On the Monopolist’s Problem Facing Consumers with Linear and Nonlinear Price Preferences

Robert J McCann

University of Toronto

www.math.toronto.edu/mccann

with Kelvin Shuangjian Zhang (ENS Paris / Waterloo / Fudan)

CPAM ’19 + work in progress

21 October 2021
Outline

1. Monopolist’s problem
2. Examples and History
3. Hypotheses
4. Results
5. Proofs
6. A new duality certifying solutions
7. A free boundary problem hidden in Rochet-Choné’s example
8. Conclusions
Monopolist’s problem

Given compact sets $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ and $Z = [z, \infty) \subset \mathbb{R}$ and

$G(x, y, z) =$ value of product $y \in Y$ to buyer $x \in X$ at price $z \in Z$

$d\mu(x) =$ relative frequency of buyer $x \in X$ (as compared to $x' \in X$)

$\pi(x, y, z) =$ value to monopolist of selling $y$ to $x$ at price $z$

Monopolist’s problem: choose price menu $\nu : Y \rightarrow Z$ to maximize profits
Monopolist’s problem

Given compact sets $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ and $Z = [z, \infty) \subset \mathbb{R}$ and $G(x, y, z) = \text{value of product } y \in Y \text{ to buyer } x \in X \text{ at price } z \in Z$

$d\mu(x) = \text{relative frequency of buyer } x \in X \text{ (as compared to } x' \in X)$

$\pi(x, y, z) = \text{value to monopolist of selling } y \text{ to } x \text{ at price } z$

Monopolist’s problem: choose price menu $\nu: Y \rightarrow Z$ to maximize profits

$$\tilde{\Pi}(\nu) := \int_X \pi(x, y_{\nu}(x), \nu(y_{\nu}(x)))d\mu(x), \quad \text{where}$$

Agent $x$’s problem: choose $y_{\nu}(x)$ to maximize

$$y_{\nu}(x) \in \arg \max_{y \in Y} G(x, y, \nu(y))$$

Constraints: $\nu$ lower semicontinuous, $(0, 0) \in Y \times Z \text{ and } \nu(0) = 0.$
Examples

- airline ticket pricing

- insurance: monopolist’s profit $\pi(x, y, z)$ may depend strongly on buyer’s identity $x$, even if regulation/ignorance prohibits price $v(y)$ from doing so

- $z$-dependence of $G(x, y, z)$ reflects different buyers price sensitivity/risk non-neutrality

- educational signaling

- optimal taxation: replace profit maximization with a budget constraint for providing services
Some history: $G(x, y, z) = b(x, y) - z$

Miryles ’71, Spence ’73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$

Rochet-Choné ’98 ($n = m > 1$): $b(x, y) = x \cdot y$ bilinear implies $y_v(x) = Dv^*(x)$ convex gradient; bunching
Some history: $G(x, y, z) = b(x, y) - z$

Mirrlees ’71, Spence ’73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$

Rochet-Choné ’98 ($n = m > 1$): $b(x, y) = x \cdot y$ bilinear implies $y_v(x) = Dv^*(x)$ convex gradient; bunching for $\pi(x, y, z) = z - \frac{1}{2}|y|^2$
Carlier-Lachand-Robert ’03: $\nu^* \in C^1(\text{supp } \mu)$; Caffarelli-Lions $\nu^* \in C^{1,1}$

Carlier ’01: $b(x, y)$ general implies existence of optimizer $\nu = \nu^{b\tilde{b}}$

Chen ’13: $u \in C^1$ under Ma-Trudinger-Wang (MTW) conditions, where

$$u(x) = \nu^b(x) := \max_{y \in Y} b(x, y) - \nu(y)$$
Carlier-Lachand-Robert ’03: $v^* \in C^1(spt \mu)$; CaffARElli-Lions $v^* \in C^{1,1}$

Carlier ’01: $b(x, y)$ general implies existence of optimizer $v = v^{b\tilde{b}}$

Chen ’13: $u \in C^1$ under Ma-Trudinger-Wang (MTW) conditions, where

$$u(x) = v^b(x) := \max_{y \in Y} b(x, y) - v(y)$$

Figalli-Kim-M. ’11: convexity of principal’s problem under strengthening of (MTW) on $b(x, y)$

Noldeke-Samuelson (ECMA ’18), Zhang (ET ’19): existence of maximizing $v$ for general $G \in C^0$

Daskalakis-Dekelbaum-Tzamos (ECMA ’17), Kleiner-Manelli (ECMA ’19): duality for multigood auctions
Hypothesis (c.f. Trudinger’s generated Jacobian equations)

(G0) $G \in C^1(X \times Y \times Z)$, $m \geq n$, and for each $x, x_0 \in X \subset \mathbb{R}^m$:

(G1) $(y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z)$ is a homeomorphism

(G2) with convex range $(Y \times Z)_x := (D_x G, G)(x, Y, Z)$ and inverse $\bar{y}_G$.

