Inscribed radius bounds for lower Ricci bounded metric measure spaces with mean convex boundary

A. Burtscher, C. Ketterer, R. McCann, E. Woolgar


University of Toronto

Slides: click on ‘Talk’ at www.math.toronto.edu/mccann

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Inscribed radius bounds

**Inscribed radius** $r = r(\Omega)$ of an open subset $\Omega$ of a metric space $(X, d)$ is

$$r(\Omega) := \sup \{ r \geq 0 \mid B(x; r) \subset \Omega \text{ for some } x \in \Omega \}$$
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**Theorem (classical)**

If $(X, d) = (\mathbb{R}^n, |\cdot|)$ and the mean-curvature $H_{\partial \Omega} \geq (n - 1)/R$ then

$$r(\Omega) \leq R,$$

with equality iff $\Omega$ isometric to the Euclidean ball $B(0; R) \subset \mathbb{R}^n$ (‘rigidity’)

- this classical bound may be proved using the maximum principle
- it has well-known Lorentzian (Hawking ’68) and Riemannian (Kasue ’83, Li ’14, ...) analogs
Theorem (Kasue ’83; mean convex Riemannian mflds with boundary)

If \((X, d) = (M^n, g)\) has \(\text{Ric}_g \geq Kg\) and \(\Omega \subset M\) with \(\partial \Omega \neq \emptyset\) is \(C^2\), open, connected, and has \(H_{\partial \Omega}(x) \geq H \in \mathbb{R}\) with \(\max \left\{ K, \frac{H}{n-1} - \sqrt{\frac{|K|}{n-1}} \right\} > 0\), then

\[
 r(\Omega) \leq r_{K,H,n} = r_{\frac{K}{n-1}, \frac{H}{n-1}, 2} < \infty
\]

with equality iff \(\Omega\) is isometric to an open ball whose boundary has mean curvature \(H\) in a spaceform of constant (sectional) curvature \(\frac{K}{n-1}\).

- here ‘spaceform’ refers to a spherical, Euclidean or hyperbolic \(n\)-space
- but what if \(\Omega\) and/or \(X\) is less smooth?
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- but what if \(\Omega\) and/or \(X\) is less smooth?
- Lott-Villani and Sturm used optimal transport to define lower Ricci bounds in metric spaces \((X, d)\) equipped with a reference measure \(m\);
- we extend Kasue’s theorem to this metric-measure space (‘mms’) setting
- our rigidity statement requires the more restrictive \(\text{RCD}\) (‘Riemannian curvature dimension’) condition of Ambrosio, Gigli, and Savare ’14, and equality is attained by truncated cones as well as by balls.
Terminology

• $(X, d, m)$ is a metric space with Borel reference measure s.t. $X = \text{spt } m$

• geodesic refers to a curve $\{x_t\}_{t \in [0,1]} \subset X$ satisfying

$$d(x_s, x_t) = |t - s|d(x_0, x_1) \quad \forall s, t \in [0, 1]$$

• assume geodesics do not branch (or essentially non-branching)
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  \[d(x_s, x_t) = |t - s| d(x_0, x_1) \quad \forall s, t \in [0, 1]\]
• assume geodesics do not branch (or essentially non-branching)
• \(\mathcal{P}(X, d) := \{\text{Borel probability measures } \mu \text{ on } X\}\), metrized by the
  \(L^2\)-Kantorovich-Rubinstein-Wasserstein distance from optimal transport
  \[d_2(\mu, \nu) := \left( \inf_{\{\gamma \in \mathcal{P}(X^2, d \otimes d) \text{ with marginals } \mu \text{ and } \nu\}} \int_{X^2} d(x, y)^2 d\gamma(x, y) \right)^{1/2}\]
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• $\mathcal{P}_2(X, d) := \{\mu \in \mathcal{P}(X, d) \mid d_2(\mu, \delta_x) < \infty\}$ for some (hence all) $x \in X$
• $\mathcal{P}_2^*(X, d, m) := \{\mu \in \mathcal{P}_2(X, d, m) \mid \text{finite rel. entropy } E(\mu \mid m) < \infty\}$
Curvature Dimension (& Measure Contraction) Properties

- **Sturm '06**: fix curvature and dimension parameters \( K \in \mathbb{R} \) and \( N \geq 1 \)
- \( K = 0 \) and/or \( N = \infty \) considered also in Lott-Villani '09

