Week 3: May 26$^{\text{th}}$ - June 1$^{\text{st}}$

Suggested Problems

Problems you may find instructive, or that I find interesting.

§2.4 #21, 24, 35, 41, 42, 45, 46, 52(C) & 56

(Only evaluate limits by the methods of Chapter 2.)

Limits toward ∞ Notes are on the website.

Answers for the following exercises are given below.

Compute the given limits, if they exist. Use only theorems or definitions discussed in lecture.

X1 $\lim_{x\to-\infty} \ln\left(1-\frac{1}{x}\right)$

X2 $\lim_{\theta\to\infty} \frac{\sin\theta}{\theta}$

X3 $\lim_{z\to\infty} \frac{7z^2+5z}{z^2-3}$

X4 $\lim_{x\to\infty} \frac{3(x-1)^2}{x+1}$

X5 $\lim_{x\to\infty} (2^x+5^x)^{\frac{1}{x}}$

Limits diverging to ∞ Notes are on the website.

Answers for the following exercises are given below.

X6 Give an $M-\delta$ proof that $\lim_{x\to1} \frac{1}{(x-1)^2} = +\infty$.

X7 Explain in two or three sentences why $\lim_{x\to1} \frac{1}{(x-1)^2}$ does not exist.

X8 Use theorems discussed in lecture to prove that $\lim_{\theta\to\pi} |\cot(\theta)| = +\infty$.

X9 Let $\lim_{x\to3} g(x) = +\infty$, and let $f(x)$ be continuous with $domain(f) = (-\infty,\infty)$. Prove that,

$$\lim_{x\to+\infty} f(x) = L \Longrightarrow \lim_{x\to3} (f\circ g)(x) = L.$$

Is the reverse implication $\Longleftarrow$ true?

§1.2 #44, 45, 46 & 47 (about boundedness)

§11.1 #9, 12, 13, 15, 18, 19, 27 & 30

Answers for the following exercises are given below.

X10 Give the GLB (if it exists) and LUB (if it exists) of,

$$S = \left\{ \ln\left(\frac{1}{x^2}\right) : x \in (1, \infty) \right\}$$

X11 Give the GLB (if it exists) and LUB (if it exists) of,

$$T = \left\{ \sec(\theta) : \theta \in (\frac{\pi}{2}, \pi) \right\}$$

X12 Prove that GLB of $\{\left(\frac{1}{2}\right)^n : n \in \mathbb{N}\}$ is zero.

X13 Prove that $\{x : \frac{1}{x} \in (0, 1)\}$ has no LUB.

§2.6 #3, 7, 9, 15, 19, 21, 26, 32 & 35 (For #35, create a function of θ and use EVT.)

Assigned Problems

None. Midterm #1 is Monday **June 1st**.

Answers for X exercises

X1 Answer: 0

Change $\lim_{x \to -\infty} f(x)$ to $\lim_{h \to 0^-} f(\frac{1}{h})$ and use continuity.

X2 Answer: 0

Change into $\lim_{\varphi \to 0^+} \varphi \sin\left(\frac{1}{\varphi}\right)$ and use Squeeze Thm.

X3 Answer: 7

Divide through by z^2, and change into $\lim_{h \to 0^+} \frac{7+5h}{1-3h^2}$. Then use continuity of polynomials and Reciprocal Thm.

X4 Answer: DNE

By long division, $\frac{3(x-1)^2}{x+1} = 3\left(x - 3 + \frac{4}{x+1}\right)$. Note that $\frac{x}{2} - 3 + \frac{4}{x+1} > 0$ for $x > \frac{5+\sqrt{17}}{2}$. Therefore,

$$\frac{3(x-1)^2}{x+1} > \frac{3}{2}x \quad \forall x \in \left(\frac{5+\sqrt{17}}{2}, \infty\right).$$

Assume the limit exists, choose any ϵ (say, $\epsilon = 1$), then look at any $x >> N(\epsilon)$.

X5 Answer:

Factor, $(2^x + 5^x)^{\frac{1}{x}} = 5\left(\left(\frac{2}{5}\right)^x + 1\right)^{\frac{1}{x}}$, and change $\lim_{x \to \infty} f(x)$ to $\lim_{h \to 0^+} f(\frac{1}{h})$,

$$\lim_{h \to 0^+} 5\left(\left(\frac{2}{5}\right)^{\frac{1}{h}} + 1\right)^h.$$

Note, $5(1)^h < 5\left(\left(\frac{2}{5}\right)^{\frac{1}{h}} + 1\right)^h < 5(1+1)^h$. Use continuity of e^x to calculate the limit of 2^h, then use Squeeze Thm.

X6 (*Rough Work*)

$$\frac{1}{(x-1)^2} > M \iff \frac{1}{M} > (x-1)^2 \iff \frac{1}{\sqrt{M}} > |x-1|$$

(*Proof*)

Let $M > 0$ be given. Choose $\delta = \frac{1}{\sqrt{M}}$. If $0 < |x-1| < \delta$ then,

$$\frac{1}{(x-1)^2} > \frac{1}{\delta^2} = \frac{1}{\left(\frac{1}{\sqrt{M}}\right)^2} = M.$$

Since $M > 0$ was arbitrary, this proves $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

X7 If $\lim_{x\to 1} \frac{1}{(x-1)^2}$ existed, then it would be a (finite) number L. Since $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$, we could let $M = L+1$ and find $\delta(M)$ so that if $0 < |x-1| < \delta$ then $\left|\frac{1}{(x-1)^2} - L\right| > 1$. This makes $\lim_{x\to 1} \frac{1}{(x-1)^2} = L$ impossible for any $\epsilon \leq 1$.

