|_E$

**HYPERBOLIC:** $\mathbb{R}^n$ equipped with the Minkowski ‘norm’

$$\|v\|_F := \begin{cases} (v_1^2 - \sum_{i \geq 2} v_i^2)^{1/2} & v \in F := \{v \in \mathbb{R}^n \mid v_1 \geq (\sum_{i \geq 2} v_i^2)^{1/2}\} \\ -\infty & \text{else} \end{cases}$$

- $\|v + w\|_F \geq \|v\|_F + \|w\|_F$

$F \subset \mathbb{R}^n$ is a convex cone called the *future*

- $v$ is *timelike* if $v \in F \setminus \partial F$
- $v$ is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$
  - $v$ is *spacelike* iff $\pm v \not\in F$ and *past-directed* if $-v \in F$
- smooth curves are called *timelike (etc.)* if all tangents are timelike (etc.)
A crash course in differential geometry: action principles

Manifold $M^n$ with symmetric nondegenerate smooth tensor field $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space

• its geometry is also encoded in the (symmetric) distance function

$$d(x, y)^q := \inf_{\sigma(0) = x, \sigma(1) = y} \int \|\dot{\sigma}_t\|_E^q \, dt \quad q > 1$$

LORENTZIAN: $g \sim (+1, -1, \ldots, -1)$ defines Minkowski norm on $T_x M$

• its asymmetric geometry is also encoded in the time-separation function

$$\ell(x, y)^q := \sup_{\sigma(0) = x, \sigma(1) = y} \int \|\dot{\sigma}_t\|_F^q \, dt \quad 0 < q < 1$$

• our convention $(-\infty)^q := -\infty$ implies

$$\ell(x, y) = -\infty \text{ unless there is a future-directed curve from } x \text{ to } y$$

• extremizers are independent of $q$; they are called geodesics
Concave $p$-energy: trading linearity for ellipticity

Additional conditions are imposed to ensure $\ell \neq +\infty$ and extremizers exist

- complete and/or proper (boundedly compact) in the Riemannian case
- global hyperbolicity in the Lorentzian case (i.e. compact diamonds, future $F$ varies continuously over $M$, no closed future-directed curves)
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Concave Hamiltonian $H(w) = \frac{1}{p} \|w\|_{F^*}^p$ and Lagrangian $L(v) = \frac{1}{q} \|v\|_{F^*}^q$
satisfy $DH = (DL)^{-1}$ if $p^{-1} + q^{-1} = 1$ (here $p < 0$ since $0 < q < 1$)

- note $-L = (-H)^*$ jumps to $+\infty$ across future cone boundary $\partial F$
  (but $-H$ diverges continuously at the boundary of the dual cone $F^*$)
- nonsmooth already on smooth Lorentzian manifolds
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Beran Braun Calisti Gigli M. Ohanyan Rott Sämann (octet):
extremizers of $p$-Dirichlet energy $u \mapsto \int_M H(du)d\text{vol}_g$ rel. to compactly supported perturbations satisfy a new degenerate elliptic nonlinear PDE
- trade linearity of d’Alembertian for ellipticity of $p$-d’Alembertian!
- gives new (elliptic) approach to Eschenburg-Galloway splitting theorem
The Riemann curvature tensor

Given (timelike) geodesics \((\sigma_s)_{s \in [0,1]}\) and \((\tau_t)_{t \in [0,1]}\) with \(\sigma_0 = \tau_0\) and \(\dot{\sigma}_0 - \dot{\tau}_0 \in F \setminus \partial F\),

\[
\ell(\sigma_s, \tau_t)^2 = \| s\dot{\sigma}_0 - t\dot{\tau}_0 \|^2_{F_g} - \frac{\text{Sec}}{6} s^2 t^2 + O((|s| + |t|)^5)
\]

where sectional curvature \(\text{Sec} = R(\dot{\sigma}_0, \dot{\tau}_0, \dot{\sigma}_0, \dot{\tau}_0)\) is quadratic in \(\dot{\sigma}_0 \wedge \dot{\tau}_0\)

