THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

SIMMER

The University of Toronto
Mathematics Network

March 1998 Presentation Topic
Hands-On Algebraic Topology
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This is a summary of what was presented and discussed at the March 26th **SIMMER** meeting, along with some problems and questions to think about.

Surfaces

We will be investigating objects called *surfaces*. Rather than giving a precise definition, we'll just give some examples. Note that in studying topology, any surface that can be bent, stretched, or otherwise deformed into a given surface X, and back again, is considered to be the same thing as X (or topologically equivalent, or *homeomorphic*).

Examples

1. *the cylinder.* This is the surface formed by taking a rectangular sheet and joining two opposite edges.
2. *the Möbius band.* This surface is created much like the cylinder, but the sheet is given a half-twist before joining the opposite edges.
3. *the sphere.* This is the surface of a ball.
4. *the torus.* The torus is a doughnut-shaped surface formed by taking a cylinder and joining the two circular ends together.

Stranger surfaces can be created which cannot be embedded in three-dimensional space; examples are the Klein bottle and the projective plane, which we'll see later. Also, two surfaces may be joined together to form their *connected sum.* For example, the connected sum of two tori is the two-holed torus.

How can we predict what will happen if we cut a surface apart, join two surfaces together, or perform any other such "surgery"? How can we tell when it is possible to transform one surface into another? One way of approaching such questions is through the use of *plane diagrams* to represent the surfaces.

Plane diagrams

We will give plane diagrams for the cylinder, the Möbius band, and the torus. Then two new surfaces, the Klein bottle and the projective plane, will be introduced via their diagrams.

As an example of the utility of plane diagrams, we use them to represent the slicing of a surface along a curve on the surface.

Problem 1: Using the plane diagram for the Möbius band, predict the outcome of cutting the surface down the middle (laterally). Verify experimentally. Then repeat the experiment, this time cutting the surface starting one-third of the way across.

Problem 2: Find a way to cut the Klein bottle to get (i) a Möbius band, and (ii) two Möbius bands.

We next show how to join two surfaces together, (i.e. perform connected sums) by way of their plane diagrams.

Problem 3: Find a plane diagram for the two-holed torus, then the three-holed torus. Try to find a pattern, and find a plane diagram for the general n-holed torus.

Problem 4: Show that the Klein bottle is the connected sum of two projective planes.

In fact, the last problem is just a consequence of the *classification theorem*, which states that any surface is either a sphere, or the connected sum of tori, or the connected sum of projective planes.

The Euler characteristic

On any surface, we can draw graphs, or networks. Edges meet at vertices, and divide the surface up into faces. For a graph on a surface, let V be the number of vertices, E be the number of edges, and F be the number of faces. It turns out that for a given surface, the quantity $V - E + F$ is constant, for *any* graph on the surface. This quantity is what is called an *invariant*, and has been dubbed the *Euler characteristic*, χ. It is important in the study of surfaces not only because it is an invariant (that is, if two surfaces are topologically equivalent, then their Euler characteristics are equal), but also because it is easily computed, in contrast to some other invariants.

We can calculate the Euler characteristic of a surface by looking at its plane diagram. For example, the characteristic of a sphere is 2, while that of a torus is 0.

Problem 5: Find the Euler characteristic for the projective plane, the Klein bottle, and the two-holed torus, the n-holed torus, and the connected sum of n projective planes.

Problem 6: Show that the two-holed torus and the connected sum of the one-holed torus and the projective plane are not topologically equivalent.

Problem 7: Compare the Euler characteristics of the torus to that of the Klein bottle. Are the two surfaces topologically equivalent? Why or why not?