Inscribed radius bounds for lower Ricci bounded metric measure spaces with mean convex boundary

A. Burtscher, C. Ketterer, R. McCann, E. Woolgar

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University of Toronto

Slides: click on ‘Talk’ at www.math.toronto.edu/mccann

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Inscribed radius $r = r(\Omega)$ of an open subset $\Omega$ of a metric space $(X, d)$ is

$$r(\Omega) := \sup\{ r \geq 0 \mid B(x; r) \subset \Omega \text{ for some } x \in \Omega \}$$
**Inscribed radius bounds**

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\]

**Theorem (classical)**

If \((X, d) = (\mathbb{R}^n, |\cdot|)\) and the mean-curvature \( H_{\partial \Omega} \geq (n - 1)/R \) then

\[
r(\Omega) \leq R,
\]

with equality iff \( \Omega \) isometric to the Euclidean ball \( B(0; R) \subset \mathbb{R}^n \) (‘rigidity’)

- this classical bound may be proved using the maximum principle
- it has well-known Lorentzian (Hawking ’68) and Riemannian (Kasue ’83, Li ’14, …) analogs
Theorem (Kasue ’83; mean convex Riemannian mflds with boundary)

If \((X, d) = (M^n, g)\) has \(\text{Ric}_g \geq Kg\) and \(\Omega \subset M\) with \(\partial \Omega \neq \emptyset\) is \(C^2\), open, connected, and has \(H_{\partial \Omega}(x) \geq H \in \mathbb{R}\) with \(\max \left\{ K, \frac{H}{n-1} - \sqrt{\frac{|K|}{n-1}} \right\} > 0\), then

\[ r(\Omega) \leq r_{K, H, n} = r \frac{K}{n-1}, \frac{H}{n-1}, 2 < \infty \]

with equality iff \(\Omega\) is isometric to an open ball whose boundary has mean curvature \(H\) in a spaceform of constant (sectional) curvature \(\frac{K}{n-1}\).

- here ‘spaceform’ refers to a spherical, Euclidean or hyperbolic \(n\)-space
- but what if \(\Omega\) and/or \(X\) is less smooth?
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- but what if \(\Omega\) and/or \(X\) is less smooth?
- Lott-Villani and Sturm used optimal transport to define lower Ricci bounds in metric spaces \((X, d)\) equipped with a reference measure \(m\);
- we extend Kasue’s theorem to this metric-measure space (‘mms’) setting
- our rigidity statement requires the more restrictive \(RCD\) (‘Riemannian curvature dimension’) condition of Ambrosio, Gigli, and Savare ’14, and equality is attained by truncated cones as well as by balls
Terminology

- \((X, d, m)\) is a metric space with Borel reference measure s.t. \(X = \text{spt } m\)
- **geodesic** refers to a curve \(\{x_t\}_{t \in [0,1]} \subset X\) satisfying
  \[
d(x_s, x_t) = |t - s|d(x_0, x_1) \quad \forall s, t \in [0, 1]
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- assume geodesics do not branch (or essentially non-branching)
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• \(\mathcal{P}(X, d) := \{\text{Borel probability measures } \mu \text{ on } X\}\), metrized by the \(L^2\)-Kantorovich-Rubinstein-Wasserstein distance from optimal transport

\[
d_2(\mu, \nu) := \left( \inf_{\\{\gamma \in \mathcal{P}(X^2, d \otimes d) \text{ with marginals } \mu \text{ and } \nu\}} \int_{X^2} d(x, y)^2 d\gamma(x, y) \right)^{1/2}
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- $\mathcal{P}_2(X, d) := \{\mu \in \mathcal{P}(X, d) \mid d_2(\mu, \delta_x) < \infty\}$ for some (hence all) $x \in X$
- $\mathcal{P}^*_2(X, d, m) := \{\mu \in \mathcal{P}_2(X, d, m) \mid \text{finite rel. entropy } E(\mu \mid m) < \infty\}$
Curvature Dimension (\& Measure Contraction) Properties