and measures the leading order correction to Pythagoras

- polarization of this quadratic form gives the \textit{Riemann} tensor \(R(\cdot, \cdot, \cdot, \cdot)\)
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- polarization of this quadratic form gives the \textit{Riemann} tensor \(R(\cdot, \cdot, \cdot, \cdot)\)
- its trace \(\text{Ric}_{ik} = g^{jl} R_{ijkl}\) yields the \textit{Ricci} tensor; \(\text{Ric}(v, v)\) measures the correction to Pythagoras averaged over all triangles including side \(v\)
- second trace \(R = g^{ik} \text{Ric}_{ik}\) yields the \textit{scalar curvature}; in the elliptic case it gives leading order correction to the area of a sphere of radius \(r\) (and the volume of a ball of radius \(r\))
- \(d\text{vol}_g(x) = \sqrt{|\det(g)|} d^n x\) in coordinates; (in the Riemannian case it coincides with the \(n\)-dimensional Hausdorff measure associated to \(d\))
Gravity not a force; rather a manifestation of curvature of spacetime
“Spacetime tells matter how to move” (along timelike/null geodesics...)

Field equation “Matter tells spacetime how to bend”

\[ \text{geometry} = \text{physics} \]
\[ \text{curvature} = \text{flux of energy and momentum} \]
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Field equation “Matter tells spacetime how to bend”

\[
\text{geometry} \quad = \quad \text{physics} \\
\text{curvature} \quad = \quad \text{flux of energy and momentum} \\
Ric_{ij} - \frac{1}{2} R g_{ij} \quad = \quad 8\pi T_{ij}
\]

- just integrate this local conservation law for \( T_{ij}(x) \) to find \( g_{ij} \)…

What if matter distribution is unknown?
Energy conditions and singularity theorems

**WEC** (weak energy condition): \( T(v, v) \geq 0 \) for all future \( v \in F \) (physical)

**SEC** (strong energy condition): \( \text{Ric}(v, v) \geq 0 \) for all future \( v \in F \) (less "")

**NEC** (null energy condition): \( " \geq 0 \) for all lightlike \( v \in \partial F \)
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[Cosmological constant (dark matter): $\geq (n-1)Kg(v,v)$]

**Hawking ’66** (big bang type) singularity theorem:
SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface $\Sigma$ implies finite-time singularities along all timelike geodesics through $\Sigma$

**Cavalletti-Mondino ’20+:** genuinely nonsmooth version
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**Cavalletti-Mondino '20+:** genuinely nonsmooth version

**Kasue '83:** \( \Omega \subset \mathbb{R}^n \) with \( H_{\partial \Omega} > h > 0 \) bounds radius of largest ball in \( \Omega \)

**Burtscher-Ketterer-M.-Woolgar '20:** extend to \( CD(K, N) \) setting; in \( RCD(K, N) \) setting equality only if \( \Omega = \text{ball or cone} \)

**Penrose '65 (stellar collapse type) singularity theorem**

NEC + trapped codimension-2 compact surface \( S + \) suitable noncompact hypersurface \( \Sigma \) imply finite-time singularity along some null geodesic

**Open:** genuinely nonsmooth version?
A nonsmooth null energy condition

(Idea: reformulate the null energy condition in a timelike way)

Lemma

Any smooth Riemannian manifold admits $k \in C(M)$ such that $\text{Ric}(v, v) \geq k(x)g(v, v)$ for all $v \in T_xM$.

Proof.

$$k(x) = \inf_{v \in T_xM} \frac{\text{Ric}(v, v)}{g(v, v)}.$$

RMK: Can’t hold in Lorentzian setting, even for $v \in F$, unless NEC holds.
A nonsmooth null energy condition

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Theorem ((M. 23+) Not only sufficient, but necessary)

$\text{NEC} \iff \exists k \in C(M)$ such that every timelike vector $v \in T_xM$ satisfies

$\text{Ric}(v, v) \geq k(x)g(v, v)$.

i.e. NEC holds iff manifold admits a variable lower bound on timelike Ricci
What about the nonsmooth setting?

