The Fields Institute for Research in Mathematical Sciences

SIMMER

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Linear Programming: The Cornerstone of Optimization

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In 1949 George B. Dantzig published the "simplex" method for solving linear programming problems. He is also usually credited with recognizing the importance of the general linear programming problem and identifying its early applications. Simply stated, the problem is one of optimizing a linear objective function on a domain defined by linear constraints.

In an example of three or four variables, the linear programming problem does not reveal the depth of some of the concepts needed to solve it, nor does it foretell the enormous importance the problem has achieved in industry. Its significance becomes more apparent with the realization that the simplex method is capable of solving optimization problems with hundreds of thousands of variables and tens of thousands of constraints.

Stories of improvements in efficiency and cost saving abound in industry. When Standard Oil introduced linear programming in its refinery control systems in the early sixties, the savings amounted to millions of dollars each month. More recently, in 1992 American Airlines designed a system of rare structures, overbooking and flight coordination which uses linear programming and other techniques of operations research. The annual increase in revenue attributed to the system exceeds $500 million.

Since linear programming is one of the few optimization techniques that can routinely solve problems with an enormous number of variables, it is not surprising that considerable effort goes into translating problems which do not at first look like LP-s into LP form. This translation in many cases requires considerable mathematical insight.

These notes present some examples ranging from the obvious to the obscure to illustrate my point. A solution is given for the hardest and most important problem.

Problems

Question 1. A craftsman can make an ordinary widget in 15 minutes using 10 m of wire and sell it at a profit of $10. A deluxe widget takes 20 minutes to make, uses 5 m of wire and brings a profit of $15. The supply of wire is limited to 200 m per day and 8 hours are available for work.

How many widgets of each kind should the craftsman make each day to maximize profit?

[1] Edited for **SIMMER** by Philip Spencer.

Question 2. A distributor keeps 15 cans of paint in Oshawa and 25 cans in Oakville. Orders come in for 23 cans to Toronto and 17 cans to Hamilton. Shipping costs per can are given in the table below:

	Toronto	Hamilton
Oshawa	\$1	\$3
Oakville	\$2	\$1

Find the cheapest shipping schedule.

Question 3. An oil company requires a fleet of tankers for shipping crude and product between six ports. Four routes must be served daily as follows:

Route#	From	To	No. ships needed per day
1	Dhahran	New York	3
2	Marseilles	Istanbul	2
3	Naples	Bombay	1
4	New York	Marseilles	1

Assume all tankers are identical. The travel time table between ports is shown below. Assume travel time is independent of direction.

	Na.	Ma.	Is.	NY.	Dh.	Bo.
Naples		1	2	14	7	7
Marseilles			3	13	8	8
Istanbul				15	5	5
New York					17	20
Dhahran						3

One day must be added to each end of any route where cargo is carried for loading and unloading. You may find that the solution requires some routes to be travelled empty.

What is the smallest fleet that will do the job? Mark on a sketch where the ships should be located at a typical instant of time.

Question 4. Boatco must determine how many how many boats to produce during each of the coming four quarters (3 month periods). The demand is forecast as 40, 60, 75, and 25 during the next 1st to 4th quarters resp. The number to be produced is decided at the beginning of each quarter. Up to 40 boats can be produced in any quarter without overtime at a cost of \$400 per boat. Beyond 40 boats overtime costs drive up the cost to \$450 per boat. Any boats unsold at the end of a quarter incur a storage and handling cost of \$20 each. Devise a production schedule to minimize the total cost of meeting the demand.

Question 5. Machinco has four machines and four jobs to be completed. Each machine must be assigned to complete one job and each assignment incurs a set-up time. The set-up times are shown below. Find the assignment which minimizes total setup time.

	Job 1	Job 2	Job 3	Job 4
Machine 1	14	15	8	7
Machine 2	2	12	6	5
Machine 3	7	8	3	9
Machine 4	2	4	6	10

Question 6. A company is taking bids on four construction jobs. Three people have placed bids on the jobs as shown in the following table ("*" indicates that the person did not bid on that job).

