Complex Numbers

The introduction of complex numbers in the 16th century made it possible to solve the equation $x^2 + 1 = 0$. These notes[1] present one way of defining complex numbers.

1. The Complex Plane

A *complex number* z is given by a pair of real numbers x and y and is written in the form $z = x + iy$, where i satisfies $i^2 = -1$. The complex numbers may be represented as points in the plane (sometimes called the Argand diagram). The real number 1 is represented by the point $(1, 0)$, and the complex number i is represented by the point $(0, 1)$. The x-axis is called the "real axis", and the y-axis is called the "imaginary axis". For example, the complex numbers $3+4i$ and $3-4i$ are illustrated in Figure 1a.

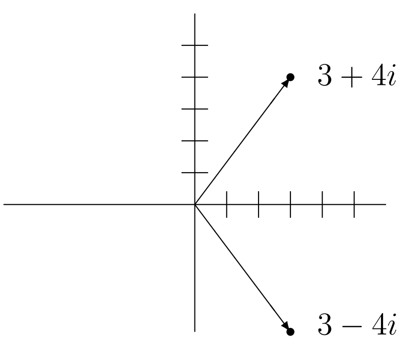

Figure 1a

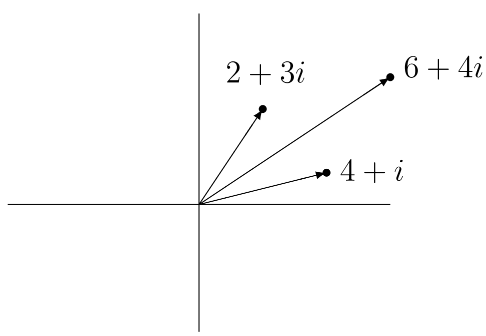

Figure 1b

Complex numbers are added in a natural way: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{1}$$

Figure 1b illustrates the addition $(4 + i) + (2 + 3i) = (6 + 4i)$. Multiplication is given by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2 + i)(-2 + 4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i$$

[1] These notes are based on notes written at the University of Washington by Bob Phelps, with modifications by Tom Duchamp. Further modifications were made by Peter Garfield.

We call x the *real part* of z and y the *imaginary part*, and we write $x = \text{Re}(z)$, $y = \text{Im}(z)$. (**Remember**: $\text{Im}(z)$ is a *real* number.) The term "imaginary" is an historical holdover; it took mathematicians some time to accept the fact that i (for "imaginary", naturally) was a perfectly good mathematical object. Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for $\sqrt{-1}$.

There is only one way we can have $z_1 = z_2$, namely, if $x_1 = x_2$ *and* $y_1 = y_2$. An equivalent statement (one that is important to keep in mind) is that $z = 0$ if and only if $\text{Re}(z) = 0$ and $\text{Im}(z) = 0$. If a is a real number and $z = x + iy$ is complex, then $az = ax + iay$ (which is exactly what we would get from the multiplication rule above if z_2 were of the form $z_2 = a + i0$). Division is more complicated (although we will show later that the *polar representation* of complex numbers makes it easy). To find z_1/z_2 it suffices to find $1/z_2$ and then multiply by z_1. The rule for finding the reciprocal of $z = x + iy$ is given by:

$$\frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

The expression $x - iy$ appears so often and is so useful that it is given a name. It is called the *complex conjugate* of $z = x + iy$ and a shorthand notation for it is $\overline{z}$; that is, if $z = x + iy$, then $\overline{z} = x - iy$. For example, $\overline{3+4i} = 3 - 4i$, as illustrated in the FIG 1A . Note that $\overline{\overline{z}} = z$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$. Another important quantity associated with a given complex number z is its *modulus*

$$|z| = (z\overline{z})^{1/2} = \sqrt{x^2+y^2} = \left((\text{Re}(z))^2 + (\text{Im}(z))^2\right)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$. There are obvious connections between these two notions: $z\overline{z} = |z|^2$ (this was already used in the denominator in equation (1) above) and $|\overline{z}| = |z|$. We can also write $|z|^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$ This leads to the inequality

$$\text{(2)} \qquad \text{Re}(z) \leq |\text{Re}(z)| = \sqrt{(\text{Re}(z))^2} \leq \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} = |z|$$

Similarly, $\text{Im}(z) \leq |\text{Im}(z)| \leq |z|$.

Exercises 1.

(1) Prove that the product of $z = x + iy$ and the expression in (1) (above) equals 1.

(2) Verify each of the following:

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$ (b) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5}$

(c) $\dfrac{5}{(1-i)(2-i)(3-i)} = \dfrac{1}{2}i$ (d) $(1-i)^4 = -4$

(3) Prove the following:

(a) $z + \overline{z} = 2\,\mathrm{Re}(z)$ and z is a real number if and only if $\overline{z} = z$.
(b) $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$.

