On the Monopolist's Problem Facing Consumers with Linear and Nonlinear Price Preferences

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CPAM '19 + work in progress

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Outline

Monopolist's problem

Given compact sets $X \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^n$ and $Z = [\underline{z}, \infty) \subset \mathbf{R}$ and
$G(x, y, z)$ = value of product $y \in Y$ to buyer $x \in X$ at price $z \in Z$
$d\mu(x)$ = relative frequency of buyer $x \in X$ (as compared to $x' \in X$)
$\pi(x, y, z)$ = value to monopolist of selling y to x at price z

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$$\tilde{\Pi}(v) := \int_X \pi(x, y_v(x), v(y_v(x))d\mu(x), \qquad \text{where}$$

Agent x's problem: choose $y_v(x)$ to maximize

$$y_v(x) \in \arg\max_{y \in Y} G(x, y, v(y))$$

Constraints: v lower semicontinuous, $(0, 0) \in Y \times Z$ and $v(0) = 0$.

Examples

• airline ticket pricing

• insurance: monopolist's profit $\pi(x, y, z)$ may depend strongly on buyer's identity x, even if regulation/ ignorance prohibits price $v(y)$ from doing so

• z-dependence of $G(x, y, z)$ reflects different buyers price sensitivity / risk non-neutrality

• educational signaling

• optimal taxation: replace profit maximization with a budget constraint for providing services

Some history: $G(x, y, z) = b(x, y) - z$

Mirrlees '71, Spence '73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$
Rochet-Choné '98 ($n = m > 1$): $b(x, y) = x \cdot y$ *bilinear* implies $y_v(x) = Dv^*(x)$ convex gradient; bunching

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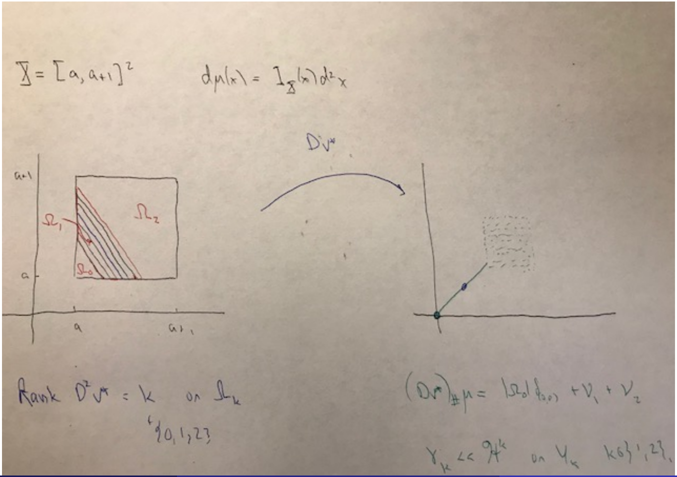

Carlier-Lachand-Robert '03: $v^* \in C^1(\operatorname{spt}\mu)$; Caffarelli-Lions $v^* \in C^{1,1}$

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Chen '13: $u \in C^1$ under Ma-Trudinger-Wang (MTW) conditions, where

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Figalli-Kim-M. '11:
convexity of principal's problem under strengthening of (MTW) on $b(x,y)$

Noldeke-Samuelson (ECMA '18), Zhang (ET '19):
existence of maximizing v for general $G \in C^0$

Daskalakis-Dekelbaum-Tzamos (ECMA '17), Kleiner-Manelli (ECMA '19):
duality for multigood auctions

Hypothesis (c.f. Trudinger's generated Jacobian equations)

(G0) $G \in C^1(X \times Y \times Z)$, $m \geq n$, and for each $x, x_0 \in X \subset \mathbf{R}^m$:

(G1) $(y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z)$ is a homeomorphism

(G2) with convex range $(Y \times Z)_x := (D_x G, G)(x, Y, Z)$ and inverse $\bar{y}_G$.