(please forget the foregoing)
Definition (Time-separation function)

On a set $M$ of events, a **time-separation function** refers to $\ell : M \times M \rightarrow \{-\infty\} \cup [0, \infty)$ satisfying the reverse triangle inequality and antisymmetry: $\forall x, y, z \in M$

$$\ell(x, y) \geq \ell(x, z) + \ell(z, y)$$  \hfill (1)

$$\min\{\ell(x, y), \ell(y, x)\} > -\infty \iff x = y.$$  \hfill (2)

Remark: (1) + (2) $\Rightarrow \ell(x, x) = 0$; (2) gives the arrow of time

Example (Minkowski space)

$M = R^{1,3}$ with $\ell(x, y) = \|y - x\|_F$

Example (Smooth globally hyperbolic Lorentzian manifolds)
Example (Causal spaces \((M, \leq, \ll) \text{ à la Kronheimer and Penrose '67}\))

A time-separation function gives a nested partial order \(\leq\) and preorder \(\ll\)
\[
M^2_{\leq} = \{(x, y) \in M^2 \mid \ell(x, y) \geq 0\}
\]
\[
M^2_{\ll} = \{(x, y) \in M^2 \mid \ell(x, y) > 0\}
\]

Definition (Causal & timelike futures; causal diamonds and emeralds)

We say \(y\) lies in the **causal future** of \(x\) and write \(x \leq y\) if \(\ell(x, y) \geq 0\); we say \(y\) lies in the **timelike future** of \(x\) and write \(x \ll y\) if \(\ell(x, y) > 0\). Also

\[
J^+(x) = \{y \in M \mid \ell(x, y) \geq 0\} = \text{future}
\]
\[
J^-(z) = \{y \in M \mid \ell(y, z) \geq 0\} = \text{past}
\]
\[
J(x, z) = J^+(x) \cap J^-(z) = \text{diamond}
\]
\[
J^+(X) = \bigcup_{x \in X} J^+(x)
\]
\[
J^-(Z) = \bigcup_{z \in Z} J^-(z)
\]
\[
J(X, Z) = J^+(X) \cap J^-(Z) = \text{emerald}
\]

and similarly \(I^\pm(y)\) and \(I(X, Z)\) but with strict inequalities \(\ell > 0\).
Definition (Causal and timelike paths)
A path \( s \mapsto \sigma(s) \in M \) is called \textit{causal} if and only if \( \ell(\sigma(s), \sigma(t)) \geq 0 \) for all \( s \leq t \), and \textit{timelike} if and only if \( \ell(\sigma(s), \sigma(t)) > 0 \) for all \( s < t \).

Definition (Lorentzian length of a causal path)
The (negative) \( \ell \)-length of a causal path \( \sigma : [a, b] \to M \) is defined by
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**Definition (Lorentzian length of a causal path)**

The **(negative) \( \ell \)-length** of a causal path \( \sigma : [a, b] \to M \) is defined by

\[
L_{-\ell}(\sigma) := \sup_{k \in \mathbb{N}} \sup_{a = t_0 \leq t_1 \leq \cdots \leq t_k = b} - \sum_{i=1}^{k} \ell(\sigma(t_{i-1}), \sigma(t_i)) \geq -\ell(\sigma(a), \sigma(b))
\]

by the triangle inequality.
Definition (\(\ell\)-path)

A path \(\sigma : [0, 1] \rightarrow M\) is called an \(\ell\)-path if and only if

\[ \ell(\sigma(s), \sigma(t)) = (t - s)\ell(\sigma(0), \sigma(1)) > 0 \quad \forall 0 \leq s < t \leq 1. \]

We denote the set of \(\ell\)-paths by \(\text{TPath}^{\ell}(M)\).

- the above shows each \(\ell\)-path minimizes \(L_{-\ell}\) relative to its endpoints
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- not all $L_{-\ell}$ minimizers are timelike, nor affinely parameterized
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Definition

\((M, \ell)\) is **timelike \(\ell\)-path space** if each \(x \ll y\) are endpoints of an \(\ell\)-path.

- Kunzinger and Sämann's **regular globally hyperbolic Lorentzian length spaces** provide a rich class of examples of timelike \(\ell\)-path spaces
- to achieve this, they need a (metrizable) topology
A metric space \((M, d)\) equipped with its metric topology and a time-separation function \(\ell\) is called a \textit{metric spacetime}.

A nonconstant causal \textit{path} is called a causal \textit{curve} if it is \textit{d-Lipschitz}.
a variation on Kunzinger & Sämann (hereafter K-S)

Definition (Metric spacetime)
A metric space \((M, d)\) equipped with its metric topology and a time-separation function \(\ell\) is called a metric spacetime.

Definition (Causal curve)
A nonconstant causal path is called a causal curve if it is \(d\)-Lipschitz.