(G3) Assume $t \mapsto G(x_0, y_t, z_t)$ is convex along each $G$-segment $(x, y_t, z_t)$

(G4) $\partial G/\partial z < 0$ throughout $X \times Y \times Z$ (i.e. buyers prefer lower prices)

(G5) $\inf_{z \in Z} G(x, y, z) < G(x, 0, 0)$ for all $(x, y) \in X \times Y$ (i.e. high enough prices force all buyers out of market)

(G6) $\pi \in C_0(X \times Y \times Z)$

Robert J McCann (Toronto)
Hypothesis (c.f. Trudinger’s generated Jacobian equations)

(G0) \( G \in C^1(X \times Y \times Z), \ m \geq n, \) and for each \( x, x_0 \in X \subset \mathbb{R}^m: \)

(G1) \( (y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z) \) is a homeomorphism

(G2) with convex range \( (Y \times Z)_x := (D_x G, G)(x, Y, Z) \) and inverse \( \bar{y}_G. \)

DEFN: \( t \in [0, 1] \mapsto (x, y_t, z_t) \in X \times Y \times Z \) is called a \( G\)-segment if

\[
(D_x G, G)(x, y_t, z_t) = (1 - t)(D_x G, G)(x, y_0, z_0) + t(D_x G, G)(x, y_1, z_1)
\]

(G3) Assume \( t \mapsto G(x_0, y_t, z_t) \) is convex along each \( G\)-segment \( (x, y_t, z_t) \)
Hypothesis (c.f. Trudinger’s generated Jacobian equations)

(G0) $G \in C^1(X \times Y \times Z)$, $m \geq n$, and for each $x, x_0 \in X \subset \mathbb{R}^m$:

(G1) $(y, z) \in Y \times Z \mapsto (D_x G, G)(x, y, z)$ is a homeomorphism

(G2) with convex range $(Y \times Z)_x := (D_x G, G)(x, Y, Z)$ and inverse $\tilde{y}_G$.

DEFN: $t \in [0, 1] \mapsto (x, y_t, z_t) \in X \times Y \times Z$ is called a $G$-segment if

$$(D_x G, G)(x, y_t, z_t) = (1 - t)(D_x G, G)(x, y_0, z_0) + t(D_x G, G)(x, y_1, z_1)$$

(G3) Assume $t \mapsto G(x_0, y_t, z_t)$ is convex along each $G$-segment $(x, y_t, z_t)$

(G4) $\frac{\partial G}{\partial z} < 0$ throughout $X \times Y \times Z$ (i.e. buyers prefer lower prices)

(G5) $\inf_{z \in Z} G(x, y, z) < G(x, 0, 0)$ for all $(x, y) \in X \times Y$

(i.e. high enough prices force all buyers out of market)

(G6) $\pi \in C^0(X \times Y \times Z)$
Monopolists problem in terms of buyers’ indirect utilities \( u \)

\[
\begin{align*}
  u(x) &:= v^G(y) := \max_{y \in Y} G(x, y, v(y)) \\
  \text{implies} \\
  (Du, u)(x) &:= (D_x G, G)(x, y_v(x), v(y_v(x))) \\
  \text{so we identify} \\
  (y_v(x), v(y_v(x)))
\end{align*}
\]
Monopolists problem in terms of buyers’ indirect utilities \( u \)

\[
u(x) := v^G(y) := \max_{y \in Y} G(x, y, v(y))
\]

implies
\[
(Du, u)(x) = (D_x G, G)(x, y_v(x), v(y_v(x))
\]

so we identify
\[
(y_v(x), v(y_v(x))) = \bar{y}_G(Du(x), u(x), x)
\]

and minimize
\[
\Pi(v) = \int_X G(x, \bar{y}_G(Du(x), u(x), x))d\mu(x)
\]

among \( u \) of form (1) (i.e. among so called \( G \)-convex \( u(\cdot) \geq G(\cdot, 0, 0) \))
\[
\max_{G(\cdot,0,0) \leq u \in \mathcal{U}} \Pi(u)
\]
where
\[
\mathcal{U} := \{ u \mid u(\cdot) = \sup_{y \in Y} G(\cdot, y, \nu(y)) \text{ on } X \text{ for some } \nu : Y \to Z \}
\]

