These solutions are written at the request of students. Please let me know if you don't understand these solutions; I'm happy to expand on them if necessary.

1. (d) Let $F(n)$ be the nth Fibonacci number. That is, $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$. Show that

$$\sum_{k=1}^{n} F(k)^2 = F(n)F(n+1). \tag{1}$$

We first prove that equation (1) holds for $n = 1$. This is simply showing that $\sum_{k=1}^{1} F(k)^2 = F(1)F(1+1)$, or $F(1)^2 = F(1)F(2)$. Since $F(1) = F(2) = 1$, this is clearly true.

Now we move on to the inductive step: we assume that equation (1) holds for n and prove that it holds for $n+1$. That is, we wish to prove that equation (1) implies that

$$\sum_{k=1}^{n+1} F(k)^2 = F(n+1)F(n+2). \tag{2}$$

The left-hand side of this equation simplifies to

$$\begin{aligned}
\sum_{k=1}^{n+1} F(k)^2 &= \sum_{k=1}^{n} F(k)^2 + F(n+1)^2 \\
&= F(n)F(n+1) + F(n+1)^2 && \text{by equation (1)} \\
&= [F(n) + F(n+1)]\, F(n+1) \\
&= F(n+2)F(n+1) && \text{since } F(n+2) = F(n+1) + F(n).
\end{aligned}$$

This is equation (2), which is what we wish to prove.

(e) Show that $2^{2^n} + 3^{2^n} + 5^{2^n}$ is divisible by 19 for every $n \geq 1$. (Hint: prove that if this is true for n then it is also true for $n + 2$. You will need two base cases here – one for odd n and the other for even n.)

We use the hint and establish two base cases. That is, we show that $2^{2^1} + 3^{2^1} + 5^{2^1}$ ($n = 1$) and $2^{2^2} + 3^{2^2} + 5^{2^2}$ ($n = 2$) are both divisble by 19. This is (again) simply computation: $2^{2^1} + 3^{2^1} + 5^{2^1} = 4 + 9 + 25 = 38$ and $2^{2^2} + 3^{2^2} + 5^{2^2} = 16 + 81 + 625 = 722$. Both these numbers are divisible by 19.

We know proceed with the inductive step. That is, we assume that 19 divides $\left(2^{2^n} + 3^{2^n} + 5^{2^n}\right)$ and try to prove that 19 also divides $\left(2^{2^{n+2}} + 3^{2^{n+2}} + 5^{2^{n+2}}\right)$.

Now we compute modulo 19, using the fact that $2^{2^n} + 3^{2^n} + 5^{2^n} \equiv 0 \bmod 19$:

$$\begin{aligned}
2^{2^{n+2}} + 3^{2^{n+2}} + 5^{2^{n+2}} &= 2^{4\cdot 2^n} + 3^{4\cdot 2^n} + 5^{4\cdot 2^n} \\
&= \left(2^4\right)^{2^n} + \left(3^4\right)^{2^n} + \left(5^4\right)^{2^n} \\
&= 16^{2^n} + 81^{2^n} + 625^{2^n} \\
&\equiv (16-19)^{2^n} + (81 - 19\cdot 4)^{2^n} + (625 - 33\cdot 19)^{2^n} \bmod 19 \\
&= (-3)^{2^n} + (5)^{2^n} + (-4)^{2^n} \\
&= 3^{2^n} + 5^{2^n} + 4^{2^n} \\
&\equiv 0 \bmod 19.
\end{aligned}$$

We've also used the fact that $(-3)^{2^n} = 3^{2^n}$ (and similarly for $(-4)^{2^n} = 4^{2^n}$).

This problem is originally from the web page `http://www.geocities.com/jespinos57/induction.htm`. This is a nice page of problems, all of which may be solved using induction. Check it out!