

# Chapter 2

## Topological Spaces

### 2.1 Metric spaces

**Definition 2.1.1** A metric space consists of a set  $X$  together with a function  $d : X \times X \rightarrow \mathbb{R}^+$  s.t.

1.  $d(x, y) = 0 \Leftrightarrow x = y$
2.  $d(x, y) = d(y, x) \quad \forall x, y$
3.  $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z$       *triangle inequality*

**Example 2.1.2** Examples

1.  $X = \mathbb{R}^n$
2.  $X = \{\text{continuous real-valued functions on } [0, 1]\}$   
 $d(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$
3.  $X = \{\text{bounded linear operators on a Hilbert space } H\}$   
 $d(f, g) = \sup_{x \in H} \|A(x) - B(x)\| =: \|A - B\|$
4.  $X$  any  
$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Notation:

$$\begin{aligned} N_r(a) &= \{x \in X \mid d(x, a) < r\} && \text{open } r\text{-ball centred at } a \\ N_r[a] &= \{x \in X \mid d(x, a) \leq r\} && \text{closed } r\text{-ball centred at } a \end{aligned}$$

**Definition 2.1.3** A map  $\phi : X \rightarrow Y$  is **continuous at**  $a$  if  $\forall \epsilon > 0 \exists \delta > 0$  s.t.  $d(x, a) < \delta \Rightarrow d(\phi(x), \phi(a)) < \epsilon$ .  $\phi$  is called **continuous** if  $\phi$  is continuous at  $a$  for all  $a \in X$ .

Equivalently,  $\phi$  is continuous if  $\forall \epsilon \exists \delta$  such that  $\phi(N_\delta(a)) \subset N_\epsilon(\phi(a))$ .

**Definition 2.1.4** A sequence  $(x_i)_{i \in \mathbb{N}}$  of points in  $X$  **converges** to  $\bar{x} \in X$  if  $\forall \epsilon, \exists M$  s.t.  $n \geq M \Rightarrow x_i \in N_\epsilon(\bar{x})$

We write  $(x_i) \rightarrow \bar{x}$ .

Exercise:  $(x_i) \rightarrow x$  in  $X \Leftrightarrow d(x_i, x) \rightarrow 0$  in  $\mathbb{R}$ .

**Proposition 2.1.5** If  $(x_i)$  converges to  $\bar{x}$  and  $(x_i)$  converges to  $\bar{y}$  then  $x = y$ .

**Proof:** Show  $d(x, y) < \epsilon \forall \epsilon$ . □

**Proposition 2.1.6**  $f : X \rightarrow Y$  is continuous  $\Leftrightarrow ((x_i) \rightarrow \bar{x} \Rightarrow (f(x_i)) \rightarrow f(\bar{x}))$

**Proof:**  $\Rightarrow$  Suppose  $f$  continuous. Let  $(x_i) \rightarrow \bar{x}$ .

Given  $\epsilon > 0, \exists \delta$  s.t.  $f(N_\delta(\bar{x})) \subset N_\epsilon(\bar{x})$

Since  $(x_i) \rightarrow \bar{x}, \exists M$  s.t.  $n \geq M \Rightarrow x_i \in N_\delta(\bar{x}) \therefore n \geq M \Rightarrow f(x_i) \in N_\epsilon(\bar{x})$ .

$\Leftarrow$  Suppose that  $((x_i) \rightarrow \bar{x} \Rightarrow (f(x_i)) \rightarrow f(\bar{x}))$

Assume  $f$  not cont. at  $a$  for some  $a \in X$ . Then  $\exists \epsilon > 0$  s.t. there is no  $\delta$  s.t.  $f(N_\delta(\bar{x})) \subset N_\epsilon(\bar{x})$ . Thus  $\exists \epsilon > 0$  s.t. for every  $\delta$  there is an  $x \in N_\delta(\bar{x})$  s.t.  $f(x) \notin N_\epsilon(\bar{x})$ . Therefore we can select, for each integer  $n$ , an  $x_n \in N_{1/n}(\bar{x})$  s.t.  $f(x_n) \notin N_\epsilon(\bar{x})$ . Then  $(x_n) \rightarrow \bar{x}$  but  $f(x_n) \not\rightarrow f(\bar{x})$ .  $\Rightarrow \Leftarrow$

**Definition 2.1.7** An **open set** is a subset  $U$  of  $X$  s.t.  $\forall x \in U$  exist  $\epsilon$  s.t.  $N_\epsilon \subset U$ .