**Definition (\( CD(K, N) \) after Erbar-Kuwada-Sturm '15)**

\[(X, d, m) \in CD(K, N) \iff \forall \mu_0, \mu_1 \in \mathcal{P}^*_2(X, d, m) \exists \text{geodesic } \{\mu_t\}_{t \in [0,1]} \text{ s.t.} \]

\[e''(t) - \frac{e'(t)^2}{N} \geq Kd_2(\mu_0, \mu_1)^2\]
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$$e''(t) - \frac{e'(t)^2}{N} \geq Kd_2(\mu_0, \mu_1)^2 \quad \text{distributionally on } t \in (0,1), \text{ where}$$

$$e(t) = E(\mu_t \mid m) := \begin{cases} \int_X \frac{d\mu_t}{dm} \log \frac{d\mu_t}{dm} dm & \text{if } \mu_t \ll m, \\ +\infty & \text{else}, \end{cases}$$

is the Boltzmann-Shannon entropy along the $(\mathcal{P}(X, d), d_2)$ geodesic.

- $(X, d, m) \in MCP(K, N) \iff$ the same $\forall \mu_0 \in \mathcal{P}_2^*(X, d, m)$ and $\mu_1 = \delta_x$.  

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• Riemannian mfd \((M^n, d_g, \text{vol}_g) \in CD(K, N) \Leftrightarrow \text{Ric}_g \geq Kg \text{ and } n \leq N;\)
e.g. \(M = \overline{B(0, 1)} \subset \mathbb{R}^n\) is \(CD(0, n)\), while \(\partial M \in CD(n - 2, n - 1)\).
• \(CD(K, N) \subset MCP(K, N)\) (Sturm '06, Ohta '07)
• Riemannian mfd \((M^n, d_g, \text{vol}_g) \in CD(K, N) \iff \text{Ric}_g \geq Kg\) and \(n \leq N\);
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• \(CD(K, N) \subset MCP(K, N)\) (Sturm '06, Ohta '07)

• If \((X, d, m) \in CD(K', N')\) and \(\Omega \subset X\), we say \(\Omega \in CD_r(K, N)\) if \(\mu_0[\Omega] = 1 = \mu_1[\Omega]\) in the previous construction implies

\[
e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2.
\]
• Riemannian mfd \((M^n, d_g, \text{vol}_g) \in CD(K, N) \iff \text{Ric}_g \geq Kg\) and \(n \leq N\);
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\[
e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2.
\]

• restriction \(MCP_r(K, N)\) is defined analogously relative to \(MCP(K', N')\)

• the **signed distance** to the boundary of \(\Omega\) is defined by

\[
d_{\Omega}^\pm(x) := \begin{cases} 
d_{\Omega}(x) & \text{if } x \notin \Omega, \\
-d_{X \setminus \Omega}(x) & \text{if } x \in \Omega,
\end{cases}
\]

where

\[
d_{\Omega}(x) := \inf_{y \in \Omega} d(x, y)
\]

• \(\text{Lip}(d_{\Omega}^\pm) \leq 1\)
After discarding a (carefully chosen!) measure zero set from the non-branching MCP space \((X, d, m)\),

\[ x \sim y \iff |d^\pm_\Omega(y) - d^\pm_\Omega(x)| = d(x, y) \]

defines an equivalence relation, whose equivalence classes consist of geodesic segments called needles, heuristically ‘normal’ to \(\partial \Omega\).
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Let \(\tilde{x}\) denote the equivalence class of \(x \in X\); the quotient space \(\tilde{X} := X/\sim\) can be identified with \(\partial \Omega\), and inherits the quotient measure \(\tilde{m}\) from \((X, d, m)\).

Viewing \((\tilde{x}, d^\pm_\Omega(x)) \in \partial \Omega \times \mathbb{R}\) as ‘coordinates’ on \(X\),

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Viewing \((\tilde{x}, d^\pm_\Omega(x)) \in \partial \Omega \times \mathbb{R}\) as ‘coordinates’ on \(X\), Cavalletti and coauthors re-express \(m\) relative to these coordinates, by providing a measure \(m_{\tilde{x}}\) on each equivalence class \(\tilde{x} \in \partial \Omega\) s.t.

\[ m(E) = \int_{\partial \Omega} d\tilde{m}(\tilde{x}) \int_{\tilde{x} \cap E \subset \mathbb{R}} dm_{\tilde{x}}(s) \quad \forall \text{ Borel } E \subset X. \]
Mean convexity and mean curvature

