These solutions are written at the request of students. Please let me know if you don't understand these solutions; I'm happy to expand on them if necessary.

2. One of the skipped parts of a proof from today's class was the following claim: that x can be written as
$$x = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} \qquad (*)$$
where $x = [a_1, a_2, a_3, \ldots]$, x_n is defined by $x = [a_1, a_2, \ldots, a_{n-1}, x_n]$, and the nth convergent is $C_n = \frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n]$.

   (a) Prove that $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$ for $n = 1$, $n = 2$, and $n = 3$. By convention, (that is, so the formulas work) we assume $p_0 = 1$ and $q_0 = 0$.

   Proof. Let's begin with computing the convergents C_1, C_2, C_3, and C_4. (We'll need all of these for the computations through $n = 3$.) The equations are
$$C_1 = [a_1] = a_1 = \frac{a_1}{1},$$
so $p_1 = a_1$ and $q_1 = 1$. Slightly more involved,
$$\begin{aligned} C_2 = [a_1, a_2] &= a_1 + \frac{1}{a_2} \\ &= \frac{a_1 a_2 + 1}{a_2}, \end{aligned}$$
so $p_2 = a_1 a_2 + 1$ and $q_2 = a_2$. Next,
$$\begin{aligned} C_3 = [a_1, a_2, a_3] &= a_1 + \frac{1}{a_2 + \frac{1}{a_3}} \\ &= a_1 + \frac{1}{\frac{a_2 a_3 + 1}{a_3}} \\ &= a_1 + \frac{a_3}{a_2 a_3 + 1} \\ &= \frac{a_1 a_2 a_3 + a_1 + a_3}{a_2 a_3 + 1}, \end{aligned}$$

so $p_3 = a_1a_2a_3 + a_1 + a_3$ and $q_3 = a_2a_3 + 1$. Finally,

$$\begin{aligned} C_4 = [a_1, a_2, a_3, a_4] &= a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}} \\ &= a_1 + \frac{1}{a_2 + \frac{1}{\frac{a_3a_4+1}{a_4}}} \\ &= a_1 + \frac{1}{a_2 + \frac{a_4}{a_3a_4+1}} \\ &= a_1 + \frac{1}{\frac{a_2a_3a_4+a_2+a_4}{a_3a_4+1}} \\ &= a_1 + \frac{a_3a_4+1}{a_2a_3a_4+a_2+a_4} \\ &= \frac{a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1}{a_2a_3a_4 + a_2 + a_4}, \end{aligned}$$

so

$$\begin{aligned} p_4 &= a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1 \\ q_4 &= a_2a_3a_4 + a_2 + a_4. \end{aligned}$$

Now we verify that

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1} \\ q_{n+1} &= a_{n+1}q_n + q_{n-1} \end{aligned}$$

for $n = 1$, 2, and 3. We begin with $n = 1$. Then

$$\begin{aligned} a_2p_1 + p_0 &= a_2a_1 + 1 = p_2, \\ a_2q_1 + q_0 &= a_21 + 0 = a_2 = q_2, \end{aligned}$$

which is as desired. For $n = 2$, we have

$$\begin{aligned} a_3p_2 + p_1 &= a_3\left(a_2a_1 + 1\right) + a_1 \\ &= a_3a_2a_1 + a_3 + a_1 \\ &= p_3, \end{aligned}$$

and

$$\begin{aligned} a_3q_2 + q_1 &= a_3\left(a_2\right) + 1 \\ &= a_3a_2 + 1 \\ &= q_3, \end{aligned}$$

again as desired. Finally, for $n = 3$,

$$\begin{aligned} a_4p_3 + p_2 &= a_4\left(a_3a_2a_1 + a_3 + a_1\right) + \left(a_2a_1 + 1\right) \\ &= a_4a_3a_2a_1 + a_4a_3 + a_4a_1 + a_2a_1 + 1 \\ &= p_4, \end{aligned}$$

and

$$
\begin{aligned}
a_4 q_3 + q_2 &= a_4 \left(a_3 a_2 + 1\right) + a_2 \\
&= a_4 a_3 a_2 + a_4 + a_2 \\
&= q_4.
\end{aligned}
$$

Thus we've verified that

$$
\begin{aligned}
p_{n+1} &= a_{n+1} p_n + p_{n-1} \\
q_{n+1} &= a_{n+1} q_n + q_{n-1}
\end{aligned}
$$

for $n = 1$, 2, and 3. □

(b) Prove equation $(*)$ for $n = 1$, $n = 2$, and $n = 3$.