**Proposition 2.1.8**

1.  $U_\alpha$  open  $\forall \alpha \Rightarrow \cup_{\alpha \in I} U_\alpha$  is open

2.  $U_\alpha$  open  $\forall \alpha, |I| < \infty \Rightarrow \cup_{\alpha \in I} U_\alpha$  is open

**Proof:**

1. Let  $x \in V = \cup_{\alpha \in I} U_\alpha$ . So  $x \in U_\alpha$  for some  $\alpha$ .

$\therefore N_\epsilon \subset U_\alpha \subset V$  for some  $\epsilon$ .

2. Number the sets  $U_1, \dots, U_n$ .

Let  $x \in V = \cap_{j=1}^n U_j$ . So  $\forall j \exists \epsilon_j$  s.t.  $N_{\epsilon_j}(x) \subset U_j$ .

Let  $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$ . Then  $N_\epsilon(x) \subset V$ .

□

Note: An infinite intersecon of open sets need not be open. For example,  $\cap_{n \geq 1} (-1/n, 1/n) = \{0\}$  in  $\mathbb{R}$ .

**Lemma 2.1.9**  $N_r(x)$  is open  $\forall x$  and  $\forall r > 0$ .

**Proof:** Let  $y \in N_r(x)$ . Set  $d = d(x, y)$ . Then  $N_{r-d}(y) \subset N_r(x)$  (and  $r - d > 0$  since  $y \in N_r(x)$ ). □

**Corollary 2.1.10**  $U$  is open  $\Leftrightarrow U = \cup N_\alpha$  where each  $N_\alpha$  is an open ball

**Proof:**  $\Leftarrow N_\alpha$  open  $\forall \alpha$  so  $\cup N_\alpha$  is open.  $\Rightarrow$  If  $U$  open then for each  $x \in U, \exists \epsilon_x$  s.t.  $N_{\epsilon_x} \subset U$ .  $U = \cup_{x \in U} N_{\epsilon_x}(x)$ . □

**Proposition 2.1.11**  $f : X \rightarrow Y$  is continuous  $\Leftrightarrow \forall$  open  $U \subset Y, f^{-1}(U)$  is open in  $X$

**Proof:**  $\Rightarrow$  Suppose  $f$  continuous. Let  $U \subset Y$  be open.

Given  $x \in f^{-1}(U), f(x) \in U$  so  $\exists \epsilon > 0$  s.t.  $N_\epsilon(f(x)) \subset U$ . Find  $\delta > 0$  s.t.  $f(N_\delta(x)) \subset N_\epsilon(f(x))$ . Then  $N_\delta(x) \subset f^{-1}(N_\epsilon(f(x))) \subset f^{-1}(U)$ .

$\Leftarrow$  Suppose that the inverse image of every open set is open.

Let  $x \in X$  and assume  $\epsilon > 0$ .

Then  $x \in f^{-1}(N_\epsilon(f(x)))$  and  $f^{-1}(N_\epsilon(f(x)))$  is open so  $\exists \delta$  s.t.  $N_\delta(x) \subset f^{-1}(N_\epsilon(f(x)))$

That is,  $f(N_\delta(x)) \subset N_\epsilon(f(x))$

$\therefore f$  continuous at  $x$ . □

Note: Although the previous Prop. shows that knowledge of the open sets of a metric space is sufficient to determine which functions are cont., it is not sufficient to determine the metric. That is, different metrics may give rise to the same collection of open sets.

## 2.2 Norms

Let  $V$  be a vector space of  $F$  where  $F = \mathbb{R}$  or  $F = \mathbb{C}$ .

**Definition 2.2.1** A **norm** on  $V$  is a function  $V \rightarrow \mathbb{R}$ , written  $x \mapsto \|x\|$ , which satisfies

1.  $\|x\| \geq 0$  and  $\|x\| = 0 \Leftrightarrow x = 0$ .
2.  $\|x+y\| \leq \|x\| + \|y\|$
3.  $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in F, x \in V$

Given a normed vector space  $V$ , define metric by  $d(x, y) = \|x - y\|$ .

