These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. In class we proved that $\sqrt{2}$ is not rational. This problem provides another proof via a method called *Fermat's infinite descent.*

   (a) Suppose that a and b are positive integers with $a/b = \sqrt{2}$. Show that

   $$\frac{2b - a}{a - b} = \sqrt{2}$$

   as well.

   (b) Show that $b > a - b > 0$ and that $2b - a$ and $a - b$ are integers.

   (c) Deduce that if we can write $\sqrt{2}$ as a ratio of integers, we can always make the denominator a *smaller* integer. Conclude that we may repeat this process (the *descent*) until the denominator is 1, so $c/1 = \sqrt{2}$ for some integer c. This is a contradiction ($\sqrt{2}$ is *not* an integer), so our original assumption (that $\sqrt{2}$ is a fraction) must be false.

Recall that in class we defined a *simple continued fraction* as

$$[a_1, a_2, a_3, a_4, \ldots] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

where the a_i are integers and $a_i > 0$ for $i \geq 2$. (That is, all the a_i are integers and only a_1 may be negative or zero.) This fraction may continue forever, but if it terminates then we say it is a *finite* simple continued fraction.

2. Let $a = 119$ and $b = 37$.

   (a) Find the greatest common divisor $\gcd(a, b)$.

   (b) Find the continued fraction representation of a/b.

   (c) Repeat (a) and (b) with numbers of your own choosing. Make them as interesting as possible.

3. Find the ordinary fraction representations of

   (a) $[1, 2]$

   (b) $[1, 2, 3]$

   (c) $[1, 2, 3, 4]$

   (d) $[1, 2, 3, 4, 5]$

   (e) a finite simple continued fraction $[a_1, a_2, a_3, a_4, a_5]$ of your choosing.

4. We saw in class that $\sqrt{2} = (1,\overline{2})$ and $\sqrt{3} = (1,\overline{1,2})$.

   (a) Find the simple continued fraction for $\sqrt{5}$.

   (b) Find the simple continued fraction for $\sqrt{6}$.

   (c) Formulate a conjecture about $\sqrt{k}$, where k is not a perfect square (so $\sqrt{k}$ is not an integer).

5. Let $\phi = [1,1,1,1,1,\ldots] = [\overline{1}]$. Find an expression for ϕ that doesn't involve continued fractions.

6. In class we saw that, while any integer can be factored uniquely into prime factors, this is not the case in every possible situation. This problem presents another situation where unique factorization fails. (You may skip part (b) if you like and just assume that it is true.)

   (a) Let us write $\mathbf{Z}[\sqrt{-5}]$ for the set of numbers $a + b\sqrt{-5}$, where a and b are integers. We can multiply these numbers together: show that

   $$(a + b\sqrt{-5}) \cdot (c + d\sqrt{-5}) = (ac - 5bd) + (bc + ad)\sqrt{-5}.$$

   (Use the fact that $(\sqrt{-5})^2 = -5$.)

   We remark that $\mathbf{Z}$ is the usual notation for the integers, and $\sqrt{-5}$ is just shorthand for "a number about which all we know is that its square is -5." Thus $\mathbf{Z}[\sqrt{-5}]$ is notation that means the integers together with this $\sqrt{-5}$.

   (b) [Hard] Show that $1 + \sqrt{-5}$ is "prime" in the sense that if $(a + b\sqrt{-5}) \cdot (c + d\sqrt{-5}) = 1 + \sqrt{-5}$, then either $a + b\sqrt{-5} = \pm 1$ or $c + d\sqrt{-5} = \pm 1$.

   You have just shown that multiplying two elements of $\mathbf{Z}[\sqrt{-5}]$ produces another element of $\mathbf{Z}[\sqrt{-5}]$. We say that $\mathbf{Z}[\sqrt{-5}]$ is closed under multiplication.

   (c) Finally, show that $(1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}) = 2 \cdot 3$, so 6 may be factored into "primes" in two different ways. (You may assume that 2, 3, and $1 - \sqrt{-5}$ are "primes" here.)