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DEFN: $t \in [0, 1] \mapsto (x, y_t, z_t) \in X \times Y \times Z$ is called a G-segment if

$$(D_x G, G)(x, y_t, z_t) = (1 - t)(D_x G, G)(x, y_0, z_0) + t(D_x G, G)(x, y_1, z_1)$$

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(G3) Assume $t \mapsto G(x_0, y_t, z_t)$ is convex along each G-segment (x, y_t, z_t)

(G4) $\frac{\partial G}{\partial z} < 0$ throughout $X \times Y \times Z$ (i.e. buyers prefer lower prices)

(G5) $\inf_{z \in Z} G(x, y, z) < G(x, 0, 0)$ for all $(x, y) \in X \times Y$

(i.e. high enough prices force all buyers out of market)

(G6) $\pi \in C^0(X \times Y \times Z)$

Monopolists problem in terms of buyers' indirect utilities u

$$u(x) := v^G(y) := \max_{y \in Y} G(x, y, v(y)) \tag{1}$$

implies

$$(Du, u)(x) = (D_x G, G)(x, y_v(x), v(y_v(x))$$

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so we identify

$$(y_v(x), v(y_v(x))) = \bar{y}_G(Du(x), u(x), x)$$

and minimize

$$\begin{aligned} \tilde{\Pi}(v) &= \int_X G(x, \bar{y}_G(Du(x,), u(x), x)) d\mu(x) \\ &=: \Pi(u) \end{aligned}$$

among u of form (1) (i.e. among so called G-convex $u(\cdot) \geq G(\cdot, 0, 0)$)

Results

$$\max_{G(\cdot,0,0)\leq u\in\mathcal{U}} \Pi(u)$$

where

$$\mathcal{U} \text{``} := \text{''} \{u \mid u(\cdot) = \sup_{y\in Y} G(\cdot, y, v(y)) \text{ on } X \text{ for some } v : Y \longrightarrow Z\}$$

THM 0: Given (G0-G1, G4-G6) the maximum above is attained. If $\mu << \mathcal{L}^m$ the map $x \to \bar{y}_G(Du(x), u(x), x)$ gives the consumer to (product,price) correspondence.

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THM 1: If (G0-G2, G4-G5) hold then $\mathcal{U}$ is convex if and only if (G3) holds.

THM 2: If (G0-G6) hold then Π is concave on $\mathcal{U}$ for all $\mu << \mathcal{L}^m$ if and only if $t \in [0,1] \mapsto \pi(x, y_t, z_t)$ is concave on every G-segment (x, y_t, z_t).

THM 2': same statement with both concaves replaced by convex.

• π is 2-uniformly concave along all G-segments if and only if Π is 2-uniformly concave on $\mathcal{U} \subset W^{1,2}(X, d\mu)$.

• alternately, strict concavity of π implies that of Π.

• in either case above, when $\mu << \mathcal{L}^m$ the hypotheses of THM 2 imply the principal's optimal strategy u is unique μ-a.e. and stable:

i.e. $(G_i, \pi_i, \mu_i) \to (G_\infty, \pi_\infty, \mu_\infty)$ in $C^2 \times C^0 \times (C^0)^*$ implies $u_i \to u_\infty$ in $L^\infty(d\mu_\infty)$

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- the Rochet-Choné $G(x, y, z) = x \cdot y - z$ lies on the boundary of the set of preferences satisfying (G3)

- if $\|A\|_{C^1} \leq 1, \|B\|_{C^1} \leq 1$ with A convex, $G(x, y) = x \cdot y - z - A(x)B(y)$ satisfies (G3) if and only if B is convex

Proof of THM 1 (convexity of space $\mathcal{U}$ of utilities on X)

Given $u_0, u_1 \in \mathcal{U}$ and $x_0 \in X$, since $u_0(\cdot) \geq \max_{y \in Y} G(\cdot, y, v_0(y))$ there exists $(y_0, z_0) \in Y \times Z$ such that

$$u_0(\cdot) \geq G(\cdot, y_0, z_0) \quad \text{with equality at} \quad x_0$$

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Similarly

$$u_1(\cdot) \geq G(\cdot, y_1, z_1) \quad \text{with equality at} \quad x_0$$

We'd like to deduce the same for $\frac{1}{2}(u_0 + u_1)$.

Adding the preceding yields

$$\begin{aligned}\frac{1}{2}(u_0 + u_1)(\cdot) &\geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\ &\geq G(\cdot, y_{\frac{1}{2}}, z_{\frac{1}{2}})\end{aligned}$$

by (G3), provided $(y_{\frac{1}{2}}, z_{\frac{1}{2}})$

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$$\begin{aligned}\frac{1}{2}(u_0+u_1)(\cdot) &\geq \frac{1}{2}(G(\cdot,y_0,z_0)+G(\cdot,y_1,z_1))\\ &\geq G(\cdot,y_{\frac{1}{2}},z_{\frac{1}{2}})\end{aligned}$$

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Moreover, both inequalities are saturated at $\cdot = x_0$.

Thus $\frac{1}{2}(u_0 + u_1) \in \mathcal{U}$.

