The Fields Institute for Research in Mathematical Sciences

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November 1997 Presentation Topic
Wave Propagation and the Formation of Caustics

Antônio Sá Barreto

Department of Mathematics
Purdue University
West Lafayette, IN 47907,
U.S.A

sabarre@math.purdue.edu

1 Brief Description

In this lecture we will discuss one example of a mathematical problem which arises in the theory of geometric optics. When light passes through, or is reflected by, a smooth surface, the image formed is in general a surface with singularities, that is, kinks or corners. These singularities are called caustics. There is a classical example of a cusp formed in a tea cup, which we hope we can demonstrate. We will state the general mathematical question, but we will concentrate our discussion on this particular example.

2 History of the Problem

The study of propagation of light in different media has been a major branch of physics and many people have contributed to it. In this lecture we will study one mathematical question arising from the so-called Huygens' principle. While Huygens was motivated by propagation of light, his principle holds for more general types of wave propagation.

The particular question we are interested is the following: Given a wavefront, or the shape of a wave, at a certain time T, how do we predict the wavefront at later times $t > T$? Huygens' principle states that every point on any wavefront may be considered as a new source of waves, and all the points on any wavefront may be used alternatively with the source to predict any later wavefront. For simplicity we will only consider wave propagation in two dimensions. In this case we can find the wavefront at a later time by drawing a circle of radius $t - T$ centered at each point of the wavefront at time T. The new fronts are the curves which are tangent to the all these circles.

The simplest application of Huygens' principle is when the initial front is just a point. In this case a unique circular wave is formed from that point. This can be seen, for example, when a stone is dropped on a water surface; see figure 1. In this case only one wavefront is formed. The second easiest example is the wavefront corresponding to a straight line. We can imagine the waves that are formed when a rod is dropped on a water surface. Two linear wavefronts are going to form, propagating away and in opposite directions, from the rod. This behavior is illustrated in figure 2. We also illustrate the wavefront evolution in other cases in figures 3 and 4. These pictures correspond to evolutions for short times, i.e., when $t-T$ is small. As we can imagine, the behavior of these wavefronts for large times, except for the first and second ones, can be rather complicated. Our goal is to discuss the behavior of the fronts in figure 3 for large times.

