Displacement Convexity of Boltzmann’s Entropy Characterizes Positive Energy in General Relativity

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Two famous sign laws in physics

- gravity is always attractive, never repulsive
- entropy always goes up, never down

 Might these be related?
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- **Bekenstein ’73**: 2nd law of black hole dynamics
  area of horizons can only increase
- **Jacobson ’95**: Einstein’s equation follows from Entropy := Horizon Area
- **E Verlinde ’11**: Gravity as an emergent entropic (i.e. statistical) force

My purpose today is to describe a new connection between gravity and entropy using optimal transport leading toward a nonsmooth theory of gravity
General Relativity
(Einstein’s theory of gravity)

“gravity not a force, merely a manifestation of curvature in the underlying geometry of spacetime”

Flat Earth Society
Einstein’s Tensor Equation

“geometry = physics”

\[ G_{ab} = 8\pi T_{ab} \]

average sectional curvature in a given direction MINUS (a multiple of) the same quantity averaged over all directions

energy and momentum fluxes of matter in system

\( a, b = t, x, y, z \)

Signature (i.e. dimensions) of space+time = 3+1
Spaceship near a black hole

\[ 0 = 16\pi T_{tt} = G_{tt} = \]

\[ = R_{txtx} + R_{tyty} + R_{tztz} \]

- side to side (squeeze)
- front to back (squeeze)
- top to bottom (head to toe stretch)
Lorentzian manifold

future cone

past cone

\( T_xM \)

\((M^n, g_{ij})\)

\[ g = \text{diag} (+1, -1, -1, \ldots, -1) \]

\( n-1 \)
Terminology and conventions

$0 \neq v \in T_x M$ is

(a) timelike if $g(v, v) > 0$
(b) lightlike (or null) if $g(v, v) = 0$
(c) spacelike if $g(v, v) < 0$

(d) causal if (a) or (b) hold, in which case

(e) future-directed if it lies in the green cone
(f) past-directed if it lies in the red cone

A $C^1$ curve $\sigma : (a, b) \to M$ is said to have the property (a-f) if each of its tangent vectors does.

Particles with mass follow timelike future-directed curves on $M$. 
**Weak** energy condition: \( G_{ij} v^i v^j \geq 0 \) for all timelike \((v, x) \in TM\) (believed to be satisfied in all physical geometries)

**Strong** energy condition: \( R_{ij} v^i v^j \geq 0 \) for all timelike \((v, x) \in TM\), where

\[
G_{ij} = R_{ij} - \frac{1}{n-2} R g_{ij}
\]

here \( R_{ij} \) is the *Ricci curvature* tensor and \( R = g^{ij} R_{ij} \) is its scalar *trace*.

- less universally satisfied
- does not imply weak energy condition
- implies gravity is attractive
- was used by Hawking and Penrose to show “trapped” spacelike surfaces (whose areas decrease instantaneously in all possible futures) imply singularities
We’ll assume *global hyperbolicity* of \((M, g)\), meaning

- \((M, g)\) is smooth, connected, Hausdorff, timelike-orientable
- has no closed future-directed curves (i.e. no ‘back to the future’)
- \(J^+(x) \cap J^-(y)\) is compact for all \(x, y \in M\), where 
  \(J^+(x)\) is the set of points reached from \(x\) along future-directed curves 
  \(J^-(y)\) is the set of points reached from \(y\) along past-directed curves

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For \(q \in (0, 1]\) introduce the **convex(!) Lagrangian** on \(TM\)

\[
L(v, x; q) := \begin{cases} 
- \frac{1}{q} g(v, v)^{q/2} & \text{if } v \text{ is future-directed} \\
+\infty & \text{else}.
\end{cases}
\]

and its convex dual **Hamiltonian** \(H = L^*\) given by \(\frac{1}{q} + \frac{1}{q'} = 1\) (if \(q \neq 1\)) and

\[
H(p, x; q) := \begin{cases} 
- \frac{1}{q'} g(p, p)^{q'/2} & \text{if } p \in T^*_xM \text{ is past-directed} \\
+\infty & \text{else}.
\end{cases}
\]

- notice \(q' < 0\)
associated convex action on $C^1$ curves $\sigma : [0, 1] \rightarrow M,$

$$A[\sigma; q] := \int_0^1 L(\sigma'(s), \sigma(s); q) ds.$$ 

The ($q$-dependent) Lorentz distance is defined by least action

$$\ell(x, y; q) := - \inf_{\sigma(0) = x, \sigma(1) = y} A[\sigma; q]$$
associated convex \textit{action} on $C^1$ curves $\sigma : [0, 1] \rightarrow M$,