**Proposition 2.2.2**  $(V, d)$  is a metric space

**Proof:** Check definitions. □

## 2.3 Topological spaces

**Definition 2.3.1** A *topological space* consists of a set  $X$  and a set  $\mathcal{T}$  of subsets of  $X$  s.t.

1.  $\emptyset \in \mathcal{T}, X \in \mathcal{T}$
2. For any index set  $I$ , if  $U_\alpha \in \mathcal{T} \forall \alpha \in I$ , then  $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$ .
3.  $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$ .

**Definition 2.3.2** **Open sets**

The subsets of  $X$  which belong to  $\mathcal{T}$  are called open.

If  $x \in U$  and  $U$  is open then  $U$  is called a *neighbourhood* of  $x$ .

If  $\mathcal{S} \subset \mathcal{T}$  has the property that each  $V \in \mathcal{T}$  can be written as a union of sets from  $\mathcal{S}$ , then  $\mathcal{S}$  is called a *basis* for the topology  $\mathcal{T}$ .

If  $\mathcal{S} \subset \mathcal{T}$  has the property that each  $V \in \mathcal{T}$  can be written as the union of finite intersections of sets in  $\mathcal{S}$  then  $\mathcal{S}$  is called a *subbasis*, in other words  $V = \cup_\alpha (\cap_{i_1, \dots, i_\alpha} S_{i_\alpha})$

Given a set  $X$  and a set  $\mathcal{S} \subset 2^X$  (the set of subsets of  $X$ ),  $\exists!$  topology  $\mathcal{T}$  on  $X$  for which  $\mathcal{S}$  is a subbasis. Namely,  $\mathcal{T}$  consists of all sets formed by taking arbitrary unions of finite intersections of all sets in  $\mathcal{S}$ .

(Have to check that the resulting collection is closed under unions and finite intersections — exercise)

In contracts, a set  $\mathcal{S} \subset 2^X$  need not form a basis for any topology on  $X$ .  $\mathcal{S}$  will form a basis iff the intersection of 2 sets in  $\mathcal{S}$  can be written as the union of sets in  $\mathcal{S}$ .

**Definition 2.3.3 Continuous** Let  $f : X \rightarrow Y$  be a function between topological spaces.  $f$  is continuous if  $U$  open in  $Y \Rightarrow f^{-1}(U)$  open in  $X$ .

Note: In general  $f(\text{open set})$  is not open. For example,  $f = \text{constant map} : \mathbb{R} \rightarrow \mathbb{R}$ .

**Proposition 2.3.4** Composition of continuous functions is continuous.

**Proof:** Trivial □

**Proposition 2.3.5** If  $\mathcal{S}$  is a subbasis for the topology on  $Y$  and  $f^{-1}(U)$  is open in  $X$  for each  $U \in \mathcal{S}$  then  $f$  is continuous.

**Proof:** Check definitions. □

## 2.4 Equivalence of Topological Spaces

Recall that a *category* consists of objects and morphisms between the objects.

For example, sets, groups, vector spaces, topological spaces with morphisms given respectively by functions, group homomorphisms, linear transformations, and continuous functions.

(We will give a precise definition of category later.)

In a any category, a morphism  $f : X \rightarrow Y$  is said to have a **left inverse** if  $\exists g : Y \rightarrow X$  s.t.  $g \circ f = 1_X$ .

A morphism  $f : X \rightarrow Y$  is said to have a **right inverse** if  $\exists g : Y \rightarrow X$  s.t.  $f \circ g = 1_Y$ .

A morphism  $g : Y \rightarrow X$  is said to be an **inverse** to  $f$  if it is both a left and a right inverse. In this case  $f$  is called **invertible** or an **isomorphism**.

**Proposition 2.4.1**

1. If  $f$  has a left inverse  $g$  and a right inverse  $h$  then  $g = h$  (so  $f$  is invertible)
2. A morphism has at most one inverse.

**Proof:**

1. Suppose  $g \circ f = 1_X$  and  $f \circ h = 1_Y$ .

Part of the definition of category requires that composition of morphisms be associative.

Therefore  $h = 1_X \circ h = g \circ f \circ h = g \circ 1_Y = g$ .

2. Let  $g, h$  be inverses to  $f$ . Then in particular  $g$  is a left inverse and  $h$  a right inverse so  $g = h$  by (1).

□

Intuitively, isomorphic objects in a category are equivalent with regard to all properties in that category.

Some categories assign special names to their isomorphisms. For example, in the category of Sets they are called “bijections”. In the category of topological spaces, the isomorphisms are called “homeomorphisms”.