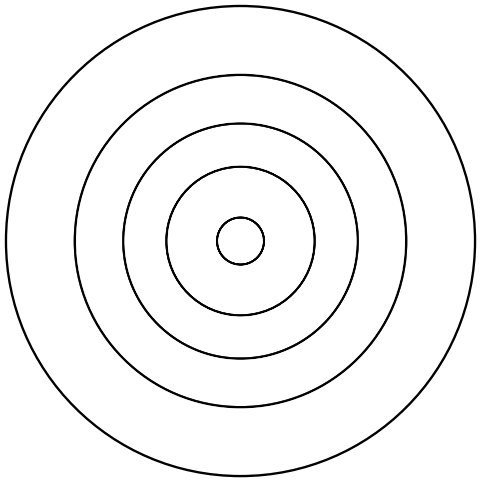

Figure 1: wavefront evolution of a point

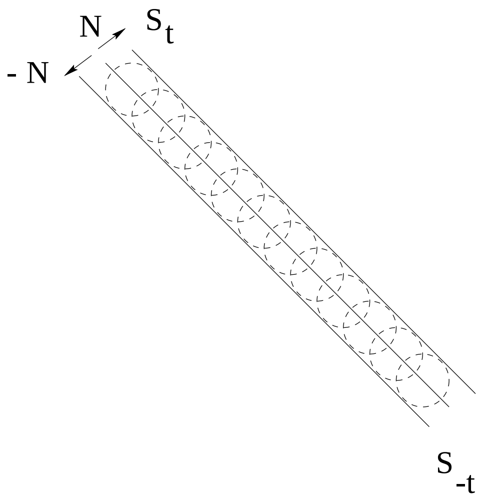

Figure 2: wavefront evolution of a line

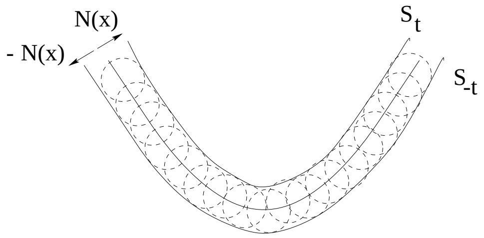

Figure 3: Wavefront evolution of a parabola

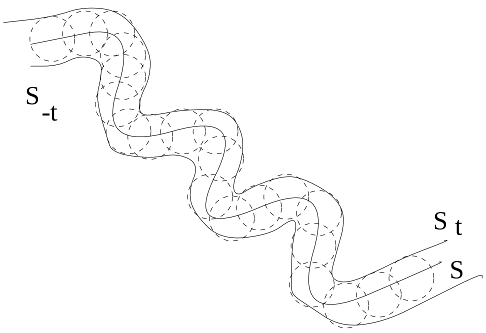

Figure 4: Wavefront evolution of a more complicated curve

3 Problems

Question 1

Let $S \subset \mathbf{R}^2$ be a smooth curve. Assuming uniqueness of the fronts, show that the wave fronts $S_{\pm t}$ defined by S at time $t > 0$ are obtained in the following way: For $(x, y) \in S$, let $\pm N(x, y)$ be the unit normal vectors to S at the point (x, y). Then

$$S_{\pm t} = \{(x, y) \pm tN(x, y) : (x, y) \in S\}.$$

Question 2

Let $S \subset \mathbf{R}^2$ be a straight line. Find $S_{\pm t}$ for $t > 0$.

Question 3

Let S be the parabola defined by $S = \{(x, y) \in \mathbf{R}^2 : y = x^2\}$. Sketch the graphs of $S_{\pm t}$ for $t > 0$. Figure 3 shows $S_{\pm t}$ for small values of t.

Question 4

Let S be the parabola defined in problem 3. Prove that the front that corresponds to the unit normal with negative y-coordinate is a smooth curve in $\mathbf{R}^2$ for all times. Also prove that the front corresponding to the normal vector with positive y-coordinate is smooth for t small, but develops singularities for t large. Find the first value of t for which S_t has a singularity.

Question 5

Find the set of points on the plane $\mathbf{R}^2$ which is the union of the sets of singular points of $S_{\pm t}$, $t > 0$.

Question 6

After this discussion do you know why the cusp formed in the tea cup is brighter?

4 Some Tools to Solve the Problems

Recall that for a curve $S = \{(x, y) \in \mathbf{R}^2 : y = f(x)\}$, the vectors

$$T(x) = \frac{1}{[1 + (f'(x))^2]^{\frac{1}{2}}}(1, f'(x))$$

$$\text{and} \qquad N(x) = \frac{1}{[1 + (f'(x))^2]^{\frac{1}{2}}}(-f'(x), 1)$$

are, respectively, tangent and normal to S at the point $(x, f(x))$. See figure 5.

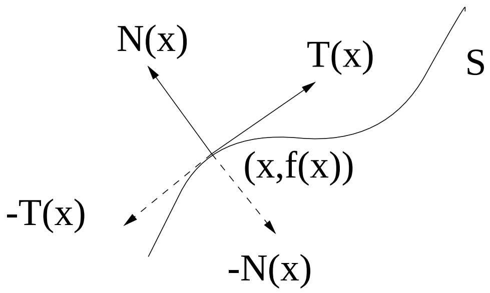

Figure 5: Tangent and normal vectors to a curve

A parametrization of a curve $S \subset \mathbf{R}^2$ is a map from an interval $I \subset \mathbf{R}$ onto $S \subset \mathbf{R}^2$ given by

$$\phi : I \longrightarrow S \subset \mathbf{R}^2$$
$$t \longmapsto \phi(t) = (\phi_1(t), \phi_2(t)).$$

The curve S is said to be smooth at a point $p \in S$ if $p = \phi(t_0)$, ϕ_1 and ϕ_2 are smooth at t_0 and the tangent vector $T(t_0) = (\phi_1'(t_0), \phi_2'(t_0))$ does not vanish at $t = t_0$, i.e.,

$$(\phi_1'(t_0))^2 + (\phi_2'(t_0))^2 \neq 0. \qquad (*)$$

When $(*)$ fails, the curve S has a singularity at p.