- **Sturm ’06**: fix curvature and dimension parameters $K \in \mathbb{R}$ and $N \geq 1$
- $K = 0$ and/or $N = \infty$ considered also in Lott-Villani ’09

**Definition (CD($K, N$) after Erbar-Kuwada-Sturm ’15)**

$(X, d, m) \in \text{CD}(K, N) \iff \forall \mu_0, \mu_1 \in \mathcal{P}_2^*(X, d, m) \exists$ geodesic $\{\mu_t\}_{t \in [0,1]}$ s.t.

$$e''(t) - \frac{e'(t)^2}{N} \geq Kd_2(\mu_0, \mu_1)^2$$
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$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2 \quad \text{distributionally on } t \in (0, 1),$$

where

$$e(t) = E(\mu_t \mid m) := \begin{cases} \int_X \frac{d\mu_t}{dm} \log \frac{d\mu_t}{dm} \, dm & \text{if } \mu_t \ll m, \\ +\infty & \text{else,} \end{cases}$$

is the Boltzmann-Shannon entropy along the $(\mathcal{P}(X, d), d_2)$ geodesic.

- $(X, d, m) \in MCP(K, N) \iff \text{the same } \forall \mu_0 \in \mathcal{P}_2^*(X, d, m) \text{ and } \mu_1 = \delta_x.$
• Riemannian mfld $(M^n, d_g, \text{vol}_g) \in CD(K, N) \iff \text{Ric}_g \geq Kg$ and $n \leq N$;
• $CD(K, N) \subset MCP(K, N)$ (Sturm '06, Ohta '07)
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• If \((X, d, m) \in CD(K', N')\) and \(\Omega \subset X\), we say \(\Omega \in CD_r(K, N)\) if \(\mu_0[\Omega] = 1 = \mu_1[\Omega]\) in the previous construction implies
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• \(\text{CD}(K, N) \subset \text{MCP}(K, N)\) (Sturm '06, Ohta '07)
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• restriction \(\text{MCP}_r(K, N)\) is defined analogously relative to \(\text{MCP}(K', N')\)
• the \textit{signed distance} to the boundary of \(\Omega\) is defined by

\[
d_{\pm}(x) := \begin{cases} d_{\Omega}(x) & \text{if } x \not\in \Omega, \\ -d_{X \setminus \Omega}(x) & \text{if } x \in \Omega, \end{cases}
\]

where

\[
d_{\Omega}(x) := \inf_{y \in \Omega} d(x, y)
\]

• \(\text{Lip}(d_{\pm}) \leq 1\)
Cavalletti et al’s Needle Decomposition / 1d Localization

After discarding a (carefully chosen!) measure zero set from the non-branching MCP space \((X, d, m)\),

\[
x \sim y \iff |d^\pm_\Omega(y) - d^\pm_\Omega(x)| = d(x, y)
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defines an equivalence relation, whose equivalence classes consist of geodesic segments called needles, heuristically ‘normal’ to \(\partial \Omega\).
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Let \(\tilde{x}\) denote the equivalence class of \(x \in X\); the quotient space \(\tilde{X} := X/\sim\) can be identified with \(\partial \Omega\), and inherits the quotient measure \(\tilde{m}\) from \((X, d, m)\).

Viewing \((\tilde{x}, d_\Omega^\pm(x)) \in \partial \Omega \times \mathbb{R}\) as ‘coordinates’ on \(X\),
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Viewing \((\tilde{x}, d_\Omega^\pm(x)) \in \partial \Omega \times \mathbb{R}\) as ‘coordinates’ on \(X\), Cavalletti and coauthors re-express \(m\) relative to these coordinates, by providing a measure \(m_{\tilde{x}}\) on each equivalence class \(\tilde{x} \in \partial \Omega\) s.t.