Conversely... $\square$

Proof of THM 2 (concavity of $\Pi(u)$)

Proof: For $u_t := (1-t)u_0 + tu_1 \in \mathcal{U}$, we've assumed concavity (in t) of

$$\pi(x, \bar{y}_G((1-t)Du_0 + tDu_1, (1-t)u_0 + tu_1, x)) \qquad (2)$$

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$$\Pi(u_t) := \int_X \pi(x, \bar{y}_G(Du_t(x), u_t(x), x))d\mu(x) \tag{3}$$

inherits this concavity.

Conversely,

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Conversely, if concavity of (2) fails for some t, x, u_0 and u_1, it also fails in (3) for μ concentrated uniformly on a small enough ball around x. $\square$

Differential condition for (G3)

When $n = m$ set $\bar{x} = (x_0, x)$, $\bar{y} = (y, z)$ and $\bar{G}(\bar{x}, \bar{y}) := x_0 G(x, y, z)$.

Assume

(G7) $\det D^2_{\bar{x}^i \bar{y}^j} \bar{G}(\bar{x}, \bar{y}) \neq 0$ throughout $\{-1\} \times X \times Y \times Z$

(G8) $H(x, y, \cdot) = G^{-1}(x, y, \cdot)$ also satisfies hypotheses (G1-G2)

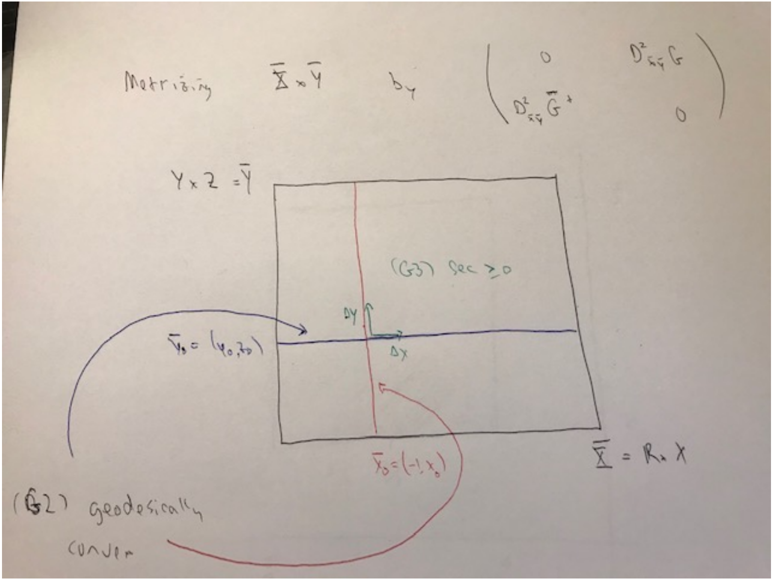
THM 3: If $G \in C^4$ satisfies (G0-G2) and (G4-G8), then (G3) is equivalent to

$$\frac{\partial^4}{\partial s^2 \partial t^2} \bar{G}(\bar{x}_s, \bar{y}_t)\Big|_{(s,t)=(s_0,t_0)} \geq 0$$

holding along all C^2 curves $\bar{x}_s$ and $\bar{y}_t$ for which $t \in [0, 1] \to (x_{s_0}, \bar{y}_t)$ forms a G-segment.

Remark: (G3) is a curvature condition on $(-\infty, 0) \times X \times Y \times Z$

Pseudo-Riemannian geometry à la Kim-McCann ’10



A new duality for bilinear preferences

Following Rochet-Choné '98 choose $G(x,y,z) = x \cdot y - z$ and $X, Y \subset \mathbf{R}^n$ convex so

$$\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$\begin{aligned} u(x) = v^*(x) &:= \sup_{y \in Y} x \cdot y - v(y) \\ &\in \mathcal{U} := \{u : X \longrightarrow [0, \infty] \textit{ convex} \mid Du(X) \subset Y\} \end{aligned}$$

THM 3:

$$\max_{u \in \mathcal{U}} \Pi(u) =$$

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$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \left\{ S : X \longrightarrow \mathbf{R}^n \mid \int_X [(x - S(x)) \cdot Du - u(x)] d\mu(x) \leq 0 \right\}$$

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where

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In words: the monopolists maximum profit coincides with the net value of a co-op able to offer its members good $y \in Y$ at price $c(y)$, minimized over possible distributions $S_{\#}\mu$ of co-op memberships satisfying

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In words: the monopolists maximum profit coincides with the net value of a co-op able to offer its members good $y \in Y$ at price $c(y)$, minimized over possible distributions $S_{\#}\mu$ of co-op memberships satisfying the strange constraint that when members whose true type is $S(x)$ irrationally display the behaviour of x facing each monopolist price menu, the expected gross value of the resulting assignment $Du(x)$ to those co-op members dominates the monopolist's expected gross revenue $\langle x \cdot Du(x) - u(x) \rangle_\mu$.