• \((X, d, m) \in CD(K, N) \iff (\tilde{x}, | \cdot |, m_{\tilde{x}}) \in CD(K, N) \quad \forall \Omega \subset X, \tilde{m}\text{-a.e. } \tilde{x}\)
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• \((\tilde{x}, | \cdot |, m_{\tilde{x}}) \in \text{CD}(K, N) \iff \tilde{x} = \{x\} \text{ or } dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \text{ and }
\begin{align*}
(h_{\tilde{x}}^{\frac{1}{N-1}})'' &\leq -\frac{K}{N-1} h_{\tilde{x}}^{\frac{1}{N-1}}
\end{align*}
\text{distributionally, where } s = d_{\Omega}^{\pm}(x) = \text{arclength along geodesic segment } \tilde{x}
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- \((\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N) \iff \tilde{x} = \{x\}\) or \(d\tilde{m}(s) = h_{\tilde{x}}(s)ds\) and

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\]

distributionally, where \(s = d^{\pm}_\Omega(x) = \text{arclength}\) along geodesic segment \(\tilde{x}\)

- Ketterer '20: define the (inner) mean curvature \(H_{\partial \Omega}\) at \(\tilde{m}\text{-a.e. } x \in \partial \Omega\) as

\[
H_{\partial \Omega}(x) := \frac{d^+}{ds} \log h_{\tilde{x}}(0^-) = \limsup_{s \uparrow 0} \frac{\log h_{\tilde{x}}(s) - \log h_{\tilde{x}}(0)}{s}
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• we’ll write \(H_{\partial \Omega} \geq \lambda\) provided \(H_{\partial \Omega}(x) \geq \lambda\) holds \(\tilde{m}\text{-a.e.}, m[\partial \Omega] = 0\), and

\[
m[\{x \mid \tilde{x} \subset \Omega \cup \partial \Omega\}] = 0
\]
(preventing e.g. inward pointing cusps on the boundary)

• c.f. Cavalletti-Mondino ’20+ nonsmooth Hawking singularity theorem
Theorem (Extending Kasue’s results to nonsmooth spaces)

(a) If \((X, d, m) \in \text{MCP}(K', N)\) for some \(K' \in \mathbb{R}\) and \(1 < N < \infty\) and if \(\Omega \subset X\) open with \(\partial \Omega \neq \emptyset\) satisfies \(\Omega \in \text{MCP}_r(K, N)\) and \(H_{\partial \Omega} \geq H \in \mathbb{R}\) with \(\max \left\{ K, H - \sqrt{(N - 1)|K|} \right\} > 0\), then the inscribed radius of \(\Omega\)

\[
r(\Omega) \leq r_{K,H,N} := r_{\frac{K}{N-1}, \frac{H}{N-1}, 2} < \infty
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where \(r_{\kappa, \lambda, 2}\) is again the radius of a circle with curvature \(\lambda := \frac{H}{N-1}\) in a two-dimensional spaceform of constant curvature \(\kappa := \frac{K}{N-1}\).
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where \(r_{K,\lambda,2}\) is again the radius of a circle with curvature \(\lambda := \frac{H}{N-1}\) in a two-dimensional spaceform of constant curvature \(\kappa := \frac{K}{N-1}\).

(b) \textbf{(Rigidity)} If also \((X, d, m) \in RCD(K, N)\) with \(\kappa \in \{-1, 0, 1\}\) and \(\Omega\) is connected, then \(r(\Omega) = r_{K,H,N} \iff \Omega\) becomes isometric to the ball \(B(o, r_{K,H,N})\) around the cone tip in some conical warped product \(I \otimes_{\kappa}^{N-1} Y\) of an interval \(I := [0, \pi_{\kappa})\) with an \(RCD(N - 2, N - 1)\) space \((Y, d_Y, m_Y)\), when both \(\Omega\) and \(B(o, r_{K,H,N})\) are equipped with their induced intrinsic distances. (Here \(Y\) is a single point when \(N < 2\).)
Kapovitch-Ketterer '20: $RCD(K, N) \subset CD(K, N)$ refers to spaces $(X, d, m)$ for which $m$-a.e. tangent cone is isometric to Euclidean space.
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• \( I = [0, \pi \kappa) \) where \( \pi \kappa := \begin{cases} \pi & \text{if } \kappa = 1, \\ \infty & \text{if } \kappa \leq 0 \end{cases} \)

• set \( \sin_{\kappa}(t) := \begin{cases} \sin(t) & \text{if } \kappa = 1, \\ t & \text{if } \kappa = 0, \\ \sinh(t) & \text{if } \kappa = -1 \end{cases} \)

