Notes for Limits involving ∞

Exercises and solutions are included with Week #3 suggested problems.

Limits TOWARD ∞

Recall that the regular limit of $f(x)$ at c is L if (roughly speaking) "$f(x)$ is near the line $y = L$ when we restrict attention to $x \approx c$. "

Idea. *How can we restrict attention to $x \approx \infty$?*

- *choose a threshold* $0 < N$
- *insist that* $N < x$

Definition (Limit as $x \to +\infty$)**.** Let function $f(x)$ be defined on at least a ray $(p, +\infty)$.
We write "$\lim_{x\to+\infty} f(x) = L$ " if:

$$\text{"Given } \epsilon > 0, \text{ there is some } N(\epsilon) > 0 \text{ so that,}$$
$$N(\epsilon) < x \Longrightarrow |f(x) - L| < \epsilon.\text{"}$$

To avoid $\epsilon - N$ proofs, we have the following,

Theorem (Equivalence for $x \to +\infty$)**.**

$$\lim_{x\to+\infty} f(x) = L \Longleftrightarrow \lim_{h\to 0^+} f\left(\frac{1}{h}\right) = L.$$

Remark (Limit as $x \to -\infty$)**.** *There are the expected analogues:*

- **[Defn]** *Let function $f(x)$ be defined on at least a ray $(-\infty, -p)$.*
  We write "$\lim_{x\to-\infty} f(x) = L$ " if given $\epsilon > 0$ there is some $N(\epsilon) > 0$ so that,

$$x < -N(\epsilon) \Longrightarrow |f(x) - L| < \epsilon.$$

- **[Thm]** $\lim_{x\to-\infty} f(x) = L \Leftrightarrow \lim_{h\to 0^-} f\left(\frac{1}{h}\right) = L$

Example. Prove $\lim_{x\to\infty} \left(\frac{1}{x} + 1\right) = 1$.

Example. Find $\lim_{x\to-\infty} \frac{\cos^2(x)+1}{x}$.

Example. Find $\lim_{x\to\infty} e^{\frac{1}{x\cos\left(\frac{1}{x}\right)}}$.

Example. Find $\lim_{x\to-\infty} \frac{\sqrt{x+x^2}}{3x}$.

Example. Let $f(x)$ be the Dirichlet function below. Prove $\lim_{x\to\infty} f(x)$ does not exist.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

Limits DIVERGING to ∞

Recall that the regular limit of $f(x)$ at c is L if (roughly speaking) "$f(x)$ is near the line $y = L$ when we restrict attention to $x \approx c$. "

Idea. *How can $f(x)$ be "near ∞ "?*

Definition (Limit diverges to $+\infty$)**.** Let function $f(x)$ be defined on at least open interval $(c - p, c + p)$, except perhaps at c.

We write "$\lim_{x\to c} f(x) = +\infty$ " if:

$$\text{"Given } M > 0, \text{ there is some } \delta(M) > 0 \text{ so that,}$$
$$0 < |x - c| < \delta(M) \Longrightarrow M < f(x)."$$

Remark (Don't Exist)**.** *"$\lim_{x\to c} f(x) = +\infty$ " is a special way to say limit doesn't exist. Note that, "$\lim_{x\to c} f(x) \neq \infty$ "* **does not** *mean the limit exists.*

To avoid $M - \delta$ proofs, we have the following,

Theorem (Equivalence for $\lim \to \infty$)**.**

$$\lim_{x\to c} f(x) = \infty \Longleftrightarrow \begin{array}{l} \displaystyle\lim_{x\to c} \frac{1}{f(x)} = 0 \textit{ and,} \\ f(x) > 0 \textit{ on some interval } (c-p, c+p) \textit{ except perhaps at } c. \end{array}$$

Remark (Limit diverges to $-\infty$)**.** *There are the expected analogues:*

- **[Defn]** *Let function $f(x)$ be defined on at least $(c - p, c + p)$ except perhaps at c.*

  We write "$\lim_{x\to c} f(x) = -\infty$ " if given $M > 0$ there is some $\delta(M) > 0$ so that,

  $$0 < |x - c| < \delta \Longrightarrow f(x) < -M.$$

- **[Thm]** $\lim_{x\to c} f(x) = -\infty \Leftrightarrow \lim_{x\to c} \frac{1}{f(x)} = 0$ *and $f(x) < 0$ on some interval $(c - p, c + p)$.*

There are also the four expected one-sided analogues.
There are also the four combinations of "diverging to ∞ as approach ∞ ".

Example. Determine $\lim_{x\to1} \frac{x^3-1}{(x-1)^2}$.

Example. Determine $\lim_{x\to0} \frac{x^2+3x-1}{2-\sqrt{x^2+4}}$.