Proof sketch ($\leq$): $S \in \mathcal{S}$, $u \in \mathcal{U}$ and the definition of c^* imply

$$\Pi(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_\mu \leq \langle c^* \circ S \rangle_\mu$$

$\geq$: Conversely, using a convex-concave saddle argument in (S, u)

$$\sup_{u \in \mathcal{U}} \langle x \cdot Du(x) - u(x) - c(Du(x)) \rangle_\mu$$

$$= \sup_{u \in \mathcal{U}} \inf_{T : Y \longrightarrow \mathbf{R}^m} \langle x \cdot Du(x) - u(x) - T(Du(x)) \cdot Du(x) + c^*(T(Du(x))) \rangle_\mu$$

$$\geq \sup_{u \in \mathcal{U}} \inf_{S : X \longrightarrow \mathbf{R}^m} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) + c^*(S(x)) \rangle_\mu$$

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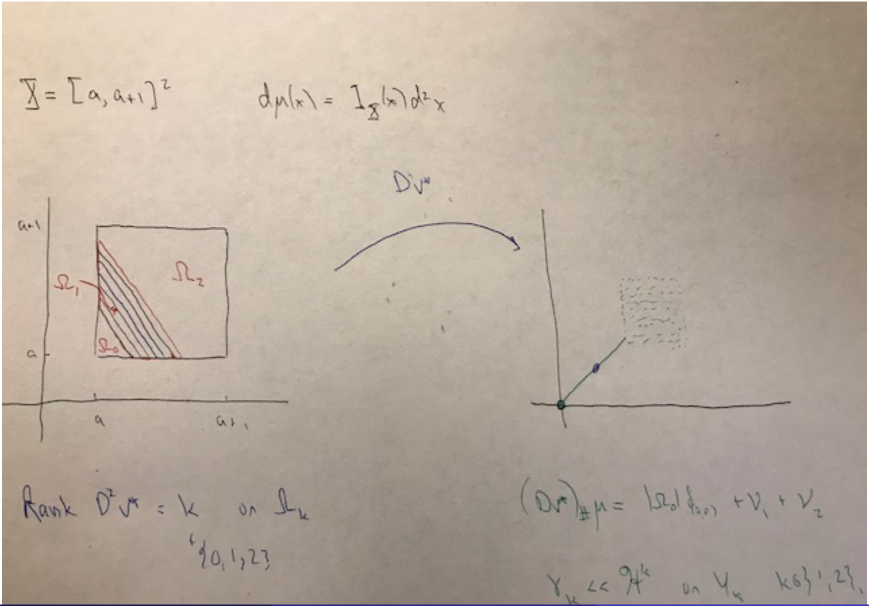
$$\geq \sup_{u\in\mathcal{U}} \inf_{S:X\longrightarrow\mathbf{R}^m} \langle x\cdot Du(x) - u(x) - S(x)\cdot Du(x) + c^*(S(x))\rangle_\mu$$

$$= \inf_{S:X\longrightarrow\mathbf{R}^m} \langle c^*(S(x))\rangle_\mu + \sup_{u\in\mathcal{U}}\langle x\cdot Du(x) - u(x) - S(x)\cdot Du(x))\rangle_\mu$$

$$= \inf_{S\in\mathcal{S}}\langle c^*\circ S\rangle_\mu.$$

(To justify this argument rigorously requires approximating both problems before applying Fenchel-Rockafellar duality to obtain an infinite-dimensional version of of the von Neumann min-max theorem.)

Rochet-Choné's example revisited



Variational calculus gives

$u = u_i$ on Ω_i where

- on Ω_0 exclusion: $u_0 = 0$
- on Ω_1, Euler-Lagrange ODE: if $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ then

$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$

subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.

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subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.
- on Ω_2 Euler-Lagrange PDE: $\Delta u_2 = 3$ subject to boundary conditions

$$\begin{aligned} (Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial X \cap \bar{\Omega}_2 \\ (Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial \Omega_2 \cap \partial \Omega_1 \quad \text{(Neumann)} \end{aligned}$$

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OVERDETERMINED!