THM 0: Given (G0-G1, G4-G6) the maximum above is attained. If \( \mu \ll \mathcal{L}^m \) the map \( x \to \tilde{y}_G(Du(x), u(x), x) \) gives the consumer to (product,price) correspondence.
\[
\max_{G(\cdot,0,0) \leq u \in U} \Pi(u)
\]

where

\[U^\prime := \{u \mid u(\cdot) = \sup_{y \in Y} G(\cdot, y, v(y)) \text{ on } X \text{ for some } v : Y \rightarrow Z\}\]

**THM 0:** Given (G0-G1, G4-G6) the maximum above is attained. If \(\mu \ll \mathcal{L}^m\) the map \(x \rightarrow \tilde{y}_G(Du(x), u(x), x)\) gives the consumer to (product, price) correspondence.

**THM 1:** If (G0-G2, G4-G5) hold then \(U\) is convex if and only if (G3) holds.
Results

\[
\max_{G(\cdot,0,0) \leq u \in \mathcal{U}} \Pi(u)
\]

where

\[
\mathcal{U} := \{ u \mid u(\cdot) = \sup_{y \in Y} G(\cdot, y, v(y)) \text{ on } X \text{ for some } v : Y \rightarrow Z \}
\]

THM 0: Given (G0-G1, G4-G6) the maximum above is attained. If \( \mu \ll L^m \) the map \( x \rightarrow \bar{y}_G(Du(x), u(x), x) \) gives the consumer to (product, price) correspondence.

THM 1: If (G0-G2, G4-G5) hold then \( \mathcal{U} \) is convex if and only if (G3) holds.

THM 2: If (G0-G6) hold then \( \Pi \) is concave on \( \mathcal{U} \) for all \( \mu \ll L^m \) if and only if \( t \in [0, 1] \mapsto \pi(x, y_t, z_t) \) is concave on every \( G \)-segment \( (x, y_t, z_t) \).

THM 2': same statement with both concaves replaced by convex.
• $\pi$ is 2-uniformly concave along all $G$-segments if and only if $\Pi$ is 2-uniformly concave on $\mathcal{U} \subset W^{1,2}(X, d\mu)$.

• alternately, strict concavity of $\pi$ implies that of $\Pi$.

• in either case above, when $\mu \ll \mathcal{L}^m$ the hypotheses of THM 2 imply the principal’s optimal strategy $u$ is unique $\mu$-a.e. and stable:

i.e. $(G_i, \pi_i, \mu_i) \rightarrow (G_\infty, \pi_\infty, \mu_\infty)$ in $C^2 \times C^0 \times (C^0)^*$ implies $u_i \rightarrow u_\infty$ in $L^\infty(d\mu_\infty)$
• \( \pi \) is 2-uniformly concave along all \( G \)-segments if and only if \( \Pi \) is 2-uniformly concave on \( \mathcal{U} \subset W^{1,2}(X, d\mu) \).

• alternately, strict concavity of \( \pi \) implies that of \( \Pi \).

• in either case above, when \( \mu \ll \mathcal{L}^m \) the hypotheses of THM 2 imply the principal’s optimal strategy \( u \) is unique \( \mu \)-a.e. and stable:

i.e. \( (G_i, \pi_i, \mu_i) \to (G_\infty, \pi_\infty, \mu_\infty) \) in \( C^2 \times C^0 \times (C^0)^* \) implies \( u_i \to u_\infty \) in \( L^\infty(d\mu_\infty) \)

• the Rochet-Choné \( G(x, y, z) = x \cdot y - z \) lies on the boundary of the set of preferences satisfying (G3)

• if \( \|A\|_{C^1} \leq 1, \|B\|_{C^1} \leq 1 \) with \( A \) convex, \( G(x, y) = x \cdot y - z - A(x)B(y) \) satisfies (G3) if and only if \( B \) is convex
Proof of THM 1 (convexity of space $\mathcal{U}$ of utilities on $X$)