Example: The curve $\phi(t) = (t^3, t^2), t \in (-1, 1)$ has a singularity at $(0, 0)$. Indeed, the tangent vector is given by $T(t) = (3t^2, 2t)$, which vanishes at $t = 0$. If we set $x = t^3$ and $y = t^2$ we see that this curve is given by $\{(x, y) : y^3 = x^2\}$. This curve is called a *cubic parabola*, and this type of singularity a *cusp*. The graph of this curve can be seen in figure 6.

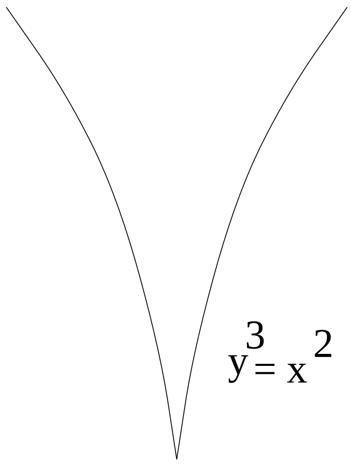

Figure 6: Curve with a cusp singularity.

5 Solutions to the problems

Question 1

Suppose that $S = \{(x, y) : y = f(x)\}$. Then, as above

$$N(x) = \frac{1}{[1 + (f'(x))^2]^{\frac{1}{2}}}(-f'(x), 1),$$
$$S_t = \{(x, f(x)) + t\frac{1}{[1 + (f'(x))^2]^{\frac{1}{2}}}(-f'(x), 1)\}.$$

Therefore the curve S_t is parametrized by $\phi(x) = (\phi_1(x), \phi_2(x))$, where

$$\phi_1(x) = x - t\frac{1}{[1 + (f'(x))^2]^{\frac{1}{2}}}f'(x),$$
$$\phi_2(x) = f(x) + t\frac{1}{[1 + (f'(x))^2]^{\frac{1}{2}}}.$$

The tangent vector to S_t at $\phi(x)$ is given by $T_t(x) = (\phi_1'(x), \phi_2'(x))$, with

$$\phi_1'(x) = 1 - tf''(x)\left(1 + (f'(x))^2\right)^{-\frac{3}{2}},$$
$$\phi_2'(x) = f'(x)\left[1 - tf''(x)\left(1 + (f'(x))^2\right)^{-\frac{3}{2}}\right].$$

Hence

$$T_t(x) = \left[1 - tf''(x)\left(1 + (f'(x))^2\right)^{-\frac{3}{2}}\right](1, f'(x)).$$

Thus we conclude that $T_t(x)$ is parallel to the tangent vector to S at $(x, f(x))$ and therefore $T_t(x)$ is tangent to the circle centered at $(x, f(x))$ with radius t at the point $(x, f(x)) + tN(x)$. See figure 7.

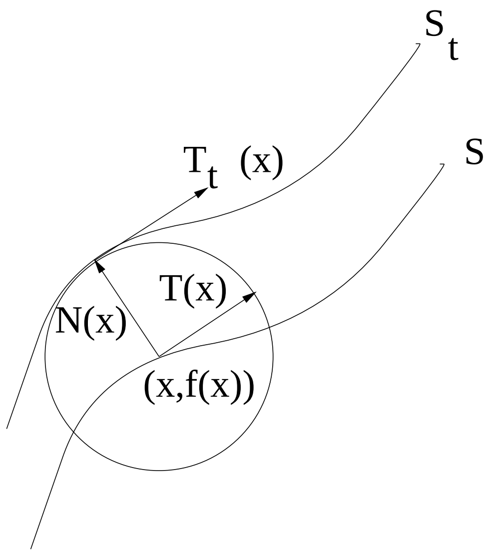

Figure 7: Tangent to S_t.

