

**MAT137Y1Y**  
**Calculus!**

**TERM TEST 3 SOLUTIONS, JULY 2009**

1. State whether the sequence converges or diverges. Justify your answer (a rigorous proof is not required). If it converges, find the limit of the sequence.

[7%]

(a)  $a_n = (-1)^n \sqrt{n}$

The sequence  $a_n = (-1)^n \sqrt{n}$  is unbounded:  $\sqrt{n} \rightarrow \infty$  as  $n \rightarrow \infty$ . Every unbounded sequence diverges. Therefore  $a_n$  diverges.

[7%]

(b)  $a_n = \ln(n+1) - \ln(n)$

The sequence converges to zero:

$$a_n = \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

[7%]

(c)  $a_n = \cos(n\pi)$

We can write down the values of this sequence:

$$\{a_n\} = \{\cos(\pi), \cos(2\pi), \cos(3\pi), \cos(4\pi), \dots\} = \{-1, 1, -1, 1, -1, \dots\}$$

and witness that  $a_n$  fails to converge to a limit. That is,  $a_n$  diverges.

[7%]

(d)  $a_n = \frac{2^n n^2}{n!}$

We will show that  $a_n$  is bounded below by zero, and above by a sequence which

converges to zero.

$$\begin{aligned}
 0 < a_n &= \frac{2^n n^2}{n!} = \frac{2 \cdot 2 \cdots 2 \cdot n \cdot n}{1 \cdot 2 \cdots (n-3)(n-2)(n-1) \cdot n} \\
 &= 2^3 \left(\frac{2}{2}\right) \left(\frac{2}{3}\right) \left(\frac{2}{4}\right) \cdots \left(\frac{2}{n-3}\right) \left(\frac{2}{n-2}\right) \left(\frac{n}{n-1}\right) \left(\frac{n}{n}\right) \\
 &= 2^3 \left(\frac{2}{3}\right) \left(\frac{2}{4}\right) \cdots \left(\frac{2}{n-3}\right) \left(\frac{2}{n-2}\right) \left(\frac{n}{n-1}\right) \\
 &< 2^3 \left(\frac{2}{n-2}\right) \left(\frac{n}{n-1}\right),
 \end{aligned}$$

since all the terms between  $2^3$  and  $\left(\frac{2}{n-2}\right)$  are less than 1. This leads to

$$\begin{aligned}
 0 < a_n &< 2^3 \left(\frac{2n}{(n-2)(n-1)}\right) \\
 &< 2^3 \left(\frac{2n}{n^2 - 3n + 2}\right) \\
 &= 2^3 \left(\frac{\frac{2}{n}}{1 - \frac{3}{n} + \frac{2}{n^2}}\right) \rightarrow 0.
 \end{aligned}$$

By the Squeeze Theorem for sequences, the sequence  $a_n$  converges to 0.

[10%] 2. Find the continuous function  $f$  and the real number  $a$  such that

$$2 + \int_a^x \frac{f(t)}{t^2} dt = \frac{x^2}{2} \text{ for all } x \in \mathbb{R}.$$

Differentiating both sides of this equation, by the Fundamental Theorem of Calculus, we find

$$\begin{aligned}
 \frac{f(x)}{x^2} &= x \\
 \Rightarrow f(x) &= x^3.
 \end{aligned}$$

Plugging this back into the equation:

$$\begin{aligned}\frac{x^2}{2} &= 2 + \int_a^x \frac{f(t)}{t^2} dt = 2 + \int_a^x t dt = 2 + \left[\frac{t^2}{2}\right]_a^x = 2 + \frac{x^2}{2} - \frac{a^2}{2} \\ \Rightarrow \frac{a^2}{2} &= 2 \\ \Rightarrow a &= \pm 2.\end{aligned}$$

3. Evaluate the following integrals. If the integral is improper, explain whether it converges or diverges.

[10%]

(a)  $\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sqrt{\sin x}} dx$

The integrand  $\frac{\cos^3 x}{\sqrt{\sin x}}$  is unbounded on every neighbourhood of zero. This is therefore an improper integral. Fix some  $b$  such that  $0 < b < \frac{\pi}{2}$ . Using  $\cos^2 x = 1 - \sin^2 x$ , we find

$$\int_b^{\frac{\pi}{2}} \frac{\cos^3 x}{\sqrt{\sin x}} dx = \int_b^{\frac{\pi}{2}} \frac{1 - \sin^2 x}{\sqrt{\sin x}} \cos x dx$$