Definition (Non-total imprisoning)
A metric spacetime \((M, d, \ell)\) is non-total imprisoning if each compact \(K \subset M\) has a bound \(\sup L_d(\sigma) < \infty\) on \(d\)-length of causal curves \(\sigma\) in \(K\).

Definition (Globally hyperbolic)
A metric spacetime \((M, d, \ell)\) is globally hyperbolic if it is non-total imprisoning and the causal diamond \(J(x, y)\) is compact for each \(x, y \in M\).
Definition (Timelike curve-connected; Lorentzian geodesic space)

A metric spacetime is *timelike curve-connected* iff each $x \ll y$ are connected by a timelike curve; it is a *Lorentzian geodesic space* iff each $x < y$ are connected by a causal curve $\sigma$ with $L_\ell(\sigma) = -\ell(\sigma(0), \sigma(1))$. 
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Theorem (M. 23+: Characterizing Lorentzian length spaces “LLS”)

Assuming globally hyperbolicity, a metric spacetime \((M, d, \ell)\) is an LLS iff it is (a) a timelike curve-connected (b) Lorentzian geodesic space; (c) \( \ell \) is upper semicontinuous; (d) \( \ell_+ = \max\{\ell, 0\} \) is continuous and (e) \( I^{\pm}(x) \) both nonempty \( \forall x \in M \).

- modelled on manifolds *without* boundary
- In such spaces, K-S showed that metric topology coincides with the order topology induced by \( \ll \); this implies g.h. LLS’s are independent of \( d \)!
- Burtscher & Garcia-Hevelling 21+ characterize global hyperbolicity of an LLS via existence of Cauchy time functions (and surfaces)
• Unfortunately, its not clear that all \( \ell \)-paths are continuous!

**Definition (Regular(ly localizable))**

An LLS is **regular** (or **regularly localizable**) if for any \( L_\ell \)-minimizing causal curve, \( L_\ell(\sigma|_{[a,b]}) = 0 \) with \( \sigma|_{[a,b]} \) non-constant implies \( L_\ell(\sigma) = 0 \).

**Lemma (M. 23+)**

In a globally hyperbolic regular LLS, each \( \ell \)-path is continuous.
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**Definition (Regular(ly localizable))**

An LLS is *regular* (or *regularly localizable*) if for any $L_\ell$-minimizing causal curve, $L_\ell(\sigma|_{[a,b]}) = 0$ with $\sigma|_{[a,b]}$ non-constant implies $L_\ell(\sigma) = 0$.

**Lemma (M. 23+)**

*In a globally hyperbolic regular LLS, each $\ell$-path is continuous.*

**Corollary (Relating $\ell$-paths to $L_\ell$-extremizers)**

*In a globally hyperbolic regularly localizable Lorentzian length space:*

(a) Every $\ell$-path becomes a $d$-Lipschitz $L_\ell$-minimizing curve after a continuous increasing (not necessarily Lipschitz) reparameterization.

(b) **K-S:** Conversely, every $L_\ell$-minimizing curve with timelike separated endpoints becomes an $\ell$-path after a similar reparameterization.
• For convenience, we deal only with metric spacetimes \((M, d, \ell)\) which are \textbf{closed Lorentzian geodesic subsets} of \textbf{globally hyperbolic regular Lorentzian length spaces} (= g.h.r. LLS).

Now that timelike geodesics exist:
• given a triple \(x \ll y \ll z\) of timelike related events, we can compare the Lorentzian length of a bisector to that of the Minkowski triangle with the same Lorentzian sidelengths

• and similarly for generalized bisectors (i.e. ratios other than 1 : 1)
• K-S define $T\text{-}\text{Sec}(M, d, \ell) \geq 0$ if our generalized bisector is longer (and $T\text{-}\text{Sec}(M, d, \ell) \leq 0$ if it is shorter) for all such timelike triangles.

• they define $\pm T\text{-}\text{Sec}(M, d, \ell) \geq k \in \mathbb{R}$ analogously by comparing to timelike triangles in constant curvature Lorentzian spaces.