Question 2

Suppose $S = \{y = ax + b\}$. The unit normal vector is given by

$$N(x) = \frac{1}{(1+a^2)^{\frac{1}{2}}}(-a, 1).$$

Hence

$$S_t(x) = \{(x, ax+b) + t\frac{1}{(1+a^2)^{\frac{1}{2}}}(-a, 1)\}.$$

Let

$$X = x - \frac{at}{(1+a^2)^{\frac{1}{2}}}, \qquad Y = ax + b + \frac{t}{(1+a^2)^{\frac{1}{2}}}.$$

Then

$$Y - aX = b + t(1+a^2)^{\frac{1}{2}}.$$

Question 3

See figure 8.

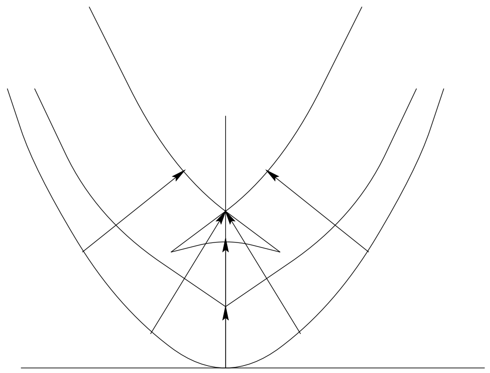

Figure 8: Evolution of a parabola along its normal.

Question 4

In this case S_t is parametrized by $\phi(x) = (\phi_1(x), \phi_2(x))$, where

$$\phi_1(x) = x - 2tx(1+4x^2)^{-\frac{1}{2}} \quad \text{and} \quad \phi_2(x) = x^2 + t(1+4x^2)^{-\frac{1}{2}}.$$

Thus

$$\begin{aligned} \phi_1'(x) &= 1 - 2t(1+4x^2)^{-\frac{3}{2}}, \\ \phi_2'(x) &= 2x\left(1 - 2t(1+4x^2)^{-\frac{3}{2}}\right). \end{aligned}$$

Hence, $\phi_1'(x) = \phi_2'(x) = 0$ if and only if

$$t = (\frac{1}{2})(1+4x^2)^{\frac{3}{2}}.$$

Only when $t > 1/2$ does this have solutions, namely $x = \pm(1/2)\sqrt{(2t)^{2/3} - 1}$.
In particular, this shows that S_t is smooth for $t < 0$, which corresponds to the evolution of the front along $-N(x)$. For $t > 0$, S_t is smooth as long as $t < 1/2$. The first value of t for which S_t has a singularity is $t = 1/2$.

Question 5

For $t = \frac{1}{2}(1+4x^2)^{\frac{3}{2}}$, the points on the curve S_t corresponding to these values of x are

$$\phi_1(x) = -4x^3 = X, \quad \phi_2(x) = \frac{1}{2} + 3x^2 = Y.$$

These points satisfy the equation

$$\left(\frac{1}{3}(Y - \frac{1}{2})\right)^3 = \left(-\frac{1}{4}X\right)^2,$$

which is curve with a cusp singularity. See figure 9.

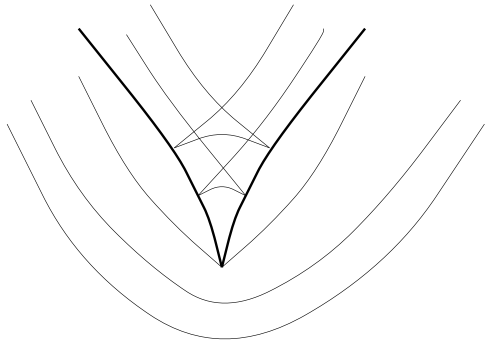

Figure 9: Union of the singular points of S_t.