Proof. We're trying to verify that

$$
x = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} \tag{$*$}
$$

where x_n is defined by $x = [a_1, a_2, \ldots, a_{n-1}, x_n]$. As with part (a), we begin with $n = 1$. In this case, x_1 is given by $x = [x_1] = x_1$ (so $x_1 = x$). Thus what we're trying to show is that

$$
x = \frac{x_1 p_0 + p_{-1}}{x_1 q_0 + q_{-1}} = \frac{x \cdot 1 + 0}{x \cdot 0 + 1},
$$

which is the case. (I guess the important point is that we assume that $p_{-1} = 0$ and $q_{-1} = 1$, which I didn't tell you. Sorry about that!)

For $n = 2$, we have x_2 is defined by $x = [a_1, x_2] = a_1 + \frac{1}{x_2}$, or $x_2 = \frac{1}{x - a_1}$. We're trying to verify that

$$
x = \frac{x_2 p_1 + p_0}{x_2 q_1 + q_0} = \frac{x_2 \cdot a_1 + 1}{x_2 \cdot 1 + 0} = \frac{a_1 x_2 + 1}{x_2}.
$$

We use our expresion for x_2 to simplify the right-hand side:

$$
\frac{a_1 x_2 + 1}{x_2} = \frac{a_1 \cdot \frac{1}{x - a_1} + 1}{\frac{1}{x - a_1}} = \left(\frac{a_1}{x - a_1} + 1\right)(x - a_1) = a_1 + x - a_1 = x,
$$

as desired.

Finally, x_3 is defined by

$$
x = [a_1, a_2, x_3] = a_1 + \frac{1}{a_2 + \frac{1}{x_3}},
$$

and so

$$
x - a_1 = \frac{1}{a_2 + \frac{1}{x_3}}.
$$

From this we have

$$\frac{1}{x-a_1} = a_2 + \frac{1}{x_3}$$

from which we get

$$\frac{1}{x_3} = \frac{1}{x-a_1} - a_2 = \frac{1-a_2x+a_1a_2}{x-a_1}$$

which implies

$$x_3 = \frac{x-a_1}{1-a_2x+a_1a_2}.$$

We're trying to verify

$$x = \frac{x_3p_2+p_1}{x_3q_2+q_1} = \frac{x_3(a_1a_2+1)+a_1}{x_3a_2+1}.$$

We use our value of x_3 and try to simplify the right-hand side:

$$\begin{aligned}
\frac{x_3(a_1a_2+1)+a_1}{x_3a_2+1} &= \frac{\frac{x-a_1}{1-a_2x+a_1a_2}(a_1a_2+1)+a_1}{\frac{x-a_1}{1-a_2x+a_1a_2}a_2+1} \\
&= \frac{\frac{(x-a_1)(a_1a_2+1)+a_1(1-a_2x+a_1a_2)}{1-a_2x+a_1a_2}}{\frac{a_2(x-a_1)+(1-a_2x+a_1a_2)}{1-a_2x+a_1a_2}} \\
&= \frac{(x-a_1)(a_1a_2+1)+a_1(1-a_2x+a_1a_2)}{1-a_2x+a_1a_2} \cdot \frac{1-a_2x+a_1a_2}{a_2(x-a_1)+(1-a_2x+a_1a_2)} \\
&= \frac{a_1a_2x+x-a_1^2a_2-a_1+a_1-a_1a_2x+a_1^2a_2}{1-a_2x+a_1a_2} \cdot \frac{1-a_2x+a_1a_2}{a_2x-a_1a_2+1-a_2x+a_1a_2} \\
&= \frac{x}{1-a_2x+a_1a_2} \cdot \frac{1-a_2x+a_1a_2}{1} \\
&= x,
\end{aligned}$$

as desired. (Yikes!)

□