SOAR Math Course Homework Two Solutions Winter, 2003

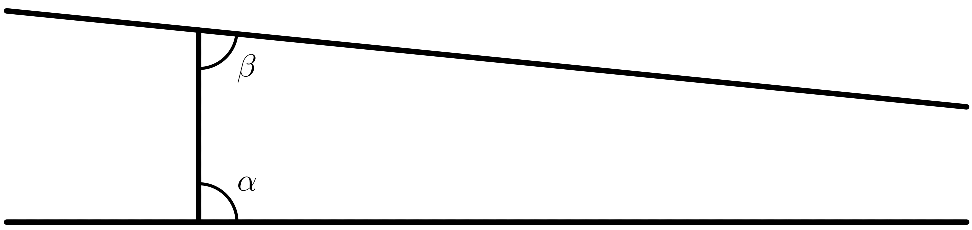
These solutions are written at the request of students. Please let me know if you don't understand these solutions; I'm happy to expand on them if necessary.

1. Prove that Euclid's Postulate 5 and Playfair's Postulate are equivalent. That is, assuming "absolute geometry" (Euclid's first four postulates), prove that...
   (a) Postulate 5 implies Playfair's Postulate, and
   (b) Playfair's Postulate implies Euclid's Postulate 5.

Before we start the proof, let me remind you of the statements of the two postulates:

> **Euclid's Postulate Five:** If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

A picture really helps to understand what Euclid is saying:

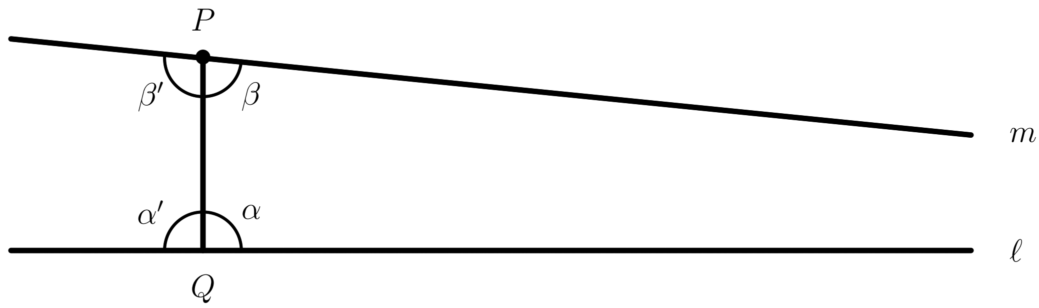


Postulate Five says that if $\alpha + \beta < 180°$, then these two lines will meet somewhere to the right.

On the other hand, Playfair's Postulate essentially says that, given a line ℓ and a point P (not on the line ℓ), there is a unique line through P parallel to ℓ:

> **Playfair's Postulate:** Through a given point not on a given line there can be drawn only one line parallel to the given line.

Proof of Part (a). We begin with part (a). That is, we assume Euclid's Postulate Five and prove Playfair's Postulate. Let's look at the same picture as before:



Begin by noticing that if $\alpha = \beta = 90°$, then these two lines are parallel (Euclid's Proposition I.27). If $\alpha + \beta < 180°$, then the lines ℓ and m meet to the right. If $\alpha + \beta > 180°$, then $\alpha' + \beta' < 180°$, and so the two lines meet to the left. Thus the line m through P is parallel to ℓ if and only if $\alpha + \beta = 180°$. □

Proof of Part (b). On the other hand, suppose Playfair's Postulate holds. This means that there is a unique line through P parallel to ℓ. But we know that if $\alpha = \beta'$, then ℓ and m are parallel (Postulate I.27), so this must be the case here. This means that $\alpha + \beta = 180°$. So if $\alpha + \beta < 180°$, then ℓ and m are *not* parallel. If ℓ and m intersect to the *left* at a point R, then the triangle ΔPQR has two of its angles α' and β', and $\alpha' + \beta' > 180°$. This contradicts Euclid's Postulate I.7 (the sum of two angles in a triangle is less than 180°), and so ℓ and m cannot meet to the left. Since they are not parallel, they must meet to the right, proving Euclid's Postulate Five. □