Given $u_0, u_1 \in \mathcal{U}$ and $x_0 \in X$, since $u_0(\cdot) \geq \max_{y \in Y} G(\cdot, y, v_0(y))$ there exists $(y_0, z_0) \in Y \times Z$ such that

$$u_0(\cdot) \geq G(\cdot, y_0, z_0) \text{ with equality at } x_0$$
Given $u_0, u_1 \in \mathcal{U}$ and $x_0 \in X$, since $u_0(\cdot) \geq \max_{y \in Y} G(\cdot, y, v_0(y))$ there exists $(y_0, z_0) \in Y \times Z$ such that

$$u_0(\cdot) \geq G(\cdot, y_0, z_0) \quad \text{with equality at} \quad x_0$$

Similarly

$$u_1(\cdot) \geq G(\cdot, y_1, z_1) \quad \text{with equality at} \quad x_0$$

We’d like to deduce the same for $\frac{1}{2}(u_0 + u_1)$. 
Adding the preceding yields

\[
\frac{1}{2}(u_0 + u_1)(\cdot) \geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\
\geq G(\cdot, y_{1/2}, z_{1/2})
\]

by (G3), provided \((y_{1/2}, z_{1/2})\).
Adding the preceding yields

\[
\frac{1}{2}(u_0 + u_1)(\cdot) \geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\
\geq G(\cdot, y_{1/2}, z_{1/2})
\]

by (G3), provided \((y_{1/2}, z_{1/2})\) defined (using (G1-G2)) by

\[
(D_x G, G)(x_0, y_t, z_t) := (1 - t)(D_x G, G)(x_0, y_0, z_0) + t(D_x G, G)(x_0, y_1, z_1)
\]

Moreover, both inequalities are saturated at \(\cdot = x_0\).
Adding the preceding yields

\[
\frac{1}{2}(u_0 + u_1)(\cdot) \geq \frac{1}{2}(G(\cdot, y_0, z_0) + G(\cdot, y_1, z_1)) \\
\geq G(\cdot, y_{1/2}, z_{1/2})
\]

by (G3), provided \((y_{1/2}, z_{1/2})\) defined (using (G1-G2)) by

\[
(D_x G, G)(x_0, y_t, z_t) := (1 - t)(D_x G, G)(x_0, y_0, z_0) + t(D_x G, G)(x_0, y_1, z_1)
\]

Moreover, both inequalities are saturated at \(\cdot = x_0\).

Thus \(\frac{1}{2}(u_0 + u_1) \in \mathcal{U}\).

Conversely…
Proof of THM 2 (concavity of $\Pi(u)$)

Proof: For $u_t := (1 - t)u_0 + tu_1 \in \mathcal{U}$, we’ve assumed concavity (in $t$) of

$$\pi(x, \bar{y}_G((1 - t)Du_0 + tDu_1, (1 - t)u_0 + tu_1, x))$$  \hspace{1cm} (2)

inherits this concavity. Conversely, if concavity of (2) fails for some $t$, $x$, $u_0$ and $u_1$, it also fails in (3) for $\mu$ concentrated uniformly on a small enough ball around $x$.  

Proof of THM 2 (concavity of $\Pi(u)$)

Proof: For $u_t := (1 - t)u_0 + tu_1 \in \mathcal{U}$, we’ve assumed concavity (in $t$) of

$$\pi(x, \bar{y}_G((1 - t)Du_0 + tDu_1, (1 - t)u_0 + tu_1, x))$$

(2)

$$\Pi(u_t) := \int_X \pi(x, \bar{y}_G(Du_t(x), u_t(x), x))d\mu(x)$$

(3)

inherits this concavity.

Conversely,
Proof of THM 2 (concavity of $\Pi(u)$)

Proof: For $u_t := (1 - t)u_0 + tu_1 \in \mathcal{U}$, we’ve assumed concavity (in $t$) of

$$\pi(x, \bar{y}_G((1 - t)Du_0 + tDu_1, (1 - t)u_0 + tu_1, x))$$  \hspace{1cm} (2)

$$\Pi(u_t) := \int_{\mathcal{X}} \pi(x, \bar{y}_G(Du_t(x), u_t(x), x))d\mu(x)$$  \hspace{1cm} (3)

inherits this concavity.

Conversely, if concavity of (2) fails for some $t, x, u_0$ and $u_1$, it also fails in (3) for $\mu$ concentrated uniformly on a small enough ball around $x$. \qed
Differential condition for (G3)

When \( n = m \) set \( \bar{x} = (x_0, x) \), \( \bar{y} = (y, z) \) and \( \bar{G}(\bar{x}, \bar{y}) := x_0 G(x, y, z) \).