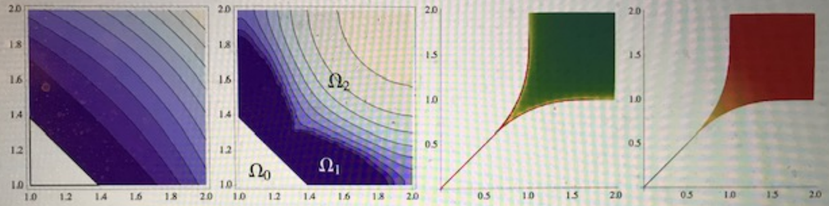

Fig. 1 Numerical approximation U of the solution of the classical Monopolist's problem (1), computed on a 50×50 grid. *Left* level sets of U, with $U = 0$ in *white*. *Center left* level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in *white*); note the degenerate region Ω_1 where $\det(\nabla^2 U) = 0$. *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). *Color scales* on Fig. 10 (color figure online)

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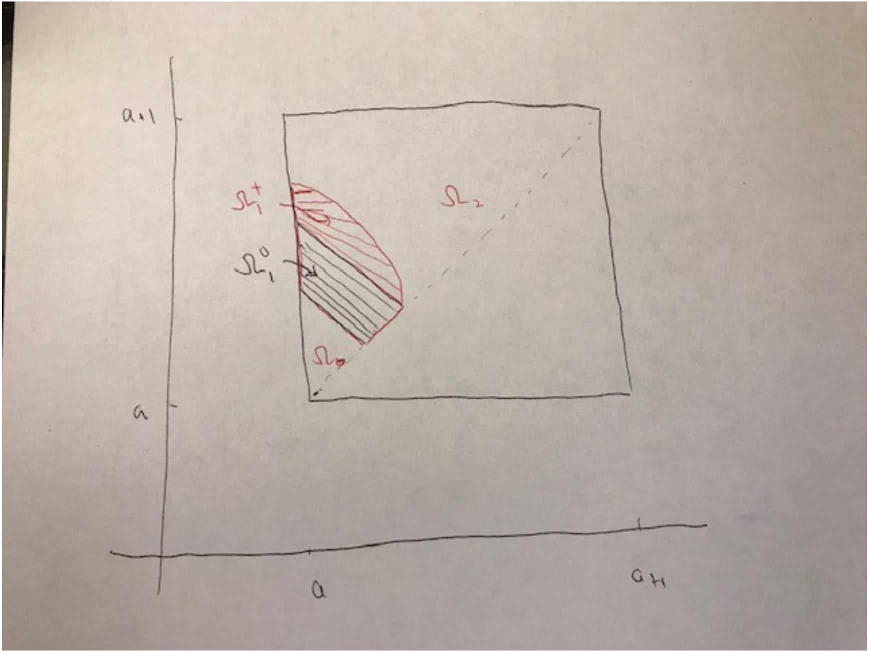
a+1
Ω₁⁺
Ω₂
Ω₁⁰
Ω₀
a
a
a+1

Free boundary problem

$u = u_i$ on Ω_i where

- on Ω_0 exclusion: $u_0 = 0$

- on Ω_1^0, Rochet-Choné's ODE: $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ where
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$
subject to boundary conditions $k = 0$ and $k' = 0$ at lower boundary.

- on Ω_1^+, $u_1 = u_1^+$ given by a NEW system of ODE (for height $h(\cdot)$ and length $R(\cdot)$ of isochoice segments together with profile of $u_1^+(\cdot)$ along them), with boundary conditions $u_1^+(x_1, x_2) = k(x_1 + x_2)$ and $Du_1^+ = (k', k')$ on $\partial\Omega_1^0 \cap \partial\Omega_1^+$