\[
m(E) = \int_{\partial \Omega} d\tilde{m}(\tilde{x}) \int_{\tilde{x} \cap E \subset \mathbb{R}} dm_{\tilde{x}}(s) \quad \forall \text{ Borel } E \subset X.
\]
Mean convexity and mean curvature

• $(X, d, m) \in \text{CD}(K, N) \iff (\tilde{x}, | \cdot |, m_{\tilde{x}}) \in \text{CD}(K, N) \forall \Omega \subset X, \tilde{m}\text{-a.e. } \tilde{x}$
Mean convexity and mean curvature

- \((X, d, m) \in CD(K, N) \iff (\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N)\) \ \forall \Omega \subset X, \tilde{m}\text{-a.e.} \ \tilde{x}
- \((\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N) \iff \tilde{x} = \{x\} \ \text{or} \ dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \ \text{and} \ \frac{1}{N-1}h_{\tilde{x}}^{\frac{1}{N-1}}' \leq -\frac{K}{N-1}h_{\tilde{x}}^{\frac{1}{N-1}}

distributionally, where \(s = d_{\Omega}^{\pm}(x) = \text{arclength} \ \text{along geodesic segment} \ \tilde{x} \)
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\[
(h_{\tilde{x}}^{\frac{1}{N-1}})'' \leq -\frac{K}{N-1} h_{\tilde{x}}^{\frac{1}{N-1}}
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- **Ketterer '20**: define the (inner) mean curvature \( H_{\partial\Omega} \) at \( \tilde{m}\text{-a.e.} \ x \in \partial\Omega \) as

\[
H_{\partial\Omega}(x) := \frac{d^+}{ds} \log h_{\tilde{x}}(0^-) = \limsup_{s \uparrow 0} \frac{\log h_{\tilde{x}}(s) - \log h_{\tilde{x}}(0)}{s}
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$$
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$$

- we’ll write $H_{\partial \Omega} \geq \lambda$ provided $H_{\partial \Omega}(x) \geq \lambda$ holds $\tilde{m}$-a.e., $m[\partial \Omega] = 0$, and

$$
m[\{x \mid \tilde{x} \subset \Omega \cup \partial \Omega\}] = 0
$$
(preventing e.g. inward pointing cusps on the boundary)

- c.f. Cavalletti-Mondino '20+ nonsmooth Hawking singularity theorem
Theorem (Extending Kasue’s results to nonsmooth spaces)

(a) If \((X, d, m) \in \text{MCP}(K', N)\) for some \(K' \in \mathbb{R}\) and \(1 < N < \infty\) and if \(\Omega \subset X\) open with \(\partial \Omega \neq \emptyset\) satisfies \(\Omega \in \text{MCP}_r(K, N)\) and \(H_{\partial \Omega} \geq H \in \mathbb{R}\) with \(\max \{K, H - \sqrt{(N - 1)|K|}\} > 0\), then the inscribed radius of \(\Omega\)

\[
r(\Omega) \leq r_{K, H, N} := r_{\frac{K}{N-1}, \frac{H}{N-1}, 2} < \infty
\]

where \(r_{\kappa, \lambda, 2}\) is again the radius of a circle with curvature \(\lambda := \frac{H}{N-1}\) in a two-dimensional spaceform of constant curvature \(\kappa := \frac{K}{N-1}\).
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(b) **(Rigidity)** If also \((X, d, m) \in \text{RCD}(K, N)\) with \(\kappa \in \{-1, 0, 1\}\) and \(\Omega\) is connected, then \(r(\Omega) = r_{K, H, N} \iff \Omega\) becomes isometric to the ball \(B(o, r_{K, H, N})\) around the cone tip in some conical warped product \(I \otimes_{\kappa}^{N-1} Y\) of an interval \(I := [0, \pi_\kappa)\) with an \(\text{RCD}(N - 2, N - 1)\) space \((Y, d_Y, m_Y)\), when both \(\Omega\) and \(B(o, r_{K, H, N})\) are equipped with their induced intrinsic distances. (Here \(Y\) is a single point when \(N < 2\).)
• Kapovitch-Ketterer '20: \( RCD(K, N) \subset CD(K, N) \) refers to spaces \((X, d, m)\) for which \(m\)-a.e. tangent cone is isometric to Euclidean space.
• Kapovitch-Ketterer '20: $RCD(K, N) \subset CD(K, N)$ refers to spaces $(X, d, m)$ for which $m$-a.e. tangent cone is isometric to Euclidean space