Assume

(G7) \( \det D^2_{\bar{x}i\bar{y}j} \bar{G}(\bar{x}, \bar{y}) \neq 0 \) throughout \( \{-1\} \times X \times Y \times Z \)

(G8) \( H(x, y, \cdot) = G^{-1}(x, y, \cdot) \) also satisfies hypotheses (G1-G2)

THM 3: If \( G \in C^4 \) satisfies (G0-G2) and (G4-G8), then \( (G3) \) is equivalent to

\[
\frac{\partial^4}{\partial s^2 \partial t^2} \bar{G}(\bar{x}_s, \bar{y}_t) \bigg|_{(s,t)=(s_0,t_0)} \geq 0
\]

holding along all \( C^2 \) curves \( \bar{x}_s \) and \( \bar{y}_t \) for which \( t \in [0, 1] \rightarrow (x_{s_0}, y_t) \) forms a \( G \)-segment.

Remark: \( (G3) \) is a curvature condition on \( (-\infty, 0) \times X \times Y \times Z \).
A new duality for bilinear preferences

Following Rochet-Choné '98 choose \( G(x, y, z) = x \cdot y - z \) and \( X, Y \subset \mathbb{R}^n \) convex so

\[
\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))]d\mu(x)
\]

with

\[
u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)
\]

\( \in \mathcal{U} := \{ u : X \rightarrow [0, \infty] \text{ convex} \mid Du(X) \subset Y \} \)

THM 3:

\[
\max_{u \in \mathcal{U}} \Pi(u) = \]
A new duality for bilinear preferences

Following Rochet-Choné '98 choose $G(x, y, z) = x \cdot y - z$ and $X, Y \subset \mathbb{R}^n$ convex so

$$
\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)
$$

with

$$
u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)
$$

$$
\in U := \{ u : X \rightarrow [0, \infty] \text{ convex} \mid Du(X) \subset Y \}
$$

THM 3:

$$
\max_{u \in U} \Pi(u) = \min_{S \in S} \int c^*(S(x)) d\mu(x)
$$

where

$$
S := \bigcap_{u \in U} \left\{ S : X \rightarrow \mathbb{R}^n \mid \int_X [(x - S(x)) \cdot Du - u(x)] d\mu(x) \leq 0 \right\}
$$
THM 3:

\[
\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)
\]

where

\[
\mathcal{S} := \bigcap_{u \in \mathcal{U}} \{ S : X \rightarrow \mathbb{R}^n | \langle x \cdot Du(x) - u(x) \rangle_\mu \leq \langle S(x) \cdot Du(x) \rangle_\mu \}
\]

In words: the monopolists maximum profit coincides with the net value of a co-op able to offer its members good \( y \in Y \) at price \( c(y) \), minimized over possible distributions \( S_\#\mu \) of co-op memberships satisfying
THM 3:
\[
\max_{u \in U} \Pi(u) = \min_{S \in S} \int c^*(S(x)) d\mu(x)
\]
where
\[
S := \bigcap_{u \in U} \{ S : X \to \mathbb{R}^n \mid \langle x \cdot Du(x) - u(x) \rangle_\mu \leq \langle S(x) \cdot Du(x) \rangle_\mu \}
\]

In words: the monopolists maximum profit coincides with the net value of a co-op able to offer its members good \( y \in Y \) at price \( c(y) \), minimized over possible distributions \( S \# \mu \) of co-op memberships satisfying the strange constraint that when members whose true type is \( S(x) \) irrationally display the behaviour of \( x \) facing each monopolist price menu, the expected gross value of the resulting assignment \( Du(x) \) to those co-op members dominates the monopolist’s expected gross revenue \( \langle x \cdot Du(x) - u(x) \rangle_\mu \).

Proof sketch (\( \leq \)): \( S \in S, \ u \in U \) and the definition of \( c^* \) imply
\[
\Pi(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_\mu \leq \langle c^* \circ S \rangle_\mu
\]
\[ \geq: \text{Conversely, using a convex-concave saddle argument in } (S, u) \]

\[
\sup_{u \in \mathcal{U}} \langle x \cdot Dv(x) - u(x) - c(Du(x)) \rangle_{\mu} \\
= \sup_{u \in \mathcal{U}} \inf_{T: Y \to \mathbb{R}^m} \langle x \cdot Du(x) - u(x) - T(Du(x)) \cdot Du(x) + c^*(T(Du(x))) \rangle_{\mu} \\
\geq \sup_{u \in \mathcal{U}} \inf_{S: X \to \mathbb{R}^m} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) + c^*(S(x)) \rangle_{\mu}
\]
$\geq$: Conversely, using a convex-concave saddle argument in $(S, u)$