(4) Prove that $|z_1 z_2| = |z_1| \cdot |z_2|$ (Hint: Use (3b).)
(5) Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

2. Polar Representation of Complex Numbers

Recall that the plane has polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates (x, y) and the polar coordinates (r, θ) is

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arctan(y/x)$$

(If $z = 0$, then $r = 0$ and θ can be anything.)

Thus, for the complex number $z = x + iy$, we can write

$$z = r(\cos\theta + i\sin\theta).$$

There is another way to rewrite this expression for z, called the *Euler Formula*. Later in life, you will see that e^x can be expressed as the following *power series* (that is, as an infinite sum of powers of x):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

For any complex number z, we *define* e^z by the power series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$$

In particular,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \end{aligned}$$

The functions $\cos(\theta)$ and $\sin(\theta)$ can also be written as power series:

$$\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + \frac{(-1)^n\theta^{2n}}{(2n)!} + \ldots$$
$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots + \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} \pm \ldots$$

Thus

(the power series for $e^{i\theta}$) = (the power series for $\cos(\theta)$)+$i\cdot$(the power series for $\sin(\theta)$)

This is the Euler Formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

(If you wish, you may simply choose to accept this formula and ignore the above diversion into infinite series.) For example,

$$e^{i\pi/2} = i, \qquad e^{\pi i} = -1 \qquad \text{and} \qquad e^{2\pi i} = +1$$

Given $z = x + iy$, then z can be written in the form $z = re^{i\theta}$, where

(3) $$r = \sqrt{x^2 + y^2} = |z| \qquad \text{and} \qquad \theta = \tan^{-1}(y/x)$$

For example the complex number $z = 8 + 6i$ may also be written as $10e^{i\theta}$, where $\theta = \arctan(.75) \approx .64$ radians. This is illustrated in Figure 2.

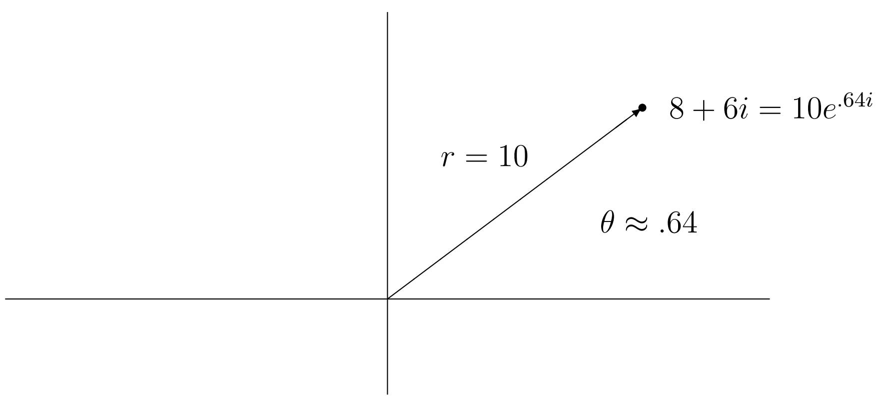

Figure 2

If $z = -4 + 4i$, then $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$, therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle which differs from $3\pi/4$ by an integer multiple of 2π will give us the same complex number. Thus, $-4+4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$. In general, if $z = re^{i\theta}$, then we also have $z = re^{i(\theta+2\pi k)}$, $k = 0, \pm 1, \pm 2, \ldots$. Moreover, there is ambiguity in equation (3) about the inverse tangent which can (and *must*) be resolved by looking at the signs of x and y, respectively, in order to determine the quadrant in which θ lies. If $x = 0$, then the formula for θ makes no sense, but $x = 0$ simply means that z lies on the imaginary axis and so θ must be $\pi/2$ or $3\pi/2$ (depending on whether y is positive or negative).

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k,\ k = 0, \pm 1, \pm 2, \ldots$. Despite this, the polar representation is very useful when it comes to multiplication:

(4) $$\text{if } z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}, \quad \text{then} \quad z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

