Inscribed radius bounds for lower Ricci bounded metric measure spaces with mean convex boundary

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Inscribed radius bounds

Inscribed radius $r = r(\Omega)$ of an open subset Ω of a metric space (X, d) is

$$r(\Omega) := \sup\{r \geq 0 \mid B(x; r) \subset \Omega \text{ for some } x \in \Omega\}$$

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Theorem (classical)

If $(X, d) = (\mathbb{R}^n, |\cdot|)$ and the mean-curvature $H_{\partial\Omega} \geq (n-1)/R$ then

$$r(\Omega) \leq R,$$

with equality iff Ω isometric to the Euclidean ball $B(0; R) \subset \mathbb{R}^n$ ('rigidity')

- this classical bound may be proved using the maximum principle
- it has well-known Lorentzian (Hawking '68) and Riemannian (Kasue '83, Li '14, ...) analogs

Theorem (Kasue '83; mean convex Riemannian mflds with boundary)

If $(X, d) = (M^n, g)$ *has* $Ric_g \geq Kg$ *and* $\Omega \subset M$ *with* $\partial\Omega \neq \emptyset$ *is* C^2*, open, connected, and has* $H_{\partial\Omega}(x) \geq H \in \mathbb{R}$ *with* $\max\left\{K, \frac{H}{n-1} - \sqrt{\frac{|K|}{n-1}}\right\} > 0$*, then*

$$r(\Omega) \leq r_{K,H,n} = r_{\frac{K}{n-1},\frac{H}{n-1},2} < \infty$$

with equality iff Ω *is isometric to an open ball whose boundary has mean curvature* H *in a spaceform of constant (sectional) curvature* $\frac{K}{n-1}$.

- here 'spaceform' refers to a spherical, Euclidean or hyperbolic n-space
- but what if Ω and/or X is less smooth?

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- Lott-Villani and Sturm used optimal transport to define lower Ricci bounds in metric spaces (X,d) equipped with a reference measure m;
- we extend Kasue's theorem to this metric-measure space ('mms') setting
- our rigidity statement requires the more restrictive *RCD* ('Riemannian curvature dimension') condition of Ambrosio, Gigli, and Savare '14, and equality is attained by truncated cones as well as by balls

Terminology

- (X, d, m) is a metric space with Borel reference measure s.t. $X = \mathrm{spt}\, m$
- *geodesic* refers to a curve $\{x_t\}_{t\in[0,1]} \subset X$ satisfying

$$d(x_s, x_t) = |t - s| d(x_0, x_1) \quad \forall s, t \in [0, 1]$$

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- $\mathcal{P}(X, d) := \{$Borel probability measures μ on $X\}$, metrized by the L^2-Kantorovich-Rubinstein-Wasserstein distance from optimal transport

$$d_2(\mu, \nu) := \left(\inf_{\{\gamma\in\mathcal{P}(X^2, d\otimes d) \text{ with marginals } \mu \text{ and } \nu\}} \int_{X^2} d(x, y)^2 \mathrm{d}\gamma(x, y) \right)^{1/2}$$

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- $\mathcal{P}_2(X, d) := \{\mu \in \mathcal{P}(X, d) \mid d_2(\mu, \delta_x) < \infty\}$ for some (hence all) $x \in X$
- $\mathcal{P}_2^*(X, d, m) := \{\mu \in \mathcal{P}_2(X, d, m) \mid$ finite rel. entropy $E(\mu \mid m) < \infty\}$

Curvature Dimension (& Measure Contraction) Properties

- Sturm '06: fix curvature and dimension parameters $K \in \mathbb{R}$ and $N \geq 1$
- $K = 0$ and/or $N = \infty$ considered also in Lott-Villani '09

Definition ($CD(K, N)$ after Erbar-Kuwada-Sturm '15)

$(X, d, m) \in CD(K, N) \Leftrightarrow \forall \mu_0, \mu_1 \in \mathcal{P}_2^*(X, d, m)$ $\exists$ geodesic $\{\mu_t\}_{t\in[0,1]}$ s.t.

