8.3 Van Kampen's theorem

Theorem 8.3.1 (Seifert-) Van Kampen *Let U and V be connected open subsets of X s.t. $U \cup V = X$ and $U \cap V$ is connected and nonempty. Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps. Choose a basepoint in $U \cap V$.*

Let $G = \pi_1(U)$, $H = \pi_1(V)$ and let $A = \pi_1(U \cap V)$. Then

$$\pi_1(X) = G *_A H$$

where $*$ denotes the amalgamated free product defined below.

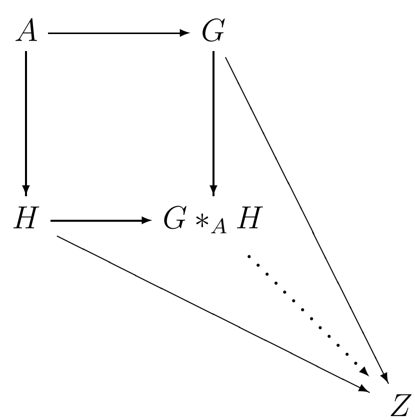
Definition 8.3.2 Amalgamated free product

If A, G, H are groups, $\alpha : A \to G$, $\beta : A \to H$ group homomorphisms, define $G *_A H$ as follows. The elements are "words" $w_1 \dots w_n$ where for each j either $w_j \in G$ or $w_j \in H$, modulo relations generated by $\big(g\alpha(a)\big)h = g\big(\beta(a)h\big)$

(Thus every element can be written as a word alternating between elements of G and H.)

Group multiplication is by juxtaposition.

Remark: $G *_A H$ is a pushout in the category of groups:



If $A = 1$ then $G * H$ is called the *free product* of G and H.

Proof: (of Theorem): Pick a basepoint x_0 for X lying in $U \cap V$. By the universal property, there exists $\phi : G *_A H \to \pi_1(X)$.

(Map $G \hookrightarrow \pi_1(X)$, $H \hookrightarrow \pi_1(X)$ and map a word in $G *_A H$ to the product of images of the elements of the word.)

Lemma 8.3.3 *ϕ is onto.*

Proof: Let $f : I \to X$ represent an element of $\pi_1(X)$. $f^{-1}(U) \cup f^{-1}(V) = I$ so by compactness $\exists N$ s.t. $J \subset I$, $\operatorname{diam} J \le 1/N \Rightarrow J \subset f^{-1}(U)$ or $J \subset f^{-1}(V)$. (i.e. $\frac{1}{N}$ is a Lebesgue number for the covering $f^{-1}(U)$, $f^{-1}(V)$.) Partition I into intervals of length $1/N$.

By discarding some division points, we may assume images of intervals alternate between U and V, so the (remaining) division points are in $U \cap V$.

Pick path α_i in $U \cap V$ joining x_0 to the i-th division point. In $\pi_1(X)$ $[f] = [f_1] \dots [f_q]$ where $f_i = \alpha_i \circ f|_{J_{i+1}} \circ \alpha_{i+1}^{-1}$. $\forall i$, $[f_i] \in G$ or $[f_i] \in H$ so $[f] \in \operatorname{Im} \phi$.

Lemma 8.3.4 *ϕ is injective.*

Proof: Notation: $A = V_0$, $U = V_1$, $V = V_2$.

Let $w = w_1 \dots w_q \in G *_A H$ s.t. $\phi(w) = 1$. For each $i = 1, \dots, q$, represent each w_i by a path f_i in either V_1 or V_2.

Reparametrize f_i so that $f_i : [(i-1)/q, i/q] \to V_1$ or V_2 in X.

Let $f : I \to X$ by $f\big|_{[(i-1)/q, i/q]} := f_i$.

$\phi(w) = 1 \Rightarrow f \cong * \operatorname{rel}\{0,1\}$ so $\exists F : I \times I \to X$ s.t. $F(s,0) = f(s)$, $F(s,1) = x_0$, $F(0,t) = F(1,t) = x_0$ $\forall t$.

By compactness $\exists$ a Lebesgue number ϵ s.t. $S \subset I \times I$ with $\operatorname{diam} S < \epsilon \Rightarrow$ either $F(S) \subset V_1$ or $F(S) \subset V_2$.

Choose partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < \cdots < t_n = 1$ of I s.t. the diameter of each rectangle on the resulting grid on $I \times I$ is less than ϵ.

Include the points k/q among the s_i.

For each ij select $\lambda(ij) = 1$ or 2 s.t. $F(R_{ij}) \subset V_{\lambda(ij)}$. (If $F(R_{ij} \subset$ both, take your pick.)

For each vertex v_{ij}, V_{ij} = intersection of $V_{\lambda(kl)}$ over the 4 (or fewer for edge vertices) rectangles having v_{ij} as vertex.

(So $\forall i, j$, $V_{ij} = V_0, V_1$ or V_2.)