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The ($q$-dependent) \textit{Lorentz distance} is defined by least action

$$\ell(x, y; q) := - \inf_{\substack{\sigma(0) = x \sigma(1) = y}} A[\sigma; q]$$

$$= \frac{1}{q} \ell(x, y; 1)^q \quad (\text{Jensen's } \neq)$$

$$\in \{ -\infty \} \cup [0, \infty)$$

where $\ell(x, y) := \ell(x, y; 1)$ is also called the \textit{time-separation} function, and denotes the maximum a particle can \textit{age} between $x$ and $y$. Throughout we adopt the convention

$$(-\infty)^q := -\infty =: (-\infty)^{1/q}$$
The time-separation function satisfies a *backwards* triangle inequality:

\[ \ell(x, z) + \ell(z, y) \leq \ell(x, y) \quad \forall x, y, z \in M. \]

Global hyperbolicity ensures the infimum defining \( \ell(x, y; q) > 0 \) is attained; for \( q < 1 \), the curve \( \sigma \) which attains it is a *geodesic segment*, i.e.
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\[ \ell(\sigma(s), \sigma(t)) = |t - s|\ell(\sigma(0), \sigma(1)) \quad \forall \ 0 \leq s < t \leq 1 \]

The resulting curve \( \sigma \) is the proper-time- (i.e. age-) maximizing trajectory between \( x \) and \( y \) (as in the twin paradox).

Apart from sign restrictions, this is quite analogous to the construction of shortest length curves achieving the Riemannian distance \( d(x, y) \) in the case of a positive definite metric \( \tilde{g} \) on \( M \), except that in that case the convex Lagrangian would be defined using an exponent \( q \geq 1 \) (e.g. \( q = 2 \)) instead of \( q \in (0, 1] \).
In the Riemannian case, given arclength-parameterized geodesics $\sigma(s)$ and $\tau(t)$ through a common point $\sigma(0) = \tau(0)$, a local Taylor expansion yields

$$d^2(\sigma(s), \tau(t)) = s^2 + t^2 - 2st\tilde{g}(\dot{\sigma}(0), \dot{\tau}(0))$$

$$- \frac{s^2 t^2}{6} \tilde{R}_{ijkl}\dot{\sigma}^i \dot{\tau}^j \dot{\sigma}^k(0) \dot{\tau}^l(0) + O(|s|^5 + |t|^5)$$

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where $\tilde{R}_{ijkl}$ is the Riemannian curvature tensor. It measures the leading correction to Pythagoras’ law, and also the failure of covariant derivatives wrt $\tilde{g}$’s Levi-Civita connection ($\tilde{\nabla}_i \tilde{g}_{jk} = 0$) to commute:

$$\tilde{R}_{ijkl} v^k = -[\tilde{\nabla}_i, \tilde{\nabla}_j] v^l$$

Its trace $\tilde{R}_{ik} := \tilde{g}^{jl} \tilde{R}_{ijkl}$ gives the Ricci tensor associated to $\tilde{g}_{ij}$. 

Robert J McCann (Toronto)
The Riemann and Ricci tensors $R_{ijkl}$ and $R_{ik}$ associated to a Lorentzian metric $g_{ij}$ can be defined analogously. Gravitational attractivity stems from positivity of $R_{ij}$ in timelike directions. But what has this to do with entropy or the second law?
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In the Riemannian setting, a line of developments starting from M. ’94 Otto & Villani ’00 Cordero-Erausquin, M., & Schmuckenschläger ’01 led von Renesse & Sturm ’04 to characterize $R_{ij} \geq 0$ via the convexity of Boltzmann’s entropy along $L^2$-Kantorovich-Rubinstein-Wasserstein geodesics given by optimal transportation of probability measures.
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This inspired Sturm ’06, Lott and Villani ’09 to adopt such convexity as the definition of lower Ricci bounds in a (non-smooth) metric-measure setting, leading to the blossoming study of curvature-dimension spaces $CD(K, N)$ developed by Ambrosio, Gigli, Savare, Erbar, Kuwada, Sturm, ...
• Highlights of the theory include contraction results for diffusion semigroups (Ambrosio et al, Carrillo-M.-Villani, Otto, Sturm)
• Bonnet-Myers diameter bounds (Lott-V., Sturm)
• Splitting (Gigli) and rigidity (Ketterer) results
• Comparison theorems for isoperimetric profiles (Cavalletti-Mondino, Milman)
• Rectifiability (Mondino-Naber)
• and presumably much more to come
Can something similar be done in the Lorentzian setting?