• they also give causal sectional curvature bounds and show such bounds prevent branching of $\ell$-geodesics:
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• they also give causal sectional curvature bounds and show such bounds prevent branching of $\ell$-geodesics:

**Definition (timelike nonbranching)**

$(M, \ell)$ timelike nonbranching if for all $\tilde{\sigma}, \sigma \in T\text{Path}^\ell$ with $\sigma|_{\left[\frac{1}{3}, \frac{2}{3}\right]} = \tilde{\sigma}|_{\left[\frac{1}{3}, \frac{2}{3}\right]}$ then $\tilde{\sigma} = \sigma$;

• Alexander-Bishop '08 shows consistency of these definitions with smooth timelike sectional curvature bounds on Lorentzian manifolds

• Minguzzi-Suhr '22+ show stability of a similar bound

• Beran-Ohanyan-Rott-Solis '22+: $T\text{-Sec}(M, d, \ell) \geq 0$ and existence of a timelike line implies geometric splitting of $(M, d, \ell)$
To pass from sectional to Ricci curvature / Einstein eq requires averaging:

**Definition (Optimal transport distance between measures)**

- Given metric spaces \((M^\pm, d^\pm)\), let \(\mathcal{P}(M)\) denote the Borel probability measures on \(M\) and \(\mathcal{P}_c(M)\) those with compact support.

- **Push-forward**: given \(G : M^- \rightarrow M^+\) Borel and \(\mu^- \in \mathcal{P}(M^-)\), define \(\mu^+ = G#\mu^- \in \mathcal{P}(M^+)\) by \(\mu^+(B) = \mu^-(G^{-1}(B))\) for all \(B \subset M^+\).

- Letting \(\pi^\pm(x^-, x^+) = x^\mp\) denote the projection from \(M^- \times M^+\) onto its left and right factors, set \(\Gamma(\mu^-, \mu^+) = \{\gamma \in \mathcal{P}(M^- \times M^+) \mid \pi^\pm # \gamma = \mu^\pm\}\).
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- Given \(p \in [1, \infty)\) and \(M = M^\pm\), the \(p\)-Kantorovich-Rubinstein-Wasserstein distance \(d_p\) between \(\mu^\pm \in \mathcal{P}(M)\) defined by

\[
d_p(\mu^-, \mu^+) := \inf_{\gamma \in \Gamma(\mu^+, \mu^-)} \left( \int_{M^2} d(x, y)^p d\gamma(x, y) \right)^{1/p} \tag{3}
\]

is well-known to metrize convergence against functions growing no faster than \(d(x, \cdot)^p\) provided \((M, d)\) is Polish (i.e. complete and separable), in which case the inf is attained.

- If \((M, d)\) is a geodesic space so is \((\mathcal{P}_c(M), d_p)\).
Definition (Causal and timelike measures)

In a Polish g.h.r LLS \((M, d, \ell)\), given \(\mu, \nu \in \mathcal{P}_c(M)\) and \(q \in (0, 1]\) set

\[
\Gamma \leq (\mu, \nu) := \{ \gamma \in \Gamma(\mu, \nu) \mid \gamma[M^2_{\leq}] = 1 \} = \{ \text{causal measures} \}
\]

\[
\Gamma \ll (\mu, \nu) := \{ \gamma \mid \gamma[M^2_{\ll}] = 1 \} = \{ \text{timelike measures} \}
\]

Lemma (Lift time-separation from events to measures)

\[
\ell_q(\mu, \nu) := \max_{\gamma \in \Gamma \leq (\mu, \nu)} \left( \int_M \ell(x, y)^q d\gamma(x, y) \right)^{1/q}
\]

makes \((\mathcal{P}_c(M), \ell_q)\) into a timelike \(\ell_q\)-path space. Not all such \(\ell_q\)-paths are \(d_1\)-continuous;
Definition (Causal and timelike measures)

In a Polish g.h.r LLS \((M, d, \ell)\), given \(\mu, \nu \in \mathcal{P}_c(M)\) and \(q \in (0, 1]\) set

\[
\Gamma \leq (\mu, \nu) := \{ \gamma \in \Gamma(\mu, \nu) \mid \gamma[M^2] = 1 \} = \{ \text{causal measures} \}
\]

\[
\Gamma \ll (\mu, \nu) := \{ \gamma \mid \gamma[M^2] = 1 \} = \{ \text{timelike measures} \}
\]