$$\sup_{u \in U} \langle x \cdot Du(x) - u(x) - c(Du(x)) \rangle_\mu$$

$$= \sup_{u \in U} \inf_{T: Y \to \mathbb{R}^m} \langle x \cdot Du(x) - u(x) - T(Du(x)) \cdot Du(x) + c^*(T(Du(x))) \rangle_\mu$$

$$\geq \sup_{u \in U} \inf_{S: X \to \mathbb{R}^m} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) + c^*(S(x)) \rangle_\mu$$

$$= \inf_{S: X \to \mathbb{R}^m} \langle c^*(S(x)) \rangle_\mu + \sup_{u \in U} \langle x \cdot Du(x) - u(x) - S(x) \cdot Du(x) \rangle_\mu$$

$$= \inf_{S \in S} \langle c^* \circ S \rangle_\mu.$$

(To justify this argument rigorously requires approximating both problems before applying Fenchel-Rockafellar duality to obtain an infinite-dimensional version of the von Neumann min-max theorem.)
\[ \bar{X} = [a, a+1]^2 \quad d\mu(x) = 1_\bar{X}(x) d^2x \]

\[ \text{Rank } D^2 u = k \text{ on } S_k \]

\[ (D^3 u)_+ \mu = 1_{(0,1,2)} + V_1 + V_2 \]

\[ V_k \leq 9k^4 \text{ on } Y_k, k \geq 1, 2, 3 \]
Variational calculus gives

\[ u = u_i \text{ on } \Omega_i \text{ where} \]

- on \( \Omega_0 \) exclusion: \( u_0 = 0 \)
- on \( \Omega_1 \), Euler-Lagrange ODE: if \( u_1(x_1, x_2) = \frac{1}{2} k(x_1 + x_2) \) then
  \[ k(s) = \frac{3}{4} s^2 - as - \log |s - 2a| + \text{const} \]
subject to boundary conditions \( u_1 = u_0 \) and \( Du_1 = Du_0 \) at lower boundary.

\[ \text{OVERDETERMINED!} \]
Variational calculus gives

\[ u = u_i \text{ on } \Omega_i \text{ where} \]

- on \( \Omega_0 \) exclusion: \( u_0 = 0 \)
- on \( \Omega_1 \), Euler-Lagrange ODE: if \( u_1(x_1, x_2) = \frac{1}{2} k(x_1 + x_2) \) then \( k(s) = \frac{3}{4} s^2 - as - \log |s - 2a| + \text{const} \)

subject to boundary conditions \( u_1 = u_0 \) and \( Du_1 = Du_0 \) at lower boundary.

- on \( \Omega_2 \) Euler-Lagrange PDE: \( \Delta u_2 = 3 \) subject to boundary conditions

\[
(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \bar{\Omega}_2
\]
\[
(Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial \Omega_2 \cap \partial \Omega_1 \quad \text{(Neumann)}
\]
Variational calculus gives

\[ u = u_i \text{ on } \Omega_i \text{ where} \]

- on \( \Omega_0 \) exclusion: \( u_0 = 0 \)
- on \( \Omega_1 \), Euler-Lagrange ODE: if \( u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2) \) then
  \[ k(s) = \frac{3}{4}s^2 - as - \log |s - 2a| + \text{const} \]
subject to boundary conditions \( u_1 = u_0 \) and \( Du_1 = Du_0 \) at lower boundary.
- on \( \Omega_2 \) Euler-Lagrange PDE: \( \Delta u_2 = 3 \) subject to boundary conditions
  \[
  (Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \tilde{\Omega}_2 \\
  (Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial \Omega_2 \cap \partial \Omega_1 \quad \text{(Neumann)} \\
  u_2 = u_1 \quad \text{on} \quad \partial \Omega_2 \cap \partial \Omega_1 \quad \text{(Dirichlet)}
  \]

OVERDETERMINED!
**Fig. 1** Numerical approximation $U$ of the solution of the classical Monopolist’s problem (1), computed on a $50 \times 50$ grid. **Left** level sets of $U$, with $U = 0$ in white. **Center left** level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in white); note the degenerate region $\Omega_1$ where $\det(\nabla^2 U) = 0$. **Center right** distribution of products sold by the monopolist. **Right** profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the bottom left). **Color scales** on Fig. 10 (color figure online)

\[ \text{ Springer } \]

\[ \text{J.-M. Mirebeau (2016)} \]
Free boundary problem

\[ u = u_i \text{ on } \Omega_i \text{ where} \]

- on \( \Omega_0 \) exclusion: \( u_0 = 0 \)

- on \( \Omega_1^0 \), Rochet-Choné’s ODE: \( u_1(x_1, x_2) = \frac{1}{2} k(x_1 + x_2) \) where
  \[
  k(s) = \frac{3}{4} s^2 - as - \log |s - 2a| + \text{const}
  \]
  subject to boundary conditions \( k = 0 \) and \( k' = 0 \) at lower boundary.