• the cone \( I \otimes_{\kappa}^{N-1} Y \) refers to the product space \( I \times Y \) endowed with measure \( \sin_{\kappa}^{N-1}(r)dr \times dm_Y \) and
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$$d((s, y), (t, z))^2 := s^2 + t^2 - 2st \cos(d_Y(y, z) \wedge \pi) \quad \text{if } \kappa = 0$$

where $s \wedge t := \min\{s, t\}$
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\]

where \( s \wedge t := \min\{s, t\} \) and otherwise

\[
\sin'_\kappa d((s, y), (t, z)) = \sin'_\kappa(s) \sin'_\kappa(t) - \kappa \sin_\kappa(s) \sin_\kappa(t) \cos(d_Y(y, z) \wedge \pi)
\]

• in each case the points \((0, y)\) and \((0, z)\) are identified (the ‘cone tip’)

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Idea of proof (a)

Assume \((K, H) = (0, N - 1)\) for simplicity, as for the unit ball in \(\mathbb{R}^N\):

- on \(\tilde{m}\)-a.e. geodesic segment \(\tilde{x}\) ‘normal’ to \(\partial \Omega\): \(\Omega \in CD_r(K, N)\) implies

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\frac{d^2 h^{\frac{1}{N-1}}_{\tilde{x}}}{ds^2} \leq 0 \quad \forall s \leq 0.
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  \frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \quad \forall s \leq 0.
  \]

- \(H_{\partial \Omega}(x) \geq H = N - 1\) implies \(\frac{d \log h_{\tilde{x}}^{\frac{1}{N-1}}}{ds} \bigg|_{s=0^-} \geq \frac{N - 1}{N - 1} = 1\)
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$$d^2 h^{\frac{1}{N-1}}_{\tilde{x}} \leq 0 \quad \forall s \leq 0.$$

- $H_{\partial \Omega}(x) \geq H = N - 1$ implies

$$\left. \frac{d \log h^{\frac{1}{N-1}}_{\tilde{x}}}{ds} \right|_{s=0^-} \geq \frac{N - 1}{N - 1} = 1$$

- concavity shows $h^{\frac{1}{N-1}}_{\tilde{x}}$ then becomes negative for $s < -1$

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- \(H_{\partial \Omega}(x) \geq H = N - 1\) implies
  
  \[
  \left. \frac{d \log h^{\frac{1}{N-1}}_{\tilde{x}}}{ds} \right|_{s=0^-} \geq \frac{N - 1}{N - 1} = 1
  \]

- concavity shows \(h^{\frac{1}{N-1}}_{\tilde{x}}\) then becomes negative for \(s < -1\)

- but \(dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0\), thus \(s := d^\pm_{\Omega}(x) \geq -1\) meaning...
Idea of proof (a)

Assume $(K, H) = (0, N - 1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ ‘normal’ to $\partial \Omega$: $\Omega \in CD_r(K, N)$ implies
  \[
  \frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \quad \forall s \leq 0.
  \]

- $H_{\partial \Omega}(x) \geq H = N - 1$ implies
  \[
  \left. \frac{d \log h_{\tilde{x}}^{\frac{1}{N-1}}}{ds} \right|_{s=0^-} \geq \frac{N - 1}{N - 1} = 1
  \]

- concavity shows $h_{\tilde{x}}^{\frac{1}{N-1}}$ then becomes negative for $s < -1$

- but $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0$, thus $s := d_{\Omega}^{\pm}(x) \geq -1$ meaning

- geodesics $\tilde{x}$ extending further than unit distance into $\Omega$ are $\tilde{m}$ negligible

- so in fact no such geodesic can exist.
Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K, N)$ balls behave \textit{volumetrically} as they would in a cone,
then the larger ball is \textit{either a metric cone or one-dimensional}
Idea of proof (b)

Relies on a de Philippis and Gigli ’16 result which asserts:

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• we use $N > 1$ to rule out one-dimensional spaces with $\#(\partial \Omega) = 2$.
• we extend Laplacian comparison estimates to $RCD(K, N)$ spaces
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* where we can combine the strong maximum principle with properties of $\sin_\kappa$ to deduce $r(\Omega) = r_{K, H, N}$ implies the volume of (a certain connected component of the punctured) inscribed ball in $\Omega$ depends on $r < r_{K, H, N}$ as it would for a ball in the model space
* (essential) nonbranchingness implies the inscribed ball cannot be disconnected by removing its center
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Thank you very much!