MAT 1060H1F
Assignment 2

Prof. McCann — Due: noon on Thursday Sept. ~~24~~ 23

Read Evans Appendix E. We have covered Chapter 1, and 2.1–2.2. If you like to read ahead I expect to get through Chapter 2.3 next week and begin 2.4 the week after.

To be handed in: Evans # 1.4, 2.5, and 2.6.

The attached exercises #1-3 on pp 116–117 of Adams and Guillemin "Measure Theory and Probability". This may require some additional reading for those of you who lack background in measure and probability theory or have not have encountered the laws of large numbers. Adams & Guillemin's book is a good place to learn.



## §2.8 The Discrete Dirichlet Problem

Let $\mathcal{O}$ be an open set in $\mathbf{R}^2$. A twice-differentiable function $f : \mathcal{O} \to \mathbf{R}$ is called *harmonic* on $\mathcal{O}$ if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad \text{on } \mathcal{O}$$

Now let $\Omega$ be a compact subset of $\mathbf{R}^2$ with a continuous boundary $\partial\Omega$. Suppose that $g : \partial\Omega \to \mathbf{R}$ is continuous. The classical *Dirichlet problem* asks one to find $f : \Omega \to \mathbf{R}$ such that $f$ is harmonic on $\operatorname{Int}\Omega$ and $f = g$ on $\partial\Omega$.

Many solutions to this problem have been discovered, some of which are quite ingenious. In particular, in *Two dimensional Brownian motion and harmonic functions* (Tokyo: Proc. Imp. Acad., 20, 706–714 [1944]), S. Kakutani showed how to construct $f$ using probabilistic methods. He used a kind of limit of the random walk in $\mathbf{R}^2$ called the Wiener process or Brownian motion. Although the theory of the Wiener process is beyond the scope of this book, we can understand the ideas behind Kakutani's construction by looking at a discrete version of the Dirichlet problem due to Courant (Courant, R., Friedrichs, K. O., and Lewy, H. *Ueber die partiellen Differenzengleichungen der mathematischen Physik.* Math. Ann. Vol. 100. pp. 32–74 [1928]). (In fact, Courant showed that the solution to the classical problem can be obtained as a limiting case of the solution of the discrete problem described below!)

Before we describe this discrete version of the Dirichlet problem, we need to translate the definition of harmonic functions into a form that is easily dealt with measure theoretically.

**Theorem.** (Mean value property) Let $\mathcal{O} \subset \mathbf{R}^2$ be open and let $f : \mathcal{O} \to \mathbf{R}$ be harmonic. Let $x_0 \in \mathcal{O}$ and assume that the circle of radius $a$ around $x_0$ lies entirely in $\mathcal{O}$. Then

$$f(x_0) = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} f(x_0 + ae^{i\theta})\, d\theta \tag{1}$$

Conversely, if $f : \mathcal{O} \to \mathbf{R}$ is continuous and equation 1 holds for all $x_0$ and $a$ such that the circle of radius $a$ around $x_0$ lies entirely in $\mathcal{O}$, then $f$ is twice-differentiable and harmonic in $\mathcal{O}$.

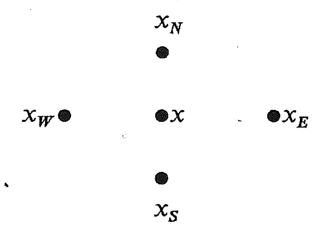
For a proof of this theorem see, for example, L. Ahlfors, *Complex Analysis* (New York: McGraw-Hill [1953]).

Using this characterization of harmonic functions, we can formulate a plausible discrete analogue of the Dirichlet problem. The space $\mathbf{R}^2$ is replaced by the integer lattice

$$\mathbf{Z}^2 = \{(m, n);\ m, n \text{ are integers}\}$$

and the compact region $\Omega$ becomes a finite subset of $\mathbf{Z}^2$.

For $x \in \mathbf{Z}^2$, there are four nearest neighbors, $x_N, x_S, x_E$, and $x_W$, as pictured below.




If $x \in \Omega$ we say $x \in \operatorname{Int}\Omega$ if $x_N, x_S, x_E$, and $x_W$ are all in $\Omega$ as well. We then define $\partial\Omega = \Omega - \operatorname{Int}\Omega$.