To see why this is true, write $z_1z_2 = re^{i\theta}$, so that $r = |z_1z_2| = |z_1||z_2| = r_1r_2$ (the next-to-last equality uses Exercise (4a)). It remains to show that $\theta = \theta_1 + \theta_2$, that is, that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, (this is Exercise (7a)). For example, let

$$z_1 = 2 + i = \sqrt{5}e^{i\theta_1}, \qquad \theta_1 \approx 0.464$$

$$z_2 = -2 + 4i = \sqrt{20}e^{i\theta_2}, \qquad \theta_2 \approx 2.034$$

Then $z_3 = z_1 z_2$, where:

$$z_3 = -8 + 6i = \sqrt{100}e^{i\theta_3} \qquad \theta_3 \approx 2.498$$

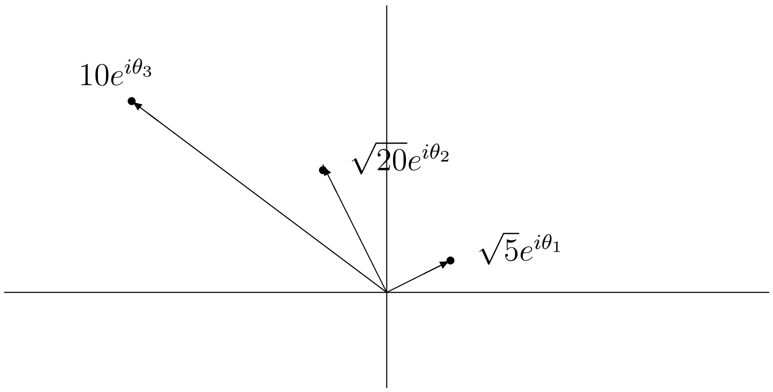

Figure 3

Applying (4) to $z_1 = z_2 = -4 + 4i = 4\sqrt{2}e^{\frac{3}{4}\pi i}$ (our earlier example), we get

$$(-4 + 4i)^2 = (4\sqrt{2}e^{\frac{3}{4}\pi i})^2 = 32e^{\frac{3}{2}\pi i} = -32i.$$

By an easy induction argument, the formula in (4) can be used to prove that for any positive integer n

$$\text{If} \qquad z = re^{i\theta}, \qquad \text{then} \qquad z^n = r^n e^{in\theta}$$

This makes it easy to solve equations like $z^3 = 1$. Indeed, writing the unknown number z as $re^{i\theta}$, we have $r^3 e^{i3\theta} = 1 \equiv e^{0i}$, hence $r^3 = 1$ (so $r = 1$) and $3\theta = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. It follows that $\theta = 2k\pi/3$, $k = 0, \pm 1, \pm 2, \ldots$. There are only three distinct complex numbers of the form $e^{2k\pi i/3}$, namely $e^0 = 1$, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The following figure illustrates $z = 8i = 8e^{i\pi/2}$ and its three cube roots $z_1 = 2e^{i\pi/6}$, $z_2 = 2e^{5i\pi/6}$, $z_3 = 2e^{9i\pi/6}$

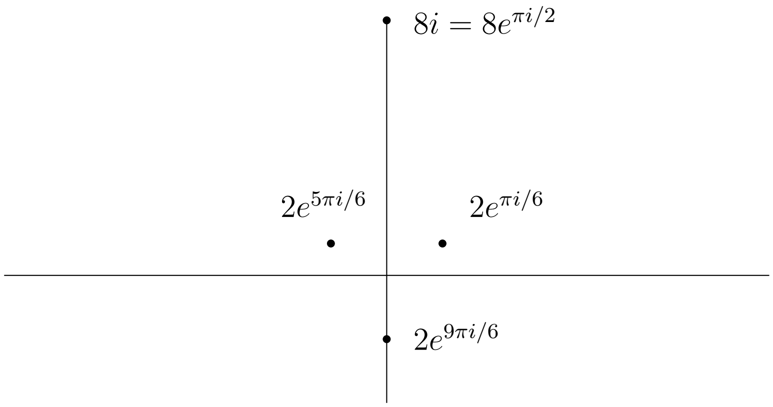

Figure 4

From the fact that $(e^{i\theta})^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

By expanding on the left and equating real and imaginary parts, this leads to trigonometric identities which can be used to express $\cos(n\theta)$ and $\sin(n\theta)$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example, taking $n = 2$ one gets $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$. For $n = 3$ one gets $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$.

Exercises 2.

(6) Let $z_1 = 3i$ and $z_2 = 2 - 2i$
   (a) Plot the points $z_1 + z_2$, $z_1 - z_2$ and $\overline{z_2}$.
   (b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
   (c) Express z_1 and z_2 in polar form.
(7) Prove the following:
   (a) $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.
   (b) Use (a) to show that $(e^{i\theta})^{-1} = e^{-i\theta}$, that is, $e^{-i\theta}e^{i\theta} = 1$.
(8) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1, z_2, z_1z_2 and z_1/z_2.
(9) Find all complex numbers z which satisfy $z^3 = -1$.
(10) Find all complex numbers $z = re^{i\theta}$ such that $z^2 = \sqrt{2}e^{i\pi/4}$.
(11) Find expressions for each of the following in terms of $\sin(\theta)$ and $\cos(\theta)$, using the technique outlined above.
   (a) $\sin(3\theta)$
   (b) $\cos(4\theta)$