Lemma (Lift time-separation from events to measures)

\[
\ell_q(\mu, \nu) := \max_{\gamma \in \Gamma \leq (\mu, \nu)} \left( \int_{M^2} \ell(x, y)^q d\gamma(x, y) \right)^{1/q}
\]  

makes \((\mathcal{P}_c(M), \ell_q)\) into a timelike \(\ell_q\)-path space. Not all such \(\ell_q\)-paths are \(d_1\)-continuous; one will be if \((\mu, \nu)\) is timelike \(q\)-dualizable:

Definition (timelike \(q\)-dualizability)

Let \(\Gamma^q = \Gamma^q(\mu, \nu)\) denote the set of maximizers. Then

- \((\mu, \nu)\) are timelike \(q\)-dualizable if \(\Gamma^q \ll := \Gamma^q \cap \Gamma \ll (\mu, \nu)\) is non-empty
- \((\mu, \nu)\) are strongly timelike \(q\)-dualizable if, in addition, \(\Gamma^q \subset \Gamma \ll (\mu, \nu)\)
Definition (Polish / proper metric-measure spacetime)

A *metric-measure spacetime* refers to a Lorentzian geodesic closed subset \((M, d, \ell)\) of a g.h.r. LLS, equipped with a Borel measure \(m \geq 0\), finite on bounded sets. It’s called *Polish* if complete and separable, and *proper* if all bounded subsets \(X \subset M\) are compact.

Example (Smooth metric-measure spacetimes)

Any smooth, connected, Hausdorff, time-oriented, \(n\)-dimensional Lorentzian manifold \((M^n, g)\) of signature \((+ - \ldots -)\) is second-countable (Ozeki-Nomizu ’61) and its topology comes from a complete Riemannian metric \(\tilde{g}\) (Geroch ’68). With the distance \(d_{\tilde{g}}\) and time-separation function \(\ell_g\) induced by \(\tilde{g}\) and \(g\) respectively, is a proper g.h.r. LLS provided it has no closed causal curves and causal diamonds \(J(x, y)\) are compact. Letting \(V \in C^{\infty}(M)\) and \(\text{vol}_g\) denote its Lorentzian volume, setting \(dm = e^{-V} \text{dvol}_g\) makes it a proper metric-measure spacetime. We call such spaces *smooth metric-measure spacetimes*. 
Synthetic timelike Ricci bounds

Desiderata:
• consistency (with the analogous smooth bounds)
• stability (preservation under suitable limits)
• consequences (e.g. Hawking-type singularity theorem)

Definition (Entropy)
We define the relative entropy by

\[ H(\mu \mid m) := \begin{cases} 
\int_M \rho \log \rho \, dm & \text{if } \mu \in \mathcal{P}^{ac}_c(M) \text{ and } \rho := \frac{d\mu}{dm}, \\
+\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}^{ac}(M).
\end{cases} \]

- our sign convention is opposite to that of the physicists’ entropy
**Definition (TCD versus \( wTCD \); e.g. \( K = 0 = 1/N \))**

For \((K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1] \) write \((M, d, \ell, m) \in wTCD^e_q(K, N) \) if and only if every strongly timelike \( q \)-dualizable finite entropy pair \( \mu_0, \mu_1 \in \mathcal{P}_c(M) \) admit a maximizer \( \gamma \in \Gamma_q^{\ll} \) and corresponding \( \ell_q \)-path \( (\mu_t)_{t \in [0,1]} \) along which the entropy \( t \in [0,1] \mapsto h(t) := H(\mu_t | m) \) is upper-semicontinuous and distributionally solves the semiconvexity inequality

\[
h''(t) \geq \frac{h'(t)^2}{N} + K \|\ell\|_{L^2(\gamma)}^2.
\]
Entropic weak timelike curvature-dimension conditions