**Definition 2.4.2 Homeomorphism** *A continuous function  $f : X \rightarrow Y$  is called a homeomorphism if there is a continuous function  $g : Y \rightarrow X$  such that  $g \circ f = 1_X$  and  $f \circ g = 1_Y$ .*

**Remark 2.4.3** *Although the word “homeomorphism” looks similar to “homomorphism” it is more closely analogous to “isomorphism”.*

Note: In groups, the set inverse to a bijective homomorphism is always a homomorphism so a bijective homomorphism is an isomorphism. In contrast, a bijective continuous map need not be a homeomorphism. That is, its inverse might not be continuous. For example

$$X = [0, 1) \quad Y = \text{unit circle in } \mathbb{R}^2 = \mathbb{C}$$

$$f : X \rightarrow Y \text{ by } f(t) = e^{2\pi it}.$$

## 2.5 Elementary Concepts

**Definition 2.5.1 Complement** *If  $A \subset X$ , the complement of  $A$  in  $X$  is denoted  $X \setminus A$  or  $A^c$ .*

**Definition 2.5.2 Closed** *A set  $A$  is closed if its complement is open.*

**Definition 2.5.3 Closure** *The closure of  $A$  (denoted  $\overline{A}$ ) is the intersection of all closed subsets of  $X$  which contain  $A$ .*

**Proposition 2.5.4** *Arbitrary intersections and finite unions of closed sets are closed.*

**Definition 2.5.5 Interior** *The interior of  $A$  (denoted  $\overset{\circ}{A}$  or  $\text{Int } A$ ) is the union of all open subsets of  $X$  which contained in  $A$ .*

**Proposition 2.5.6**  $x \in \overset{\circ}{A} \Leftrightarrow \exists U \subset A$  s.t.  $U$  is open in  $X$  and  $x \in U$ .

**Proof:**  $\Rightarrow$  If  $x \in \overset{\circ}{A}$ , let  $U = \overset{\circ}{A}$ .

$\Leftarrow$   $x \in U \subset A$ . Since  $U$  is open,  $U \subset \overset{\circ}{A}$ , so  $x \in \overset{\circ}{A}$ . □

**Proposition 2.5.7**  $(\overline{A})^c = \overset{\circ}{A^c}$

**Proof:** Exercise □

**Corollary 2.5.8** If  $x \notin \overline{A}$  then  $\exists$  open  $U$  s.t.  $x \in U$  and  $U \cap A = \emptyset$ . □

**Definition 2.5.9 Dense** A subset  $A$  of  $X$  is called dense if  $\overline{A} = X$ .

**Definition 2.5.10 Boundary** Let  $X$  be a topological space and  $A$  a subset of  $X$ . The boundary of  $A$  (written  $\partial A$ ) is

$\{x \in X \mid \text{each open set of } X \text{ containing } x \text{ contains at least one point from } A$   
 $\text{and at least one from } A^c\}$

**Proposition 2.5.11** Let  $A \subset X$

1.  $\partial A = \overline{A} \cap \overline{A^c} = \partial(A^c)$
2.  $\partial A$  is closed
3.  $A$  is closed  $\Leftrightarrow \partial A \subset A$

**Proof:**

1. Suppose  $x \in \partial A$ .  
If  $x \notin \overline{A}$  then  $\exists$  open  $U$  s.t.  $x \in U$  and  $U \cap A = \emptyset$ .  
Contradicts  $x \in \partial A \Rightarrow \Leftarrow$   
 $\therefore \partial A \subset \overline{A}$ .  
Similarly  $\partial A \subset \overline{A^c}$ .  
 $\therefore \partial A \subset \overline{A} \cap \overline{A^c}$ .  
Conversely suppose  $x \in \overline{A} \cap \overline{A^c}$ .  
If  $U$  is open and  $x \in U$  then  
 $x \in \overline{A} \Rightarrow U \cap A \neq \emptyset$  and  
 $x \in \overline{A^c} \Rightarrow U \cap A^c \neq \emptyset$   
True  $\forall$  open  $U$  so  $x \in \partial A$ .  
 $\therefore \overline{A} \cap \overline{A^c} \subset \partial A$ .

2. By (1),  $\partial A$  is the intersection of closed sets

3.  $\Rightarrow$  Suppose  $A$  closed

$$\partial A = \overline{A} \cap \overline{A^c} \subset \overline{A} = A \text{ (since } A \text{ closed)}$$

$\Leftarrow$  Suppose  $A$  closed.