X8 Since $|\cdot|$ and $\cos(\theta)$ are both continuous functions,

$$\lim_{\theta\to 0} |\cos(\theta)| = \left|\lim_{\theta\to 0} \cos(\theta)\right| = |\cos(0)| = |1| = 1.$$

For basically the same reasons,

$$\lim_{\theta\to 0} |\sin(\theta)| = \left|\lim_{\theta\to 0} \sin(\theta)\right| = |\sin(0)| = 0.$$

By the Reciprocal Thm,

$$\lim_{\theta\to 0} \left(\frac{|\sin(\theta)|}{|\cos(\theta)|}\right) = \frac{(\lim_{\theta\to 0} |\sin(\theta)|)}{(\lim_{\theta\to 0} |\cos(\theta)|)} = \frac{0}{1} = 0.$$

We proved in class that if $h(x) > 0$ and $\lim_{x\to 0} h(x) = 0$, then $\lim_{x\to 0} \frac{1}{h(x)} = +\infty$. Use $h(\theta) = \frac{1}{|\cot(\theta)|} = \frac{|\sin(\theta)|}{|\cos(\theta)|}$.

X9 Let $\epsilon > 0$ be given.

Since $\lim_{x\to\infty} f(x) = L$, there exists $N_f(\epsilon) > 0$ so that,

$$N_f < x \Longrightarrow |f(x) - L| < \epsilon.$$

Plug $N_f(\epsilon)$ into the other definition as M. Since $\lim_{x\to 3} g(x) = +\infty$, there exists $\delta_g(N_f) > 0$ so that,

$$0 < |x-3| < \delta_g \Longrightarrow N_f < g(x).$$

Therefore, if $0 < |x-3| < \delta_g$, then $N_f < g(x)$ and $|f(g(x)) - L| < \epsilon$. That is,

$$0 < |x-3| < \delta_g \Longrightarrow |(f\circ g)(x) - L| < \epsilon.$$

Reverse implication $\Longleftarrow$ is FALSE. It is possible for $\lim_{x\to 3}(f\circ g)(x)$ to exist even though $\lim_{x\to\infty} f(x)$ does not. (You can build an example using the Dirichlet function.)

X10 Answer: GLB does not exist; LUB is zero.

Logarithm is a strictly increasing function. $\frac{1}{x^2}$ is largest for x^2 smallest. Thus, LUB should be $\ln\left(\frac{1}{1}\right) = 0$. For very large x, $\frac{1}{x^2}$ is a very small positive number, which has a very negative logarithm. That is, GLB should be $\ln(0)$ which DNE.

X11 Answer: GLB does not exist; LUB is zero.

In quadrant II, $cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ is negative. The numerator is ≈ 0 for $\theta \approx \frac{\pi}{2}$, and denominator is ≈ 0 for $\theta \approx \pi$.

X12 Label the set $H = \left\{\left(\frac{1}{2}\right)^n : n \in \mathbb{N}\right\}$.

To begin, note that $\left(\frac{1}{2}\right)^n$ is positive for any $n \in \mathbb{N}$. This proves that zero is a lower bound for set H.

Assume there is a larger lower bound, $L > 0$. Since $\frac{1}{2} \in H$, it must be that $L \leq \frac{1}{2} < 1$, which means that $\ln(L) < 0$. Calculate,

$$\log_{\frac{1}{2}} L = \frac{\ln(L)}{\ln(\frac{1}{2})} = \frac{\ln(L)}{-\ln(2)} > 0.$$

This proves that $\beta = \left\lceil \frac{\ln(L)}{-\ln(2)} \right\rceil + 1$ is a natural number. That is, $\left(\frac{1}{2}\right)^\beta \in H$. We have,

$$\left(\frac{1}{2}\right)^\beta < \left(\frac{1}{2}\right)^{\left\lceil \frac{\ln(L)}{-\ln(2)} \right\rceil} \leq L.$$

This proves that L is not a lower bound for set H, which contradicts our assumption.

X13 Label the set $Q = \left\{x : \frac{1}{x} \in (0,1)\right\}$.

Assume that L is an upper bound for set Q. Note first that $\frac{1}{2} \in (0,1)$, so $2 \in Q$, which means that $L \geq 2$. Suppose $\frac{1}{x} \in (0,1)$. In particular $x > 0$, so we may divide by both x and L,

$$x \leq L \iff \frac{1}{L} \leq \frac{1}{x} \iff \frac{1}{x} \in \left[\frac{1}{L}, \infty\right)$$

The set $(0,1)$ is not contained in the set $[\frac{1}{L}, \infty)$, so there are $\frac{1}{x} \in (0,1)$ for which $x \not\leq L$. This proves L is not an upper bound for set Q, which contradicts our assumption.

Set Q has no upper bound, therefore it has no least upper bound.