$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2$$

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$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2 \quad \text{distributionally on } \mathrm{t} \in (0,1), \text{ where}$$

$$e(t) = E(\mu_t \mid m) := \begin{cases} \int_X \frac{\mathrm{d}\mu_t}{\mathrm{d}m} \log \frac{\mathrm{d}\mu_t}{\mathrm{d}m} \mathrm{d}m & \text{if } \mu_t << m, \\ +\infty & \text{else,} \end{cases}$$

is the Boltzmann-Shannon entropy along the $(\mathcal{P}(X,d), d_2)$ geodesic.

- $(X,d,m) \in MCP(K,N) \Leftrightarrow$ the same $\forall \mu_0 \in \mathcal{P}_2^*(X,d,m)$ and $\mu_1 = \delta_x$.

- Riemannian mfld $(M^n, d_g, \mathrm{vol}_g) \in CD(K,N) \Leftrightarrow Ric_g \geq Kg$ and $n \leq N$;
- $CD(K,N) \subset MCP(K,N)$ (Sturm '06, Ohta '07)

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- If $(X,d,m) \in CD(K',N')$ and $\Omega \subset X$, we say $\Omega \in CD_r(K,N)$ if $\mu_0[\Omega] = 1 = \mu_1[\Omega]$ in the previous construction implies

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- restriction $MCP_r(K,N)$ is defined analogously relative to $MCP(K',N')$
- the *signed distance* to the boundary of Ω is defined by

$$d_\Omega^\pm(x) := \begin{cases} d_\Omega(x) & \text{if } x \notin \Omega, \\ -d_{X\setminus\Omega}(x) & \text{if } x \in \Omega, \end{cases}$$

where

$$d_\Omega(x) := \inf_{y\in\Omega} d(x,y)$$

- $Lip(d_\Omega^\pm) \leq 1$

Cavalletti et al's Needle Decomposition / 1d Localization

After discarding a (carefully chosen!) measure zero set from the non-branching MCP space (X, d, m),

$$x \sim y \Leftrightarrow |d_\Omega^\pm(y) - d_\Omega^\pm(x)| = d(x, y)$$

defines an equivalence relation, whose equivalence classes consist of geodesic segments called needles, heuristically 'normal' to $\partial\Omega$.

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Let $\tilde{x}$ denote the equivalence class of $x \in X$;
the quotient space $\tilde{X} := X/\sim$ can be identified with $\partial\Omega$,
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Cavalletti and coauthors re-express m relative to these coordinates,
by providing a measure $m_{\tilde{x}}$ on each equivalence class $\tilde{x} \in \partial\Omega$ s.t.

$$m(E) = \int_{\partial\Omega} \mathrm{d}\tilde{m}(\tilde{x}) \int_{\tilde{x}\cap E \subset \mathbb{R}} \mathrm{d}m_{\tilde{x}}(s) \qquad \forall \text{ Borel } E \subset X.$$

Mean convexity and mean curvature

- $(X, d, m) \in CD(K, N) \Leftrightarrow (\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N)\ \forall \Omega \subset X, \tilde{m}$-a.e. $\tilde{x}$

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- $(\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N) \Leftrightarrow \tilde{x} = \{x\}$ or $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds$ and

$$(h_{\tilde{x}}^{\frac{1}{N-1}})'' \leq -\frac{K}{N-1} h_{\tilde{x}}^{\frac{1}{N-1}}$$

distributionally, where $s = d_{\Omega}^{\pm}(x) =$ arclength along geodesic segment $\tilde{x}$

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- Ketterer '20: define the (inner) mean curvature $H_{\partial\Omega}$ at $\tilde{m}$-a.e. $x \in \partial\Omega$ as

$$H_{\partial\Omega}(x) := \frac{\mathrm{d}^+}{\mathrm{d}s} \log h_{\tilde{x}}(0^-) = \limsup_{s \nearrow 0} \frac{\log h_{\tilde{x}}(s) - \log h_{\tilde{x}}(0)}{s}$$