Use Lorentz distance $\ell(x, y)$ to lift the geometry from $M$ to the set $\mathcal{P}_c(M)$ of (compactly supported for simplicity) Borel probability measures on $M$: Given $0 < q \leq 1$ and $\mu_0, \mu_1 \in \mathcal{P}_c(M)$, define

$$\ell_q(\mu_0, \mu_1) := \left( \sup_{\gamma \in \Gamma(\mu_0, \mu_1)} \int_{M \times M} \ell(x, y)^q \, d\gamma(x, y) \right)^{1/q},$$

where the supremum is over joint measures $\gamma \geq 0$ on $M \times M$ having $\mu_0$ and $\mu_1$ as left and right marginals

- this is a (Kantorovich ’42) optimal transport problem
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- this is a (Kantorovich ’42) optimal transport problem with lower semicontinuous cost $-\ell^q$ whose gradient diverges as the boundary of the causal set $J^+ = \ell^{-1}([0, \infty))$ is approached, and which jumps to $+\infty$ outside $J^+$.

- these singularities reflects the degeneration of both strict convexity and of smoothness for the Lagrangian $L(v, x; q)$ at the light cone

- still the supremum is attained by some $\gamma$ which will be called $\ell^q$-optimal (unless $\ell_q(\mu_0, \mu_1) = -\infty$).
A close variant of $\ell_q(\mu_0, \mu_1)$ was defined in Eckstein & Miller '17, who show $\ell_q$ inherits the reverse triangle inequality from $\ell(x, y)$:

$$\ell_q(\mu_0, \mu_1) \geq \ell_q(\mu_0, \nu) + \ell_q(\nu, \mu_1).$$

DEFN: We say $(\mu_s)_{s \in [0, 1]}$ is a $q$-geodesic in $\mathcal{P}_c(M)$ iff
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**DEFN:** We say $(\mu_s)_{s \in [0,1]}$ is a $q$-geodesic in $\mathcal{P}_c(M)$ iff

$$\ell_q(\mu_s, \mu_t) = |t-s|\ell_q(\mu_0, \mu_1) > 0 \quad \forall \quad 0 \leq s < t \leq 1.$$

- $q$-geodesics exist, if $\ell > 0$ a.e. wrt a Kantorovich maximizer $\gamma \in \Gamma(\mu_0, \mu_1)$

- When $\ell > 0$ on $\text{spt}[\mu_0 \times \mu_1]$ (:= smallest closed set of full mass), so that $\mu_1$ lies entirely in the timelike future of $\mu_0$, we’ll characterize the $q$-geodesic joining them uniquely provided $\mu_0 \in \mathcal{P}_c^{ac}(M)$, meaning $\mu_0$ is absolutely continuous wrt the Lorentzian volume

$$\text{dvol}_g(x) \quad (:= |\det g_{ij}(x)|^{1/2}d^n x \text{ in coordinates}).$$
DEFN Setting $dm(x) := e^{-V} dv_{\text{vol}_g}(x)$ where $V \in C^2(M)$ we define the relative entropy by

$$E_V(\mu) := \begin{cases} 
\int_M \rho \log \rho \, dm & \text{if } \mu \in \mathcal{P}_c^{ac}(M) \text{ and } \rho := \frac{d\mu}{dm}, \\
+\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}^{ac}(M).
\end{cases}$$

- When $V = 0$ the Boltzmann-Shannon entropy $E_0(\mu)$ results.
- our sign convention is opposite to that of the physicists' entropy
THM 1: *(Positive energy = entropic displacement convexity)*

Fix $0 < q < 1$ and a globally hyperbolic spacetime $(M^n, g)$.

(a) If $R_{ij}v^iv^j < K \in \mathbb{R}$ for SOME unit timelike vector $(v, x) \in TM$, there is a $q$-geodesic $(\mu_s)_{s \in [0,1]}$ supported arbitrarily close to $x$ and with $\ell > 0$ on $\text{spt}[\mu_0 \times \mu_1]$ along which $e(s) := E_0(\mu_s) \in C^2$ and $e''(0) < K\ell_q(\mu_0, \mu_1)^2$. 
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(b) Conversely, if $R_{ij} v^i v^j \geq Kg(v, v) \geq 0$ for EVERY timelike $(v, x) \in TM$, then

$$e''(s) \geq \frac{1}{n} e'(s)^2 + K \ell_q(\mu_0, \mu_1)^2 \geq 0$$

holds (distributionally) along ALL $q$-geodesics $(\mu_s)_{s \in [0, 1]}$ having finite entropy endpoints and $\ell > 0$ on $\text{spt}[\mu_0 \times \mu_1]$. 
Lazy Gas Experiment (M. 94, Villani 09)

16 Displacement convexity 1

Action minimizing paths satisfy pressureless Euler equation.
\( Rc > 0 \)

\( Rc \leq 0 \)
DEFN ($N$-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm’15)

Given $N \in (n, \infty]$ and $V \in C^2(M)$ define

$$ R^{(N,V)}_{ij} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N - n}(\nabla_i V)(\nabla_j V) $$

THM 1’ Fix $0 < q < 1$ and a globally hyperbolic spacetime $(M^n, g)$.