The curves S_t are the pictures of the evolution of the front S with time. We can think of this as the intersections of a surface in $\mathbf{R}^3$ with the planes defined by the time t. The surface defined in this way has a singularity called a swallowtail, because of its similarity with the tail of the bird. See figure 10.

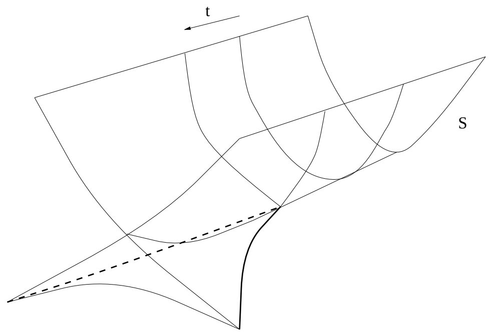

Figure 10: Surface defined by the flow of the wavefront S.

It is difficult result, which is outside the scope of this lecture, to show that, in two dimensions, the cusp and the swallowtail are the only stable caustics. Stable in the sense that by slightly moving the initial curve S, any other singularity decomposes into these two. In dimensions greater than two there are many other possible stable caustics. Formation of caustics in higher dimensions is, as we can imagine, a very complicated subject and many people have studied it. There are long lists of possible stable caustics. I will make no attempt of discussing this here. For the interested reader I would suggest the references [1] and [2] mentioned below.

I will end this lecture with the remark that a surface with a swallowtail singularity also appears as an algebraic object. Consider the polynomial of fourth degree in λ

$$p(\lambda) = \lambda^4 + z\lambda^2 + y\lambda + x,$$

with coefficients depending on $(x, y, z) \in \mathbf{R}^3$. Consider the set

$$\Sigma = \{(x, y, z) : p(\lambda) \text{ has a double real root}\}.$$

I leave as an exercise for the reader to obtain the *polynomial* equation of this surface: $\Sigma = S \setminus L$ where

$$S = \{(x, y, z) : 16xz^4 - 4y^2z^3 - 128x^2z^2 + 144xzy^2 + 256x^3 - 27y^4 = 0\}$$
$$L = \{(x, y, z) : y = 0, x = \frac{z^2}{4}, z > 0\}.$$

The set $\{(x, y, z) : y = 0, x = \frac{z^2}{4}, z > 0\}$ corresponds to the values of (x, y, z) for which $p(\lambda)$ has a double *complex* root. It also has a swallowtail singularity and looks very much like the one in figure 10. For a discussion of this see [2].

6 The Maple Program

Many of you were interested in seeing the code for the Maple program used in the demonstration. Here it is:

```
with(plots):
  animate( [x-2*0.1*t*x/sqrt(1+4*x^2), x^2 +0.1*t/sqrt(1+4*x^2),x=-1..1],
  t=0..10,frames=100, color=red);

with(plots):
  animate( [x-2*0.1*t*x/sqrt(1+4*x^2), x^2 +0.1*t/sqrt(1+4*x^2),x=0..1],
  t=0..10,frames=100,color=red);

with(plots):
  plot3d( [x-2*0.5*t*x/sqrt(1+4*x^2),2*t, x^2 +0.5*t/sqrt(1+4*x^2)],
  x=-10..10, t=0..15);

with(plots):
  animate( [x-0.12*t*cos(x)/sqrt(1+(cos(x))^2),
            sin(x) +0.12*t/sqrt(1+(cos(x))^2), x=-10..10],
  t=0..30,frames=100, color=red);
```

References

[1] V.I. Arnol'd. *Catastrophe Theory.* Springer-Verlag, Berlin, 1992, 150 pp. Third Edition.

[2] J. Britt. *The anatomy of low dimensional stable singularities.* Am. Math. Monthly 92 (1985) 183–201.