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- we'll write $H_{\partial\Omega} \geq \lambda$ provided $H_{\partial\Omega}(x) \geq \lambda$ holds $\tilde{m}$-a.e., $m[\partial\Omega] = 0$, and

$$m[\{x \mid \tilde{x} \subset \Omega \cup \partial\Omega\}] = 0$$

(preventing e.g. inward pointing cusps on the boundary)

- c.f. Cavalletti-Mondino '20+ nonsmooth Hawking singularity theorem

Theorem (Extending Kasue's results to nonsmooth spaces)

(a) If $(X, d, m) \in MCP(K', N)$ *for some* $K' \in \mathbb{R}$ *and* $1 < N < \infty$ *and if* $\Omega \subset X$ *open with* $\partial\Omega \neq \emptyset$ *satisfies* $\Omega \in MCP_r(K, N)$ *and* $H_{\partial\Omega} \geq H \in \mathbb{R}$ *with* $\max\left\{K, H - \sqrt{(N-1)|K|}\right\} > 0$, *then the inscribed radius of* Ω

$$r(\Omega) \leq r_{K,H,N} := r_{\frac{K}{N-1},\frac{H}{N-1},2} < \infty$$

where $r_{\kappa,\lambda,2}$ *is again the radius of a circle with curvature* $\lambda := \frac{H}{N-1}$ *in a two-dimensional spaceform of constant curvature* $\kappa := \frac{K}{N-1}$.

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(b) (Rigidity) If also $(X, d, m) \in RCD(K, N)$ *with* $\kappa \in \{-1, 0, 1\}$ *and* Ω *is connected, then* $r(\Omega) = r_{K,H,N} \Leftrightarrow \Omega$ *becomes isometric to the ball* $B(o, r_{K,H,N})$ *around the cone tip in some conical warped product* $I \otimes_{\kappa}^{N-1} Y$ *of an interval* $I := [0, \pi_\kappa)$ *with an* $RCD(N-2, N-1)$ *space* (Y, d_Y, m_y), *when both* Ω *and* $B(o, r_{K,H,N})$ *are equipped with their induced intrinsic distances. (Here* Y *is a single point when* $N < 2$.)

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• $I = [0, \pi_\kappa)$ where $\pi_\kappa := \begin{cases} \pi & \text{if } \kappa = 1, \\ \infty & \text{if } \kappa \leq 0 \end{cases}$

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$$\sin'_\kappa d((s, y), (t, z)) = \sin'_\kappa(s)\sin'_\kappa(t) + \kappa \sin_\kappa(s)\sin_\kappa(t)\cos(d_Y(y, z) \wedge \pi)$$

• in each case the points $(0, y)$ and $(0, z)$ are identified (the 'cone tip')

Idea of proof (a)

Assume $(K, H) = (0, N-1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ 'normal' to $\partial\Omega$: $\Omega \in CD_r(K, N)$ implies

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- but $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0$, thus $s := d_{\Omega}^{\pm}(x) \geq -1$ meaning
- geodesics $\tilde{x}$ extending further than unit distance into Ω are $\tilde{m}$ negligible
- so in fact no such geodesic can exist.

Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K,N)$ balls behave volumetrically as they would in a cone,
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• where we can combine the strong maximum principle with properties of $\sin_\kappa$ to deduce $r(\Omega) = r_{K,H,N}$ implies the volume of (a certain connected component of the punctured) inscribed ball in Ω depends on $r < r_{K,H,N}$ as it would for a ball in the model space

• (essential) nonbranchingness implies the inscribed ball cannot be disconnected by removing its center

Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K, N)$ balls behave volumetrically as they would in a cone,
then the larger ball is either a metric cone or one-dimensional

• we use $N > 1$ to rule out one-dimensional spaces with $\#(\partial\Omega) = 2$.

• we extend Laplacian comparison estimates to $RCD(K, N)$ spaces

• where we can combine the strong maximum principle with properties of $\sin_\kappa$ to deduce $r(\Omega) = r_{K,H,N}$ implies the volume of (a certain connected component of the punctured) inscribed ball in Ω depends on $r < r_{K,H,N}$ as it would for a ball in the model space

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Thank you very much!