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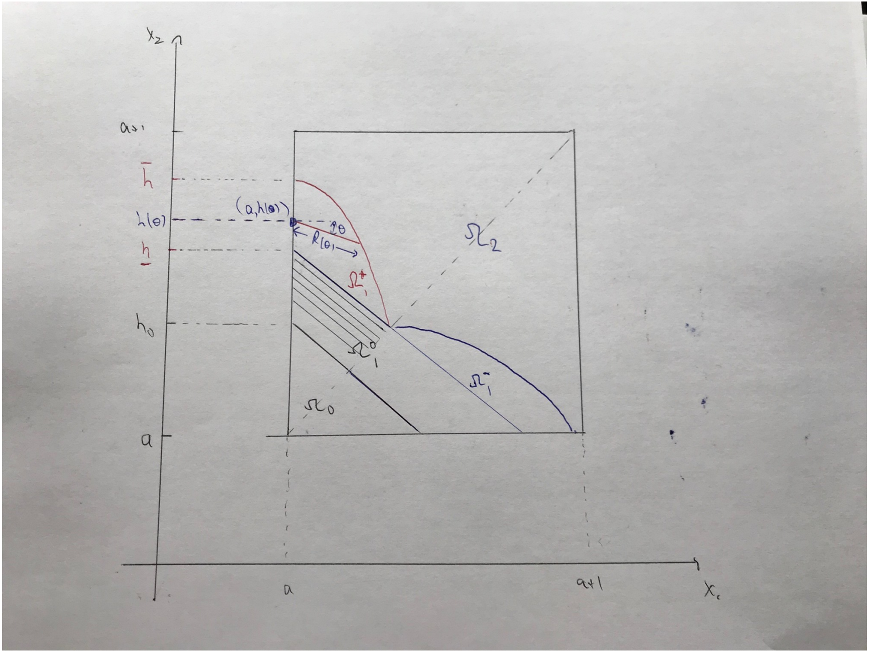
- on Ω_2, PDE: $\Delta u_2 = 3$ with Rochet-Choné's overdetermined conditions

$$\begin{aligned}
(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial X \cap \bar{\Omega}_2 \text{ and on } \{x_1 = x_2\} \\
(Du_2 - Du_1^+) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Neumann)} \\
u_2 = u_1^+ \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Dirichlet)}
\end{aligned}$$


x_2
$a+1$
$\bar{h}$
$h(\theta)$
$(a,h(\theta))$
2θ
$R(\theta)$
$\underline{h}$
h_0
Ω_2
Ω_1^+
Ω_1^0
Ω_1^-
Ω_0
a
a
$a+1$
x_1

Precise Euler-Lagrange equation in the 'missing' region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta).$$

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For $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta \quad (4)$$

$$m(-\tfrac{\pi}{4}) = 0, \qquad m'(-\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}}k'(a + \underline{h}).$$

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$$h(t) = \underline{h} + \frac{1}{3}\int_{-\pi/4}^{t} (m''(\theta) + m(\theta) - 2R(\theta))\frac{d\theta}{\cos\theta}, \quad (6)$$

$$b(t) = \frac{1}{2}k(a + \underline{h}) + \int_{-\pi/4}^{t} (m'(\theta)\cos\theta + m(\theta)\sin\theta)h'(\theta)d\theta. \quad (7)$$

• for $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ we can solve (4)–(7) to find Ω_1^+ and u_+^1.

• we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on Ω_2

• while it is not yet *rigorously* proved is that some choice of $\underline{h}$ and $R(\cdot)$ also yields $u_1 - u_2 = const$ on $\partial\Omega_2 \setminus \partial X$,

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• if such a choice exists such that, absorbing the constant into u_2, the resulting u given by $u_i^{(\pm)}$ on $\Omega_i^{(\pm)}$ for $i \in \{0, 1, 2\}$ is in $\mathcal{U}$, our new duality can be used to certify that u is the desired optimizer

WHY CAN WE NOT YET PROVE SUCH A CHOICE EXISTS?

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• a unique optimizer $\bar{u} \in \mathcal{U}$ is known to exist (Rochet-Choné) and $\bar{u} \in C^{1,1}_{loc}(X^0)$ (Caffarelli-Lions); if the sets Ω_i where its Hessian is rank i are smooth enough, and Ω_1 has the expected 3 components, then (4)–(7) and the overdetermined Poisson problem $\Delta u_3 = 0$ must be satisfied

• but maybe Ω_i are not smooth enough, or Ω_1 is not simply connected and/or has more than three components (some too small for the numerics to resolve); we seriously doubt this, but can't yet rule it out rigorously...

CONCLUSIONS

- Convexity, when present, is a powerful tool for optimization

- for numerics, uniqueness, stability, and characterization of optimum

- Duality of price menu $v(y)$ with buyers' indirect utilities $u(x) = v^G(x)$

- Necessary and sufficient conditions for convexity of monopolist's problem (as a function of u)

- Related to curvature conditions governing regularity in generated Jacobian equations (à la Ma, Trudinger and Wang) but

- adapted to payoffs $G(x, y, z)$ which may depend nonlinearly on price z

- new duality certifying solutions for $G(x, y, z) = x \cdot y - z$

- square example requires solving an unexpected free boundary problem

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THANK YOU!