By setting  $u = \sin x$ , we have  $du = \cos x dx$ . When  $x = b$ ,  $u = \sin b$ , and when  $x = \frac{\pi}{2}$ ,  $u = \sin \frac{\pi}{2} = 1$ . This gives

$$\begin{aligned}\int_b^{\frac{\pi}{2}} \frac{1 - \sin^2 x}{\sqrt{\sin x}} \cos x dx &= \int_{\sin b}^1 \frac{1 - u^2}{\sqrt{u}} du \\ &= \int_{\sin b}^1 \left(\frac{1}{\sqrt{u}} - u^{3/2}\right) du \\ &= \left[2\sqrt{u} - \frac{2}{5}u^{5/2}\right]_{\sin b}^1 \\ &= \frac{8}{5} - \left(2\sqrt{\sin b} - \frac{2}{5}\sin^{5/2} b\right) \rightarrow \frac{8}{5}, \text{ as } b \rightarrow 0^+.\end{aligned}$$

Therefore  $\int_0^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}} dx = \frac{8}{5}$ .

[10%]

$$(b) \int \frac{1}{\ln(x^x)} \arctan(\ln(\ln(x))) dx$$

Setting  $w = \ln(\ln(x))$ , we find  $dw = \frac{dx}{x \ln(x)}$ . Then using the logarithm rule  $\ln(x^x) = x \ln(x)$ ,

$$\int \frac{1}{\ln(x^x)} \arctan(\ln(\ln(x))) dx = \int \arctan w dw.$$

Now we integrate by parts. Set  $u = \arctan w$ , and  $dv = dw$ . Then  $du = \frac{1}{1+w^2} dw$ , and  $v = w$ . We find

$$\begin{aligned} \int \arctan w dw &= w \arctan w - \int \frac{w}{1+w^2} \\ &= \ln(\ln(x)) \arctan(\ln(\ln(x))) - \int \frac{w}{1+w^2}. \end{aligned}$$

Taking  $z = 1 + w^2$ , we find  $dz = 2w dw$ , and then

$$\begin{aligned} \int \frac{w}{1+w^2} &= \frac{1}{2} \int \frac{dz}{z} = \frac{1}{2} \ln |z| + C \\ &= \frac{1}{2} \ln(1+w^2) + C \\ &= \frac{1}{2} \ln(1+(\ln(\ln(x)))^2) + C. \end{aligned}$$

Therefore

$$\int \frac{1}{\ln(x^x)} \arctan(\ln(\ln(x))) dx = \ln(\ln(x)) \arctan(\ln(\ln(x))) - \frac{1}{2} \ln(1+(\ln(\ln(x)))^2) + C.$$

4. Consider the region  $\Omega$  bounded between the curve  $y = \frac{1}{x^2}$  and the  $x$ -axis, for  $1 \leq x < \infty$ .

[5%] (a) Use an improper integral to evaluate the area of  $\Omega$ .

Fix some  $c > 0$ . We find  $\int_1^c \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^c = 1 - \frac{1}{c} \rightarrow 1$  as  $c \rightarrow \infty$ . Therefore,

$$\text{Area of } \Omega = \int_1^{\infty} \frac{1}{x^2} dx = 1.$$

[5%] (b) Find the number  $a$  such that the line  $x = a$  divides  $\Omega$  into two regions with equal area.

Let  $\Gamma_a$  be the region bounded between the curve  $y = \frac{1}{x^2}$  and the  $x$ -axis, for  $1 \leq x \leq a$ . The area of  $\Gamma_a$  is given by

$$\text{Area of } \Gamma_a = \int_1^a \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^a = 1 - \frac{1}{a}.$$

We wish to find the value of  $a$  such that

$$\begin{aligned} 2(\text{Area of } \Gamma_a) &= \text{Area of } \Omega \\ \Rightarrow 2 - \frac{2}{a} &= 1 \\ \Rightarrow a &= 2. \end{aligned}$$

[5%]

- (c) Find the number  $b$  such that the line  $y = b$  divides  $\Omega$  into two regions with equal area.