• $I = [0, \pi_\kappa)$ where $\pi_\kappa := \begin{cases} \pi & \text{if } \kappa = 1, \\ \infty & \text{if } \kappa \leq 0 \end{cases}$

• set $\sin_\kappa(t) := \begin{cases} \sin(t) & \text{if } \kappa = 1, \\ t & \text{if } \kappa = 0, \\ \sinh(t) & \text{if } \kappa = -1 \end{cases}$

• the cone $I \otimes_{\kappa}^{N-1} Y$ refers to the product space $I \times Y$ endowed with measure $\sin_\kappa^{N-1}(r)dr \times dm_Y$ and
• Kapovitch-Ketterer '20: \( RCD(K, N) \subset CD(K, N) \) refers to spaces \((X, d, m)\) for which \(m\)-a.e. tangent cone is isometric to Euclidean space

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\[
d(((s, y), (t, z)))^2 := s^2 + t^2 - 2st \cos(d_Y(y, z) \wedge \pi) \quad \text{if } \kappa = 0
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where \( s \wedge t := \min\{s, t\} \)
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where $s \wedge t := \min\{s, t\}$ and otherwise

$$\sin'_\kappa d(((s, y), (t, z))) = \sin'_\kappa(s) \sin'_\kappa(t) + \kappa \sin_\kappa(s) \sin_\kappa(t) \cos(d_Y(y, z) \wedge \pi)$$

• in each case the points $(0, y)$ and $(0, z)$ are identified (the 'cone tip')
Idea of proof (a)

Assume \((K, H) = (0, N - 1)\) for simplicity, as for the unit ball in \(\mathbb{R}^N\):

- on \(\tilde{m}\)-a.e. geodesic segment \(\tilde{x}\) ‘normal’ to \(\partial \Omega\): \(\Omega \in CD_r(K, N)\) implies

\[
\frac{d^2 h^{\frac{1}{N-1}}_{\tilde{x}}}{ds^2} \leq 0 \quad \forall s \leq 0.
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- $H_{\partial \Omega}(x) \geq H = N - 1$ implies

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• but \(dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0\), thus \(s := d_{\Omega}^\pm(x) \geq -1\) meaning
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  \]

- concavity shows \(h^{\frac{1}{N-1}}_{\tilde{x}}\) then becomes negative for \(s < -1\)

- but \(dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0\), thus \(s := d^\pm_{\Omega}(x) \geq -1\) meaning

- geodesics \(\tilde{x}\) extending further than unit distance into \(\Omega\) are \(\tilde{m}\) negligible

- so in fact no such geodesic can exist.
Idea of proof (b)

Relies on a de Philippis and Gigli ’16 result which asserts:

whenever two concentric $RCD(K, N)$ balls behave \textit{volumetrically} as they would in a cone,
then the larger ball is either a metric cone or one-dimensional

• we use $N > 1$ to rule out one-dimensional spaces with $\#(\partial\Omega) = 2$.
• we extend Laplacian comparison estimates to $RCD(K, N)$ spaces
where we can combine the strong maximum principle with properties of $\sin\kappa$ to deduce $r(\Omega) = r_{K, H, N}$ implies the volume of (a certain connected component of the punctured) inscribed ball in $\Omega$ depends on $r < r_{K, H, N}$ as it would for a ball in the model space
• (essential) nonbranchingness implies the inscribed ball cannot be disconnected by removing its center

Thank you very much!
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