(a) If $R^{(N,V)}_{ij} \nu^i \nu^j < K \in \mathbb{R}$ for some unit timelike vector $(\nu, x) \in TM$, there is a $q$-geodesic $(\mu_s)_{s \in [0,1]}$ supported arbitrarily close to $x$ and with $\ell > 0$ on $\text{spt} [\mu_0 \times \mu_1]$ along which $e(s) := E_V(\mu_s) \in C^2$ and $e''(0) < K \ell q(\mu_0, \mu_1)^2$. 
DEFN \((N\)-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm’15)\)
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THM 1’ Fix \(0 < q < 1\) and a globally hyperbolic spacetime \((M^n, g)\).

(a) If \(R^{(N,V)}_{ij} v^i v^j < K \in \mathbb{R}\) for some unit timelike vector \((v, x) \in TM\), there is a \(q\)-geodesic \((\mu_s)_{s \in [0,1]}\) supported arbitrarily close to \(x\) and with \(\ell > 0\) on \(\text{spt}[\mu_0 \times \mu_1]\) along which \(e(s) := E_V(\mu_s) \in C^2\) and \(e''(0) < K \ell_q(\mu_0, \mu_1)^2\).

(b) Conversely, if \(R^{(N,V)}_{ij} v^i v^j \geq Kg(v, v) \geq 0\) for every timelike \((v, x) \in TM\), then
\[
e''(s) \geq \frac{1}{N} e'(s)^2 + K \ell_q(\mu_0, \mu_1)^2(\geq 0)
\]
holds (distributionally) along all \(q\)-geodesics \((\mu_s)_{s \in [0,1]}\) having finite entropy \(q\)-\textit{separated} endpoints.
Remarks and related developments

1. Although the Ricci tensor is only defined in the smooth setting, as in the Riemannian case the entropic characterization can be used to define the strong energy condition in a metric measure setting (future work).

2. The smooth manifolds which satisfy it do not depend on $q \in (0, 1]$; whether or not the nonsmooth spaces which satisfy it depend on $q$ is unclear, even in the analogous Riemannian problem (c.f. Kell '17; w.i.p)

3. A few months later, a related but independent local construction was announced by Mondino and Suhr '18+ which also gives the complementary lower bound — thus a weak formulation to the full Einstein equation!

4. Kunzinger & Sämann '17+ give a synthetic approach to two-sided sectional curvature bounds via triangle comparison.

5. Loeper '06 first observed the enhancement of displacement convexity by Newtonian gravity in the Euclidean setting.

6. Gomes & Senici '18+ give a heuristic proof that displacement convexity extends to the planning problem from mean fields games with local congestion interactions.
To prove this, used linear programming duality to analyze the optimal transportation problem defining

\[
\frac{1}{q} \ell_q(\mu, \nu)^q = \inf_{u \oplus v \geq \frac{1}{q} \ell_q} \int_M u \, d\mu + \int_M v \, d\nu,
\]

Unfortunately, the singularities of \( \ell \) may prevent attainment of this Kantorovich dual infimum by (lsc) potentials \((u, v)\) satisfying

\[
(*) \quad u(x) + v(y) \geq \frac{1}{q} \ell(x, y)^q \quad \forall \quad (x, y) \in \text{spt}[\mu \times \nu]
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DEFN: Fix \( q \in (0, 1] \). We say \((\mu, \nu) \in \mathcal{P}_c(M)\) is \( q \)-separated by \( \gamma \in \Gamma(\mu, \nu) \) and \((u, v)\) if \((\ast)\) is satisfied and \( \text{spt} \gamma \subset S \subset \{\ell > 0\} \) where \( S \) is the set of \((x, y) \in \text{spt}[\mu \times \nu]\) producing equality in \((\ast)\)
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**LEMMA:** (a) \( q \)-separation of \((\mu_0, \nu_0)\) implies optimality of \( \gamma \) and \((u, v)\).  
(b) \( \ell > 0 \) on \( \text{spt}[\mu_0 \times \mu_1] \) is sufficient (but not necessary) for \( q \)-separation.  
(c) \((\mu_s, \mu_t)\) inherit \( q \)-separation along the \( q \)-geodesic from \( \mu_0 \) to \( \mu_1 \).
THM (Lagrangian characterization of $q$-geodesics)

Fix $0 < q < 1$. If $(\mu_0, \mu_1) \in \mathcal{P}_c^2(M)$ is $q$-separated by $(\gamma, u, v)$ and $\mu_0 << \text{vol}_g$ then the map $F_s(x) := \exp_x sDH(Du(x); q)$ induces the unique $q$-geodesic $s \in [0, 1] \mapsto \mu_s$ in $\mathcal{P}_c(M)$ linking $\mu_0$ to $\mu_1$.