$\forall i, j$ choose a path $g_{ij} : I \to V_{ij}$ joining x_0 to $F(v_{ij})$ in $V_{\lambda(ij)}$, using that V_0, V_1, and V_2 are path connected.

Choose these g_{ij} arbitrarily except:

If $s_i = k/q$ choose $g_{i0} = c_{x_0}$

Choose $g_{0j} = c_{x_0}$ and $g_{1j} = c_{x_0}$ $\forall j$.

Choose $g_{i1} = c_{x_0}$ $\forall i$.

Let $A_{ij} = F_{a_{ij}}$, $B_{ij} = F_{b_{ij}}$.

A_{ij}, B_{ij} are not closed paths, but from them form closed paths $\alpha_{ij} = g_{i-1,j} \circ A_{ij} \circ g_{ij}^{-1}$, $\beta_{ij} = g_{i-1,j} \circ B_{ij} \circ g_{ij}^{-1}$

$\forall i, j$ either $[\alpha_{ij}]$ and $[\beta_{ij}] \in G$, or $[\alpha_{ij}]$ and $[\beta_{ij}] \in H$.

$$w_1 = [A_{01} \cdot \cdots \cdot A_{0i_1}] = [\alpha_{01} \cdot \cdots \cdot \alpha_{0i_1}]$$

(since $g_{00} = g_{0,i_1} = c_{x_0}$, because the points s/q are among the s_i).

Similarly

$$w_2 = [A_{0(i_1+1)} \cdot \dots \cdot A_{0i_2}] = [\alpha_{0(i_1+1)} \cdot \dots \cdot \alpha_{0i_2}]$$

$$\vdots$$

$$w_q = [A_{0(i_q+1)} \cdot \dots \cdot A_{0m}] = [\alpha_{0(i_q+1)} \cdot \dots \cdot \alpha_{0m}]$$

Therefore $w = [\alpha_{01} \cdot \dots \cdot \alpha_{0m}][\alpha_{01}] \dots [\alpha_{0m}] \in G *_A H$.
By Lemma 7.2.5, each $R_{i,j}$ gives $A_{i,j-1}B_{ij} \simeq B_{i-1,j}A_{ij}$ rel$\{0,1\}$.
Hence $\alpha_{i,j-1}\beta_{ij} \cong \beta_{i-1,j}\alpha_{ij}$ rel$\{0,1\}$.
So the relation
$[\alpha_{i,j-1}][\beta_{ij}] = [\beta_{i-1,j}][\alpha_{ij}]$ holds in either G or H and thus in $G \times_A H$.
Also $[\beta_{0j}] = [\beta_{mj}] = 1\ \forall j$ (again for each j it holds in one of G, H) and $[\alpha_{in}] = 1\ \forall i$.
Hence $\forall j$

$$\begin{aligned} [\alpha_{1,j-1}] \dots [\alpha_{m,j-1}] &= [\alpha_{1,j-1}] \dots [\alpha_{m,j-1}][\beta_{m,j}] \\ &= [\alpha_{1,j-1}] \dots [\alpha_{m-1,j-1}][\beta_{m-1,j}][\alpha_{m,j}] \\ &= \dots \\ &= [\beta_{0,j}][\alpha_{1,j}] \dots [\alpha_{m-1,j}][\alpha_{m,j}] \\ &= [\alpha_{1,j}] \dots [\alpha_{m-1,j}][\alpha_{m,j}]. \end{aligned}$$

Hence $w_1 \dots w_q = \prod_{i=1}^m \alpha_{i_0} = \dots = \prod_{i=1}^m \alpha_{i_n} = 1$. □

Corollary 8.3.5 *If X can be written as the union of 2 simply connected open subsets whose intersection is connected then X is simply connected.*

Corollary 8.3.6 *S^n is simply connected for $n \geq 2$.*

Proof: Write S^n = slightly enlarged upper hemisphere ∪ slightly enlarged lower hemisphere. □

Example 1: $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for $n \geq 2$.

(Our covering space argument to compute $\pi_1(\mathbb{R}P^n)$ required knowing that S^n is simply connected for $n \geq 2$.)

Example 2: X is the figure eight. Then $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$.
Proof: Circles comprising X are not open, but slightly enlarge to form U and V .Then $U \cong S^1$ and $V \cong S^1$. □

The space X is denoted $S^1 \vee S^1$. The *wedge* of pointed spaces $(Y, *)$ and $(Z, *)$ written $Y \vee Z$ is the space formed from the disjoint union of Y and Z by identifying respective basepoints and using the common basepoint as the basepoint of $Y \vee Z$. In other words, $Y \vee Z = \{(y, z) \in Y \times Z \mid y = * \text{ or } z = *\}$

$Y \simeq Y' \Rightarrow Y \vee Z \simeq Y' \vee Z$

In particular, if W is contractible then $Y \vee W \simeq Y$. So if $X \simeq Y \vee Z$ where $\exists$ contractible open $* \in U \subset Y$ and contractible open $* \in V \subset Y$ then $\pi_1(X) = \pi_1(Y) * \pi_1(Z)$.