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The Mathematics of Risk Management
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The lecture notes from this presentation are not yet available. Below are the questions used to introduce the topic. The full notes will be posted when available.

Shall we invest our money in the Stock Market, or shall we buy Canada Savings bonds? Shall we purchase full auto insurance, or just the minimum liability protection? Should vaccines for chicken pox be given to children in a systematic way? These are some of many frequently asked questions that share the same theme: shall we enter into a risky situation, or instead decide for a safe—if more expensive—alternative?

If the risky choice was made, another question arises: how should we prepare for eventualities? If we didn't purchase auto collision insurance, how much money should we reserve for accidents?

Both questions are part of the theory of *risk management*. In the first, we try to minimize risk. In the second, we assess it. And Mathematics lies at the very heart of all of this.

In order to scratch the surface of what is involved in this, consider the following example:

Example 1. Consider two stocks: one for a company called Fantastix, with present value at \$1. After one year, its value is expected to be \$3.0 with probability 40%, and \$1.00 with probability 60%. The other company, Splendix, has stock valued at \$1 today, and after one year, its value will be \$2.4 with probability 50%, and \$1.6 with probability also 50%. Now we ask: *which one of them has more risk*?

In order to answer the question, we first consider another related one. Which is more lucrative: Fantastix, or Splendix? This is easy: we just compute their *returns*:

For Fantastix: $r = \$3.0 \times 0.4 \; + \; \$1.0 \times 0.6 = \$1.8$
For Splendix: $r = \$2.4 \times 0.5 \; + \; \$1.6 \times 0.5 = \$2.0$

This settles it: Splendix is more lucrative than Fantastix.

But how about risk? This is harder. In fact, what *is* risk?

A partial answer is provided by the following: we just look at how much prices deviate from the expected return, as follows.

For Fantastix,

$$(3 - 1.8)^2 \times 0.4 \; + \; (1.0 - 1.8)^2 \times 0.6 = 0.96.$$

Note that we take the difference *squared* when we take deviations. This is funny, since it means we also consider the risk of *making too much money*. Never mind for now ...

For Splendix,

$$(2.4 - 2)^2 \times 0.5 \; + \; (1.6 - 2)^2 \times 0.5 = 0.16.$$

Conclusions are clear now: Splendix provides a higher return, with lower risk, and Fantastix provides a lower return with also higher risk. Any investor's choice would then be to buy Splendix, as much of it as desired, and avoid Fantastix ... right?

Wrong! Consider an investor that just wants to minimize risk, and is insensitive to returns. Assume that it has \$1 to invest, and decides to buy a units of Fantastic stock, and b units of Splendix, which must total \$1:

$$a + b = 1.$$

In understanding how its investment evolves, it is now crucial to know how each of the two stocks will move with respect to each other. We assume, for simplicity, that they are independent. This means that if one of them goes up, that has no effect on the other going up or down. That's not case for instance, of stocks which are similar which, due to economic reasons, will go up or down in a synchronized way. Mathematically, this means that the risk of the portfolio under consideration has a risk equal to

$$a^2 \times 0.96 + b^2 \times 0.16$$

We now note that if one invests exactly $a = 0.1429$ units of Fantastix, and $b = 0.8571$ units of Splendix, the total risk is approximately 0.136, which is lower than even the risk of investing in Splendix alone.

Question 1: Explain what is going on.

Question 2: Find the values of a and b above that minimize the total risk of the investment.

Question 3: Remove the independence condition above and create the optimal portfolio in that case.