SOAR Math Course — **Homework Three** — Spring, 2003

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. (a) Show that the continued fraction $x = [\overline{1, 2}]$ satisfies the quadratic equation $2x^2 - 2x - 1 = 0$. Find the value of x.
   (b) Show that the other root of this equation is $-1/[\overline{2, 1}]$.
   (c) Find the quadratic equation that the continued fraction $x = [\overline{a, b}]$ satisfies. Show that the other root of this equation is $-1/[\overline{b, a}]$.
   (d) Find the quadratic equation that the continued fraction $x = [\overline{a, b, c}]$ satisfies.
   (e) The other root of this equation is probably either $-1/[\overline{c, b, a}]$, $-1/[\overline{a, c, b}]$, or $-1/[\overline{b, a, c}]$. Can you tell which without solving the equation?

2. One of the skipped parts of a proof from today's class was the following claim: that x can be written as

$$x = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} \tag{$*$}$$

   where $x = [a_1, a_2, a_3, \ldots]$, x_n is defined by $x = [a_1, a_2, \ldots, a_{n-1}, x_n]$, and the nth convergent is $C_n = \frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n]$.

   (a) Prove that $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $a_{n+1} = a_{n+1}q_n + q_{n-1}$ for $n = 1$, $n = 2$, and $n = 3$. By convention, (that is, so the formulas work) we assume $p_0 = 1$ and $q_0 = 0$.
   (b) Prove equation ($*$) for $n = 1$, $n = 2$, and $n = 3$.

3. Another of the skipped parts is that equation ($*$) gives us a nice inequality for x:

$$\frac{p_{n-1}}{q_{n-1}} < x < \frac{p_{n-2}}{q_{n-2}} \tag{$**$}$$

   (provided n is even). We prove this by showing that $\frac{A}{B} < \frac{C}{D}$ implies that

$$\frac{A}{B} < \frac{xA + C}{xB + D} < \frac{C}{D} \tag{$\dagger$}$$

   and

$$\frac{A}{B} < \frac{A + xC}{B + xD} < \frac{C}{D}. \tag{$\ddagger$}$$

   (Here A, B, C, D, and x are any positive real numbers.)

   (a) For the first equation, prove that $A(xB + D) = AD + xAB < BC + xAB = B(xA + C)$.
   (b) Use the inequality from part (a) to prove that $A/B < \frac{xA+C}{xB+D}$.
   (c) Use a similar technique to prove the other inequalities involved in equations ($\dagger$) and ($\ddagger$)

(d) Use equations (†) and (‡) to prove equation (**).

4. Solve the following equations for x. That is, find a value (*not* all values) satisfying the equations. Fair warning: at least one has no solution!

   (a) $203x + 73 \equiv 0 \bmod 8$

   (b) $x^2 + 42x + 64 \equiv 0 \bmod 3$

   (c) $(x+2)^3 \equiv 1 \bmod 3$

   (d) $(x+2)^3 \equiv 2 \bmod 3$

   (e) $(x+2)^4 \equiv 2 \bmod 3$

5. In class we discussed a simple version of the Chinese Remainder Theorem, which says that if $\gcd(m, n) = 1$, then given two integers a and b there is a unique value of x with $0 \leq x < mn$ such that

$$x \equiv a \bmod m$$
$$x \equiv b \bmod n.$$

   (a) Solve for x if $m = 5$, $n = 6$, $a = 3$, and $b = 1$. That is, solve

$$x \equiv 3 \bmod 5$$
$$x \equiv 1 \bmod 6.$$

   (b) Solve for x if $m = 13$, $n = 11$, $a = 1$, and $b = 2$. That is, solve

$$x \equiv 1 \bmod 13$$
$$x \equiv 2 \bmod 11.$$

   (c) Can you write x (up to multiples of mn) in terms of a, b, m, and n?