Read Evans Appendix D. We have covered Chapter 1, 2.1–2.2 and hope to finish Chapter 2.3 this week and start 2.4 next week.

To be handed in: Evans (Second edition) # 2.8, 2.9, 2.11, 2.12, 2.14 and

1. Let given $U \subset \mathbb{R}^n$ and $g \in C(\partial U)$ let $W = \{ u \in C^2(\bar{U}) \mid u = g \text{ on } \partial U \}$. Let

$$A(u) := \int_U \sqrt{1 + |Du(x)|^2} \, dx$$

denote the $n$-dimensional area of the graph of $u$, and let $m = \inf_{w \in W} A(w)$ the minimum area of all graphs $w \in W$ satisfying the boundary conditions.

(a) Find a second order nonlinear partial differential equation satisfied by any area minimizing graph $u \in W$ such that $A(u) = m$. (HINT: Use the calculus of variations. An equation derived by finding critical points of a functional is called an Euler-Lagrange equation.) This particular PDE is also called the minimal surface equation; it represents the equilibrium shape of a soap film whose boundary lies on a wire given by the graph of the function $g \in C(\partial U)$.

(b) Show $A : W \rightarrow \mathbb{R}$ is strictly convex, meaning $w_0, w_1 \in W$ and $s \in ]0, 1[ \implies A((1 - s)w_0 + sw_1) < (1 - s)A(w_0) + sA(w_1)$ unless $w_0 = w_1$.

(c) Prove at most one function $u \in W$ satisfies the minimal surface equation. (HINT: First prove that the derivative of a convex function $a : \mathbb{R} \rightarrow \mathbb{R}$ vanishes only at the minimum of $a$.)