Let  $\Gamma_b$  be the region bounded between the curve  $y = \frac{1}{x^2}$  and the line  $y = b$ , for  $x \geq 1$ . The area of  $\Gamma_b$  is given by

$$\text{Area of } \Gamma_b = \int_b^1 \left( \frac{1}{\sqrt{y}} - 1 \right) dx = \left[ 2\sqrt{y} - y \right]_b^1 = 1 - 2\sqrt{b} + b.$$

We wish to find the value of  $b$  such that

$$\begin{aligned} 2(\text{Area of } \Gamma_b) &= \text{Area of } \Omega \\ \Rightarrow 2 - 4\sqrt{b} + 2b &= 1 \\ \Rightarrow 2b - 4\sqrt{b} + 1 &= 0 \\ \Rightarrow \sqrt{b} &= 1 \pm \frac{\sqrt{2}}{2}. \end{aligned}$$

We insist on  $b < 1$ , and therefore  $b = \left(1 - \frac{\sqrt{2}}{2}\right)^2$ .

[15%]

5. Before a party, a hemispherical punch bowl (half of a sphere) with a radius of 30 cm is filled within 3 cm of the top of the bowl. At the end of the party, there are only 5 cm of punch left at the bottom of the bowl. How much punch was consumed at the party? (in  $cm^3$ , there is no need to simplify your answer.)

We construct the solid occupied by the punch (consumed at the party) by revolving the region bounded between the circle  $x^2 + y^2 = (30)^2$  of radius 30, and the  $x$ -axis, about the  $x$ -axis, for  $3 \leq x \leq 25$ . The area of a cross-section of this solid for fixed  $x$ , with  $3 \leq x \leq 25$ , is given by  $A(x) = \pi(\sqrt{(30)^2 - x^2})^2 = \pi((30)^2 - x^2)$ . We integrate this cross-sectional area from  $x = 3$  to  $x = 25$  to obtain the volume of punch consumed:

$$\begin{aligned} V_{consumed} &= \int_3^{25} A(x) dx = \pi \int_3^{25} ((30)^2 - x^2) dx = \pi \left[ (30)^2 x - \frac{x^3}{3} \right]_3^{25} \\ &= \pi \left[ (30)^2 \cdot 22 - \frac{25^3}{3} + 3^2 \right] cm^3 \end{aligned}$$

- [12%] 6. Does the infinite series  $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$  converge or diverge? Justify your answer.

The function  $f(x) = \frac{\ln x}{x^2}$  is continuous, positive, and decreasing for  $x > 2$ . Continuity and positivity are obvious. The function is confirmed to be decreasing by observing that

$$f'(x) = \frac{1 - 2 \ln x}{x^3} < 0$$

when  $x > 2$ . By the integral test this series converges if and only if the improper integral  $\int_2^{\infty} \frac{\ln x}{x^2} dx$  converges. For fixed  $b > 2$ , we evaluate

$$\int_2^b \frac{\ln x}{x^2} dx$$

through integration by parts. Let  $u = \ln x$ ,  $dv = \frac{dx}{x^2}$ . Then  $du = \frac{dx}{x}$ ,  $v = -\frac{1}{x}$ . We find

$$\begin{aligned} \int_2^b \frac{\ln x}{x^2} dx &= \left[ -\frac{\ln x}{x} \right]_2^b + \int_2^b \frac{dx}{x^2} \\ &= \frac{\ln 2}{2} - \frac{\ln b}{b} + \left[ -\frac{1}{x} \right]_2^b \\ &= \frac{\ln 2}{2} - \frac{\ln b}{b} + \frac{1}{2} - \frac{1}{b}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_2^{\infty} \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{\ln x}{x^2} dx \\ &= \lim_{b \rightarrow \infty} \left( \frac{\ln 2}{2} - \frac{\ln b}{b} + \frac{1}{2} - \frac{1}{b} \right) \\ &= \frac{1}{2}(\ln 2 + 1), \end{aligned}$$

since by L'Hospital's rule:  $\lim_{b \rightarrow \infty} \frac{\ln b}{b} = \lim_{b \rightarrow \infty} \frac{1}{b} = 0$ . The improper integral  $\int_2^{\infty} \frac{\ln x}{x^2} dx$  therefore converges, and by the integral test  $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$  converges.