- on \( \Omega_1^+ \), \( u_1 = u_1^+ \) given by a NEW system of ODE (for height \( h(\cdot) \) and length \( R(\cdot) \) of isochoice segments together with profile of \( u_1^+(\cdot) \) along them), with boundary conditions \( u_1^+(x_1, x_2) = k(x_1 + x_2) \) and
  \[
  Du_1^+ = (k', k') \text{ on } \partial\Omega_1^0 \cap \partial\Omega_1^+
  \]
Free boundary problem

\[ u = u_i \text{ on } \Omega_i \text{ where} \]

- on \( \Omega_0 \) exclusion: \( u_0 = 0 \)
- on \( \Omega_0^0 \), Rochet-Choné's ODE: \( u_1(x_1, x_2) = \frac{1}{2} k(x_1 + x_2) \) where
  \[ k(s) = \frac{3}{4} s^2 - as - \log |s - 2a| + \text{const} \]
  subject to boundary conditions \( k = 0 \) and \( k' = 0 \) at lower boundary.
- on \( \Omega_1^+ \), \( u_1 = u_1^+ \) given by a NEW system of ODE (for height \( h(\cdot) \) and length \( R(\cdot) \) of isochoice segments together with profile of \( u_1^+(\cdot) \) along them), with boundary conditions \( u_1^+(x_1, x_2) = k(x_1 + x_2) \) and \( Du_1^+ = (k', k') \) on \( \partial \Omega_0^0 \cap \partial \Omega_1^+ \)
- on \( \Omega_2 \), PDE: \( \Delta u_2 = 3 \) with Rochet-Choné's overdetermined conditions
  \[ (Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \text{ on } \partial X \cap \tilde{\Omega}_2 \text{ and on } \{x_1 = x_2\} \]
  \[ (Du_2 - Du_1^+) \cdot \hat{n}_{\Omega_2}(x) = 0 \text{ on } \partial \Omega_2 \cap \partial \Omega_1^+ \text{ (Neumann)} \]
  \[ u_2 = u_1^+ \text{ on } \partial \Omega_2 \cap \partial \Omega_1^+ \text{ (Dirichlet)} \]
Precise Euler-Lagrange equation in the ‘missing’ region $\Omega_1^+$

Index each isochoice segment in $\Omega_1^+$ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+ \left( (a, h(\theta)) + r (\cos \theta, \sin \theta) \right) = m(\theta) r + b(\theta).$$
Precise Euler-Lagrange equation in the ‘missing’ region $\Omega_1^+$

Index each isochoice segment in $\Omega_1^+$ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+((a, h(\theta)) + r (\cos \theta, \sin \theta)) = m(\theta) r + b(\theta).$$

For $h \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(h - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta) \sin \theta - m(\theta) \cos \theta + a) = \frac{3}{2} R^2(\theta) \cos \theta \quad (4)$$

$$m(-\frac{\pi}{4}) = 0, \quad m'(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}k'(a + h).$$
Precise Euler-Lagrange equation in the ‘missing’ region $\Omega_1^+$

Index each isochoice segment in $\Omega_1^+$ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+ \left( (a, h(\theta)) + r (\cos \theta, \sin \theta) \right) = m(\theta) r + b(\theta).$$

For $h \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(h - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta) \sin \theta - m(\theta) \cos \theta + a) = \frac{3}{2} R^2(\theta) \cos \theta \quad (4)$$

$$m(-\frac{\pi}{4}) = 0, \quad m'(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} k'(a + h) \quad \text{Then set} \quad (5)$$

$$h(t) = h + \frac{1}{3} \int_{-\pi/4}^{t} (m''(\theta) + m(\theta) - 2R(\theta)) \frac{d\theta}{\cos \theta}, \quad (6)$$

$$b(t) = \frac{1}{2} k(a + h) + \int_{-\pi/4}^{t} (m'(\theta) \cos \theta + m(\theta) \sin \theta) h'(\theta) d\theta. \quad (7)$$
• for $h \in [a, a + 1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(h - a)$ we can solve (4)–(7) to find $\Omega_1^+$ and $u_1^+$.