**Definition (TCD versus $wTCD$; e.g. $K = 0 = 1/N$)**

For $(K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1]$ write $(M, d, \ell, m) \in wTCD_q^e(K, N)$ if and only if every strongly timelike $q$-dualizable finite entropy pair $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ admit a maximizer $\gamma \in \Gamma_q^q$ and corresponding $\ell_q$-path $(\mu_t)_{t \in [0, 1]}$ along which the entropy $t \in [0, 1] \mapsto h(t) := H(\mu_t | m)$ is upper-semicontinuous and distributionally solves the semiconvexity inequality

$$h''(t) \geq \frac{h'(t)^2}{N} + K \|\ell\|_{L^2(\gamma)}^2.$$

Cavalletti-Mondino '20+ prove all limits of $TCD_q^e(K, N)$ space in a suitable (pointed measured weak) sense lie in $wTCD_q^e(K, N)$ if $N < \infty$.
Fixing $x_j \in \text{spt } m_j$ where $m_j$ is a Radon measure, we say
$(M_j, d_j, \ell_j, m_j, x_j) \rightarrow_{\text{pmGL}} (M_\infty, d_\infty, \ell_\infty, m_\infty, x_\infty)$ iff all $(M_j, d_j, \ell_j, m_j, x_j)$ embed $d$-continuously and $\ell$-isometrically into a single proper g.h.r. LLS $(X, d, \ell)$ and after this embedding, $d(x_j, x_\infty) \rightarrow 0$ and the measures $m_j \rightarrow m_\infty$ converge weakly against continuous compactly supported test functions: i.e.

$$
\lim_{j \rightarrow \infty} \int_X \phi dm_j = \int_X \phi dm_\infty \quad \forall \phi \in C_c(X).
$$
Fixing $x_j \in \text{spt } m_j$ where $m_j$ is a Radon measure, we say
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$$\lim_{j \to \infty} \int_X \phi \, dm_j = \int_X \phi \, dm_\infty \quad \forall \phi \in C_c(X).$$

- although the limit of $TCD^e_q(K, N)$ spaces is only $wTCD^e_q(K, N)$,
  Braun '22+ shows (q-essentially) timelike nonbranching $wTCD^e_q(K, N)$ spaces are $TCD^e_q(K, N)$. Hence a limit of timelike nonbranching $wTCD^e_q(K, N)$ spaces is $wTCD^e_q(K, N)$. 


Positive energy $\iff$ displacement convexity of entropy

**DEF (N-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm’15)**

Given $N \neq n$ and $V \in C^\infty(M^n)$ define

$$R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N - n}(\nabla_i V)(\nabla_j V)$$

**THM (M ’20 Consistency)** Fix $(K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1)$ and a smooth metric-measure spacetime $(M^n, g)$ with $dm = e^{-V}d\text{vol}_g$. Then $(M, d_{\tilde{g}}, \ell_g, m) \in (w) \text{TCD}_q^e(K, N)$ if and only if either

(a) $N = n$, $V = \text{const}$ and $R_{ij}v^i v^j \geq K$ for all unit timelike $(v, x) \in TM$,

(b) $N > n$ and $R_{ij}^{(N,V)} v^i v^j \geq K$ for all unit timelike vectors $(v, x) \in TM$. 

Mondino-Suhr ’18+ Use entropic convexity to say also when equality holds, giving a weak (but unstable) solution concept for Einstein field equation.

Akdemir-Cavalletti-Colinet-M.-Santarcangelo ’21 CD$_p(K, N)$ $\cap \{\text{nonbranching}\}$ is independent of $p > 1$.
Positive energy ⇔ displacement convexity of entropy

DEF \((N\)-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm’15) Given \(N \neq n\) and \(V \in C^\infty(M^n)\) define

\[
R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N-n} (\nabla_i V)(\nabla_j V)
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THM (M ’20 Consistency) Fix \((K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1)\) and a smooth metric-measure spacetime \((M^n, g)\) with \(dm = e^{-V} d\text{vol}_g\). Then \((M, d_{\tilde{g}}, \ell_g, m) \in (w) \text{TCD}_q^e(K, N)\) if and only if either

(a) \(N = n\), \(V = \text{const}\) and \(R_{ij} \nu^i \nu^j \geq K\) for all unit timelike \((\nu, x) \in TM\),

(b) \(N > n\) and \(R_{ij}^{(N,V)} \nu^i \nu^j \geq K\) for all unit timelike vectors \((\nu, x) \in TM\).