To define harmonic functions on $\operatorname{Int}\Omega$, the integral in equation 1 is translated to be the average over the nearest neighbors. Namely, if $f : \Omega \to \mathbf{R}$ we say $f$ is *harmonic* on $\operatorname{Int}\Omega$ if

$$f(x) = \frac{1}{4}[f(x_N) + f(x_S) + f(x_E) + f(x_W)]$$

for all $x \in \operatorname{Int}\Omega$.

Now let's consider the following problem.

## Discrete Dirichlet Problem

Given $g : \partial\Omega \to \mathbf{R}$ find $f : \Omega \to \mathbf{R}$ such that $f$ is harmonic on $\operatorname{Int}\Omega$ and $f = g$ on $\partial\Omega$.

We ask you to solve this problem by yourself. The following three exercises should be of some help.

1. Let $\mathscr{R}_{x_0}$ denote the set of all random walks on $\mathbf{Z}^2$ with $x_0$ as the starting point. This set can be identified with the set of all sequences of $N$'s, $E$'s, $S$'s, and $W$'s (for example, N W W E S N . . .). Assign to $N$, $E$, $S$, and $W$ the numerical values 0, 1, 2, and 3. Let $I = (0, 1]$ = the half-closed unit interval. If $\omega \in I$, the quaternary expansion of $\omega$ gives rise to a sequence of 0's, 1's, 2's and 3's and hence to a sequence such as that above. Therefore, we can identify $I$ with $\mathscr{R}_{x_0}$. (For the details of this identification, see §1.2.) Now suppose $x_0 \in \Omega$. Consider the random walk $r_\omega \in \mathscr{R}_{x_0}$ indexed by $\omega \in I$. Two possibilities exist: Either $r_\omega$ stays inside $\operatorname{Int}\Omega$ forever, or it eventually gets to a boundary point $x_b(\omega)$. (For instance, if $x_0 \in \partial\Omega$, then $x_b(\omega) = x_0$.)
   a. Show that the first of these two possibilities occurs with probability zero. (See §1.4, exercise 17.)
   b. Let $f_{x_0}(\omega) = g[x_b(\omega)]$. Show that $f_{x_0}$ is a measurable function of $\omega \in I$.



2. Let $\mathscr{R}^N_{x_0}$ be the set of all random walks starting at $x_0$ that move directly to $x_N$ on the first step. Define $\mathscr{R}^E_{x_0}$, $\mathscr{R}^S_{x_0}$, and $\mathscr{R}^W_{x_0}$ similarly.
   a. Show that $\mathscr{R}_{x_0} = \mathscr{R}^N_{x_0} \cup \mathscr{R}^E_{x_0} \cup \mathscr{R}^S_{x_0} \cup \mathscr{R}^W_{x_0}$ (disjoint union) and show that, under the correspondence $\mathscr{R}_{x_0} \sim I$, $\mathscr{R}^N_{x_0}$ corresponds to the interval $(0, \frac{1}{4}]$, $\mathscr{R}^E_{x_0}$ to the interval $(\frac{1}{4}, \frac{1}{2}]$, and so on.
   b. There is an obvious bijective map $\rho : \mathscr{R}^N_{x_0} \to \mathscr{R}_{x_N}$. Namely, take the random walk whose first position after $x_0$ is $x_N$ and think of it as a random walk starting at $x_N$. Show that, if we identify $\mathscr{R}_{x_N}$ with $(0, 1]$ as in exercise 1 and identify $\mathscr{R}^N_{x_0}$ with $(0, \frac{1}{4}]$ as in part a above, the mapping $\rho$ becomes the mapping $\omega \to 4\omega$.
   c. Show that, with the identifications in parts a and b,

$$(*) \qquad f_{x_N}(\omega) = f_{x_0}\left(\frac{\omega}{4}\right)$$

   Obtain comparable identities for $f_{x_E}, f_{x_S}$, and $f_{x_W}$.
3. Define $f : \Omega \to \mathbf{R}$ by setting

$$f(x_0) = \int_I f_{x_0}(\omega)\, d\mu$$

   for all $x_0 \in \Omega$, with $\mu$ being Lebesgue measure. Prove that $f$ is harmonic and equal to $g$ on $\partial\Omega$.