• we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on $\Omega_2$

• while it is not yet rigorously proved is that some choice of $h$ and $R(\cdot)$ also yields $u_1 - u_2 = \text{const}$ on $\partial\Omega_2 \setminus \partial X$, 
• for \( h \in [a, a + 1] \), \( R : \left[ -\frac{\pi}{4}, \frac{\pi}{2} \right] \to [0, a\sqrt{2}) \) Lipschitz (say) and \( R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(h - a) \) we can solve (4)–(7) to find \( \Omega_1^+ \) and \( u_1^+ \).

• we can then solve the resulting Neumann problem for \( \Delta u_2 = 3 \) on \( \Omega_2 \)

• while it is not yet rigorously proved is that some choice of \( h \) and \( R(\cdot) \) also yields \( u_1 - u_2 = \text{const} \) on \( \partial\Omega_2 \setminus \partial X \), we hope to do this in the future

• if such a choice exists such that, absorbing the constant into \( u_2 \), the resulting \( u \) given by \( u_i^{(\pm)} \) on \( \Omega_i^{(\pm)} \) for \( i \in \{0, 1, 2\} \) is in \( \mathcal{U} \), our new duality can be used to certify that \( u \) is the desired optimizer

WHY CAN WE NOT YET PROVE SUCH A CHOICE EXISTS?
• for \( h \in [a, a+1] \), \( R : [-\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow [0, a\sqrt{2}) \) Lipschitz (say) and \( R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(h - a) \) we can solve (4)–(7) to find \( \Omega_1^+ \) and \( u_1^+ \).

• we can then solve the resulting Neumann problem for \( \Delta u_2 = 3 \) on \( \Omega_2 \)

• while it is not yet rigorously proved is that some choice of \( h \) and \( R(\cdot) \) also yields \( u_1 - u_2 = \text{const} \) on \( \partial\Omega_2 \setminus \partial X \), we hope to do this in the future

• if such a choice exists such that, absorbing the constant into \( u_2 \), the resulting \( u \) given by \( u_i^{(\pm)} \) on \( \Omega_i^{(\pm)} \) for \( i \in \{0, 1, 2\} \) is in \( \mathcal{U} \), our new duality can be used to certify that \( u \) is the desired optimizer

WHY CAN WE NOT YET PROVE SUCH A CHOICE EXISTS?

• a unique optimizer \( \bar{u} \in \mathcal{U} \) is known to exist (Rochet-Choné) and \( \bar{u} \in C^{1,1}_{loc}(X^0) \) (Caffarelli-Lions); if the sets \( \Omega_i \) where its Hessian is rank \( i \) are smooth enough, and \( \Omega_1 \) has the expected 3 components, then (4)–(7) and the overdetermined Poisson problem \( \Delta u_3 = 0 \) must be satisfied

• but maybe \( \Omega_i \) are not smooth enough, or \( \Omega_1 \) is not simply connected and/or has more than three components (some too small for the numerics to resolve); we seriously doubt this, but can’t yet rule it out rigorously...
CONCLUSIONS

• **Convexity**, when present, is a powerful tool for optimization

• for numerics, uniqueness, stability, and characterization of optimum

• **Duality** of price menu $v(y)$ with buyers’ indirect utilities $u(x) = v^G(x)$

• Necessary and sufficient conditions for **convexity** of monopolist’s problem (as a function of $u$)

• Related to **curvature conditions** governing regularity in generated Jacobian equations (à la Ma, Trudinger and Wang) but

• adapted to payoffs $G(x, y, z)$ which may depend **nonlinearly** on price $z$

• **new duality** certifying solutions for $G(x, y, z) = x \cdot y - z$

• square example requires solving an unexpected **free boundary** problem
CONCLUSIONS

• Convexity, when present, is a powerful tool for optimization

• for numerics, uniqueness, stability, and characterization of optimum

• Duality of price menu $v(y)$ with buyers’ indirect utilities $u(x) = v^G(x)$

• Necessary and sufficient conditions for convexity of monopolist’s problem (as a function of $u$)

• Related to curvature conditions governing regularity in generated Jacobian equations (à la Ma, Trudinger and Wang) but

• adapted to payoffs $G(x, y, z)$ which may depend nonlinearly on price $z$

• new duality certifying solutions for $G(x, y, z) = x \cdot y - z$

• square example requires solving an unexpected free boundary problem

THANK YOU!