Let  $x \in \overline{A}$ . Then every open  $U$  containing  $x$  contains a point of  $A$ .

If  $x \notin A$  then every open  $U$  containing  $x$  also contains a point of  $A^c$ , namely  $x$ .

In this case  $x \in \partial A \subset A \Rightarrow \Leftarrow$

$\therefore \overline{A} \subset A$  so  $A = \overline{A}$  and so  $A$  is closed.

□

## 2.6 Weak and Strong Topologies

Given a set  $X$ , topological space  $Y, \mathcal{S}$  and a collection of functions  $f_\alpha : X \rightarrow Y$  then there is a 'weakest topology on  $X$  s.t. all  $f_\alpha$  are continuous':

namely intersect all the topologies on  $X$  under which all  $f_\alpha$  are continuous.

Given a set  $X$ , a topological space  $W$  and functions  $g_\alpha : W \rightarrow X$  we can form  $\mathcal{T}$ , the strongest topology on  $X$  s.t. all  $g_\alpha$  are continuous. Define  $\mathcal{T}$  by  $U \in \mathcal{T} \Leftrightarrow g_\alpha^{-1}(U)$  is open in  $W \forall \alpha$ .

**Strong and weak topologies** Given  $X$ , a topology on  $X$  is 'strong' if it has many open sets, and is 'weak' if it has few open sets.

Extreme cases:

(a)  $\mathcal{T} = 2^X$  is the strongest possible topology on  $X$ . With this topology any function  $X \rightarrow Y$  becomes continuous.

(b)  $\mathcal{T} = \{\emptyset, X\}$  is the weakest possible topology on  $X$ . With this topology any function  $W \rightarrow X$  becomes continuous.

**Proposition 2.6.1** *If  $\mathcal{T}_\alpha$  are topologies on  $X$  then so is  $\cap_{\alpha \in I} \mathcal{T}_\alpha$ .*

Common application: Given a set  $X$ , a topological space  $(Y, \mathcal{S})$  and a collection of functions  $f_\alpha : X \rightarrow Y$  then there is a 'weakest topology on  $X$  s.t. all  $f_\alpha$  are continuous'. Namely, intersect all the topologies on  $X$  under which all  $f_\alpha$  are continuous.

Similarly, given a set  $X$ , a topological space  $(W, \mathcal{P})$  and functions  $g_\alpha : W \rightarrow X$ , we can form  $\mathcal{T}$  which is the strongest topology on  $X$  s.t. all  $g_\alpha$  are continuous. Explicitly, define  $\mathcal{T}$  by  $U \in \mathcal{T} \Leftrightarrow g_\alpha^{-1}(U)$  is open in  $W \forall \alpha$ .

Example:  $\mathcal{H}$  = Hilbert space.

$B(\mathcal{H})$  = bounded linear operators on  $\mathcal{H}$

Some common topologies on  $B(\mathcal{H})$ :

(a) Norm topology: Define

$$\|A\| = \sup_{x \in \mathcal{H}, \|x\|=1} \|A(x)\|$$

A norm determines a metric, which determines a topology.

(b) Weak topology: For each  $x, y \in \mathcal{H}$ , define a function  $f_{x,y} : B(\mathcal{H}) \rightarrow \mathbb{C}$  by

$$A \mapsto (Ax, y)$$

The weak topology on  $B(\mathcal{H})$  is the weakest topology s.t.  $f_{x,y}$  is continuous  $\forall x, y$ .

(c) Strong topology: For each  $x \in \mathcal{H}$  define a function  $g_x : B(\mathcal{H}) \rightarrow \mathbb{R}$  by

$$A \mapsto \|A(x)\|$$

The strong topology is the weakest topology on  $B(\mathcal{H})$  s.t.  $g_x$  is continuous  $\forall x \in \mathcal{H}$ .

### Definition 2.6.2 Subspace topology

Let  $X$  be a topological space, and  $A$  a subset of  $X$ . The subspace topology on  $A$  is the weakest topology on  $A$  such that the inclusion map  $A \rightarrow X$  is continuous.

Explicitly, a set  $V$  in  $A$  will be open in  $A$  iff  $V = U \cap A$  for some open  $U$  of  $X$ .

### Definition 2.6.3 Quotient spaces

If  $X$  is a topological space and  $\sim$  an equivalence relation on  $X$ , the quotient space  $X/\sim$  consists of the set  $X/\sim$  together with the strongest topology such that the canonical projection  $X \rightarrow X/\sim$  is continuous.

