SOAR Math Course   Homework Ten Solutions   Spring, 2003

These solutions are written at the request of students. Please let me know if you don't understand these solutions; I'm happy to expand on them if necessary.

6. Let F_k be the Fibonacci numbers and L_k be the Lucas numbers. That is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Similarly, $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

   To help with both solutions, I'm going to make a short table of values for L_n and F_n so that we can perhaps see the pattern in the table:

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
L_n	2	1	3	4	7	11	18	29	47	76	123

(a) Use induction to show that $L_n = F_{n+1} + F_{n-1}$ for $n \geq 1$.

Notice: I'm very sorry – there was a typo in this question. The correct question is posed above; the equation was originally stated as $L_n = (F_{n+1} + F_{n-1})/2$. This can be seen to be incorrect by looking at the table above.

Solution: Notice first that $L_1 = 1$ and $F_2 + F_0 = 1 + 0 = 1$, so the required formula holds for $n = 1$. It also holds for $n = 2$: $L_2 = 3$ and $F_3 + F_1 = 2 + 1 = 3$.

Now we assume that $L_k = F_{k+1} + F_{k-1}$ for all k with $1 \leq k \leq n$, and we prove that $L_{n+1} = F_{n+2} + F_n$. This is simply a computation:

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1} \\ &= (F_{n+1} + F_{n-1}) + (F_n + F_{n-2}) \qquad \text{by the inductive hypothesis} \\ &= (F_{n+1} + F_n + F_{n-1} + F_{n-2}) \\ &= F_{n+2} + F_n \qquad \text{since } F_{n+2} = F_{n+1} + F_n \text{ and } F_n = F_{n-1} + F_{n-2}. \end{aligned}$$

This is the desired equation. (Notice in the second line of this displayed set of equations, we needed to use the required expression for both L_n and L_{n-1}. This is why we established the formula for $n = 1$ and $n = 2$ at the outset.)

(b) Use induction to show that $F_n = (L_{n+1} + L_{n-1})/5$ for $n \geq 1$.

Solution: Again, we establish the desired formula for $n = 1$ and $n = 2$. When $n = 1$, $F_1 = 1$ and $(L_2 + L_0)/5 = (3 + 2)/5 = 1$ as well. Similarly, $F_2 = 1$ and $(L_3 + L_1)/5 = (4 + 1)/5 = 1$.

Now assume that $F_k = (L_{k+1} + L_{k-1})/5$ for all k with $1 \leq k \leq n$. We'll show that $F_{n+1} = (L_{n+2} + L_n)/5$ as well. This is a computation:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &= \frac{1}{5}(L_{n+1} + L_{n-1}) + \frac{1}{5}(L_n + L_{n-2}) \qquad \text{by the inductive hypothesis} \\ &= \frac{1}{5}(L_{n+1} + L_n + L_{n-1} + L_{n-2}) \\ &= \frac{1}{5}(L_{n+2} + L_n) \qquad \text{since } L_{n+2} = L_{n+1} + L_n \text{ and } L_n = L_{n-1} + L_{n-2}. \end{aligned}$$

This is the desired equation. (The same remarks in the solution to part (a) apply here as well.)