Moreover, $\mu_s << \text{vol}_g$ if $s < 1$ (by using uniform convexity of $L$ away from light cone to adapt Monge-Mather ‘shortening’ estimate).

Here $\mu_s := (F_s)_\# \mu_0$ is defined by

$$\mu_s[V] := \mu_0[F_s^{-1}(V)] \quad \forall V \subset M,$$
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Here $\mu_s := (F_s)_#\mu_0$ is defined by

$$\mu_s[V] := \mu_0[F_s^{-1}(V)] \quad \forall V \subset M,$$

$F_s$ is an optimal (i.e. Monge) map between $\mu_0$ and $\mu_s$, and $\gamma = (id \times F_1)_#\mu_0$ uniquely maximizes the Kantorovich problem defining $\ell_q(\mu_0, \mu_1)$.

Setting $\rho_s := \frac{d\mu_s}{d\text{vol}_g}$ yields the Monge-Ampère type equation

$$\rho_0(x) = \rho_s(F_s(x)) |JF_s(x)| \quad \rho_0 - a.e.,$$

where $JF_s(x) = \det D\tilde{F}_s(x)$ is the (approximate) Jacobian of $F_s$ and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} (\tilde{D}F_s) = D^2H|_{Du} \tilde{D}^2u.$$
\[ u(x, y) = \nabla G(x, y) \]

\[ u(x, y) = \frac{\nabla G(x, y)}{\gamma} \]

\[ \text{on } S: \quad \nabla u(x) = \frac{\nabla \log (x, y)}{\gamma} = \frac{\nabla \log (x, y)}{\gamma} \frac{\dot{\gamma}(t)}{\gamma(t)} \]

\[ \partial_h (\nabla u(x)) = -10 \nabla u(x) \gamma^{-1} \quad \nabla u(x) = \frac{\dot{\gamma}(t)}{\gamma(t)} \]

\[ \sigma(s) = \exp_x s \cdot \delta(y) \]
Remarks and related developments

1. A posteriori, it is possible to relax the $q$-separation hypothesis à la Gigli '12 to extend these results provided $\gamma \in \Gamma(\mu_0, \mu_1)$ exists with $\ell > 0$ holding $\gamma$-a.e.

2. An independent alternative approach to solving the dual problem in case $q = 1$ was announced a few months later by Kell and Suhr '18+, following Suhr '16+'s earlier study of that case (which was in turn inspired by work on the relativistic heat equation by Brenier '03, M. & Puel '09, and Bertrand & Puel '13 with Pratelli '18)

3. Optimal transport with respect to smooth, strictly convex Hamiltonians was studied in varying degrees of generality Bernard & Buffoni '06, Villani '09, Fathi & Figalli '10 and Agueh '02-, Ohta '09, Lee '13, Kell '17, Schachter '17.

4. Not much beyond #2 above on singular Lagrangians, apart from the subRiemannian case investigated by Ambrosio & Rigot '04, Agrachev & Lee '09, Figalli & Rifford '10 and Lee, Li & Zelenko '16, Balogh, Kristály & Sipos '18.
Conclusions: optimal transport relates gravity to entropy

1. Fractional powers $0 < q < 1$ of the time-separation $\ell(x, y)$ come from a Lagrangian $L$, smooth and strictly(!) convex away from the light cone.

2. Optimal transport with respect to this cost lifts the geometry from spacetime events $M$ to probability measures on $M$.

3. $q$-separation of the target and source makes this transportation problem and its dual analytically tractable.

4. Convexity properties of Boltzmann’s entropy along timelike geodesics of probability measures provide a robust formulation of the strong energy condition of Hawking and Penrose ’70 — and via Mondino & Suhr 18+’s parallel work, of Einstein’s field equations.

5. This provides a new approach to gravity without smoothness — much desired in view of the singularity theorems from general relativity.

6. Whereas the second law of thermodynamics is encoded in the first time-derivative of entropy, the Einstein equations of gravity are encoded in its second time-derivative along $q$-geodesics.
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THANK YOU!