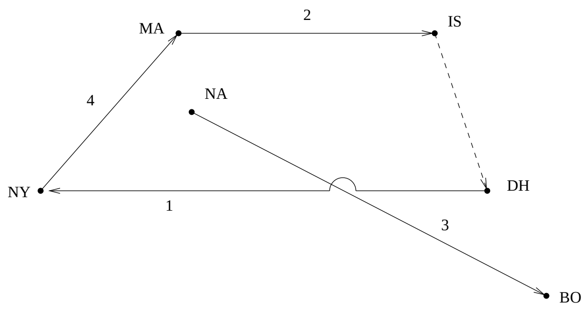
	Job 1	Job 2	Job 3	Job 4
Person 1	50	46	42	40
Person 2	51	48	44	*
Person 3	*	47	45	45

Person 1 can do only one job but persons 2 and 3 can each do up to three jobs. Determine the minimum cost assignment of persons to jobs. [Winston 379–4]

Question 7. Woodco gets an order for 25 3-ft, 20 5-ft, and 15 9-ft pieces of lumber. It fills the order by cutting up stock 8-ft and 16-ft pieces of which it has a vast supply. How should the cutting be done to minimize waste.

Solution to Question 3 (the hard problem)

A rough map will help focus our thinking:



Observe that routes have been numbered according to the statement of the problem.

Also observe that in order to make shipping a continuous steady state process, to have one ship per day arriving (or leaving) a given port, there will have to be a queue of ships spaced one day apart along the adjoining route. Furthermore, once a ship has finished a cargo route, it may have to travel empty to another port in order to begin another route.

For example, in order to have 1 ship per day shifting from route 2 to route 1, there will need to be a queue of 10 ships spaced out along the itinerary MA $\rightarrow$ IS $\rightarrow$DH. This is because it takes a ship 10 days from the time it begins route 2 until the time it is ready to begin route 1: 1 day loading cargo at Marseilles, 3 days travelling along route 2 to Istanbul, 1 day unloading at Istanbul, and 5 days travelling empty from Istanbul to Dhahran in preparation for beginning route 1.

In general, in order to have 1 ship per day shifting from route i to route j, there must always be a queue of c_{ij} ships involved in servicing route i then travelling to begin route j, where c_{ij} is 2 plus the time taken to travel route i plus the time taken to travel from the endpoint of route i to the starting point of route j.

The values of c_{ij} are tabulated below:

	1	2	3	4
1	36	32	33	19
2	10	8	7	20
3	12	17	16	29
4	23	15	16	28

If we let x_{ij} be the number of ships shifting from route i to route j daily, then the total number of ships servicing route i at any one time (where "servicing route i" includes getting ready for the next route after finishing route i) is

$$\sum_{j=1}^{4} x_{ij} c_{ij}.$$

Our goal is to choose the x_{ij} so as to minimize the total number of ships, which is

$$\sum_{i,j=1}^{4} x_{ij} c_{ij}.$$

The only constraints are that the daily number of ships completing a route must equal the daily number of ships beginning it, and this must equal the number specified in the problem (3 for route 1, 2 for route 2, and 1 for routes 3 and 4).

Under our definition for x_{ij}, $\sum_{j=1}^{4} x_{ij}$ yields the daily number of ships completing route i. This gives constraints $\sum_j x_{1j} = 3$, $\sum_j x_{2j} = 2$, $\sum_j x_{3j} = 1$, $\sum_j x_{4j} = 1$.

Similarly $\sum_{i=1}^{4} x_{ij}$ yields the daily number of ships beginning route i. This gives constraints $\sum_i x_{i1} = 3$, $\sum_i x_{i2} = 2$, $\sum_i x_{i3} = 1$, $\sum_i x_{i4} = 1$.

Eureka! a transportation problem of modest size. Its solution is

$$x_{ij} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Which we interpret as follows:

Of the three ships servicing route 1 daily, one goes back to begin route 1 again, one goes on to route 1, and one goes on to route 4. Referring back to the table for C_{ij}, this means that at any given time

- 36 ships are occupied travelling route 1 (DH $\rightarrow$ NY) then returning empty in preperation for another route 1 trip.
- 32 ships are occupied travelling route 1 then going empty to MA in preparation for a route 2 (MA $\rightarrow$ IS) cargo journey;
- 19 ships are occupied travelling route 1 then reloading in preparation for a loaded journey to MA (route 4).

Similarly, we find that, at any given time,

- 10 ships are occupied travelling route 2 (MA $\rightarrow$ IS) then going empty to DH in preparation for a route 1 (DH $\rightarrow$ NY) cargo journey;
- 7 ships are occupied travelling route 2 then going empty to NA in preparation for a route 3 (NA $\rightarrow$ BO) cargo journey;
- 12 ships are occupied travelling route 3 (NA $\rightarrow$ BO) then going empty to DH in preparation for a route 1 (DH $\rightarrow$ NY) cargo journey;
- 15 ships are occupied travelling route 4 (NY $\rightarrow$ MA) then reloading in in preparation for a route 2 (MA $\rightarrow$ IS) cargo journey.

Thus, 131 ships are needed in all.