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Akdemir-Cavalletti-Colinet-M.-Santarcangelo ’21
\(CD_p(K, N) \cap \{\text{nonbranching}\}\) is independent of \(p > 1\)
Lazy Gas Experiment \textbf{(M. 94, Villani 09)}

16 Displacement convexity 1

Action minimizing paths satisfy pressureless Euler equation.
Braun 23:

- $N = \infty$
- alternative definitions of $(w) TCD_q^{(*)}(K, N)$ based on convexity properties of a power-law entropy (instead of $H(\mu \mid m)$) along $\ell_q$-paths

$$S_N(\mu) := -N \int_M \left( \frac{d\mu}{dm} \right)^{1-\frac{1}{N}} dm$$

- equivalence of most of these various definitions to $TCD_q^e(K, N)$ assuming ($q$-essential) timelike nonbranching

Cavalletti-Mondino ’22:

- asked for a synthetic formulation of the null energy condition (NEC)
- stronger physical motivation; more widely satisfied
- forms a key hypothesis in the Penrose singularity theorem for stellar collapse
Theorem (M’ 23+)

Fix a smooth spacetime \((M^n, g)\) with signature \((+−⋯−)\) and symmetric 2-tensor field \(Q\). Then

\[Q(v, v) \geq 0 \quad \forall (v, x) \in TM \text{ with } g(v, v) = 0\]

holds if and only if each compact subdomain \(X \subset M^n\) admits a timelike lower bound \(K = K_X\) for \(Q\), i.e.

\[Q(v, v) \geq Kg(v, v) \quad \forall (v, x) \in TX \text{ with } g(v, v) > 0\]

Taking \(Q = \text{Ric}^{(N,V)}\) (or \(Q_{ab} = 8\pi T_{ab}\) if Einstein holds) motivates

Definition (A synthetic null energy-dimension condition)

Given \((N, q) \in (0, \infty] \times (0, 1)\), a metric-measure spacetime \((M, d, \ell, m)\) satisfies \(wNC_q^{(e)}(N)\) if and only if each compact subset \(X \subset M\) admits a bound \(K = K_X \in \mathbb{R}\) such that \(J(X, X) \in wTCD_q^{(e)}(K, N)\).
• in other words, the null energy condition is equivalent to a variable lower (semicontinuous) bound $k(x)$ on the timelike Ricci curvature

• Consistency with smooth (NEC) + $(n \leq N)$: follows from theorem above

• for $(q$-essentially) timelike nonbranching spaces $wNC^e_q(N) = NC^*_q(N)$

• Consequences: many of Cavalletti & Mondino’s nice properties (timelike Bishop-Gromov and Brunn-Minkowski inequalities, needle decomposition, etc) of nonsmooth $wTCD_q^{(e)}(K, N)$ spacetimes are therefore inherited directly by $wNC_q^{(e)}(N)$ spacetimes; c.f. Braun-M. (in progress)

• (In)stability: on the other hand, any stability result appears hopeless unless we are will to assume some uniformity in $j$ of the lower bound $k(\cdot)$ along the sequence $(M_j, d_j, \ell_j, m_j, x_j)$
• in other words, the null energy condition is equivalent to a variable lower (semicontinuous) bound \(k(x)\) on the timelike Ricci curvature

• Consistency with smooth (NEC) + \((n \leq N)\): follows from theorem above

• for \((q\text{-essentially})\) timelike nonbranching spaces \(wNC_q^e(N) = NC_q^*(N)\)

• Consequences: many of Cavalletti & Mondino’s nice properties (timelike Bishop-Gromov and Brunn-Minkowski inequalities, needle decomposition, etc) of nonsmooth \(wTCD_q^{(e)}(K, N)\) spacetimes are therefore inherited directly by \(wNC_q^{(e)}(N)\) spacetimes; c.f. Braun-M. (in progress)

• (In)stability: on the other hand, any stability result appears hopeless unless we are will to assume some uniformity in \(j\) of the lower bound \(k(\cdot)\) along the sequence \((M_j, d_j, \ell_j, m_j, x_j)\)

• BBCGMORS: infinitesmally Minkowski refinement of \(TCD_q^e\), analogous to Ambrosio-Gigli-Savarè’s infinitesmally Hilbertian refinement \(RCD\) of \(CD\)

• OPEN: it is natural to wonder if a Penrose singularity theorem can hold in this nonsmooth setting? (c.f. Graf ’20 on \(g \in C^1\) spacetimes \((M^n, g)\), Ketterer ’23+ entropic convexity derivation on \(g \in C^\infty\) spacetimes)
A few references

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Ketterer arXiv:2304.01853


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HAPPY BIRTHDAY, LUIGI!