Special case:  $A$  a subset of  $X$ .  $x \sim y \Leftrightarrow x, y \in A$ . In this case  $X/\sim$  is written  $X/A$ .

For example, if  $X = [0, 1]$  and  $A = \{0, 1\}$  then  $X/A \cong \text{circle}$ .

(Exercise: Prove this homeomorphism between  $X/A$  and the circle.)

### Example 2.6.4 Examples

1. Spheres:

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

2. Projective spaces:

(a) Real projective space  $\mathbb{R}P^n$ : Define an equivalence relation on  $S^n$  by  $x \sim -x$ . Then

$$\mathbb{R}P^n = S^n / \sim$$

with the quotient topology.

Thus points in  $\mathbb{R}P^n$  can be identified with lines through 0 in  $\mathbb{R}^{n+1}$ , in other words identify the equivalence class of  $x$  with the line joining  $x$  to  $-x$ .

Similarly

(b) Complex projective space  $\mathbb{C}P^n$ :

$$S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}.$$

Define an equivalence relation  $x \sim \lambda x$  for every  $\lambda \in S^1 \subset \mathbb{C}$  where  $\lambda x$  is formed by scalar multiplication of  $\mathbb{C}$  on  $\mathbb{C}^{n+1}$ . Then

$$\mathbb{C}P^n = S^{2n+1} / \sim$$

with the quotient topology. The points correspond to complex lines through the origin in  $\mathbb{C}^{n+1}$ .

(c) Quaternionic projective space  $\mathbb{H}P^n$

$$S^{4n+3} \subset \mathbb{R}^{4n+4} = \mathbb{H}^{n+1}$$

Define  $x \sim \lambda x$  for every  $\lambda \in S^3 \subset \mathbb{H}$  where  $\lambda x$  is formed by scalar multiplication of  $\mathbb{H}$  on  $\mathbb{H}^{n+1}$ .

$$\mathbb{H}P^n = S^{4n+3} / \sim$$

with the quotient topology.

3. Zariski topology:

(This is the main example in algebraic geometry.)

$R$  is a ring.

$$\text{Spec } R = \{\text{prime ideals in } R\}$$

Define Zariski topology on  $\text{Spec } R$  as follows: Given an ideal  $I$  of  $R$ , define  $V(I) = \{P \in \text{Spec}(R) \mid I \subset P\}$ .

Specify the topology by declaring the sets of the form  $V(I)$  to be closed.

To show that this gives a topology, we must show this collection is closed under finite unions and arbitrary intersections.

This follows from

**Lemma 2.6.5**

- (a)  $V(I) \cup V(J) = V(IJ)$   
 (b)  $\bigcap_{\alpha \in K} V(I_\alpha) = V(\sum_{\alpha \in K} I_\alpha)$ .

4. *Ordinals:*

Let  $\gamma$  be an ordinal.

Define  $X = \{\text{ordinals } \sigma \mid \sigma \leq \gamma\}$ , where for ordinals  $\sigma$  and  $\gamma$ ,  $\sigma < \gamma$  means that the well-ordered set representing  $\sigma$  is isomorphic to an initial interval of that representing  $\gamma$ .

Recall the Theorem: For two well-ordered sets  $X$  and  $Y$  either  $X \cong Y$  or  $X \cong$  initial interval of  $Y$  or  $Y \cong$  initial interval of  $X$ . Thus all ordinals are comparable.

Define a topology on  $X$  as follows.

For  $w_1, w_2 \in X$  define  $U_{w_1, w_2} = \{\sigma \in X \mid w_1 < \sigma < w_2\}$ . Here allow  $w_1$  or  $w_2$  to be  $\infty$ .

Take as base for the open sets all sets of the form  $U_{w_1, w_2}$  for  $w_1, w_2 \in X$ . Note that this collection of sets is the base for a topology since it is closed under intersection,

in other words  $U_{w_1, w_2} \cap U_{w'_1, w'_2} = U_{\max\{w_1, w'_1\}, \min\{w_2, w'_2\}}$ .

**Definition 2.6.6 Product spaces**

The product of a collection  $\{X_\alpha\}$  of topological spaces is the set  $X = \prod_\alpha X_\alpha$  with the topology defined by: the weakest topology such that all projection maps  $\pi_\alpha : X \rightarrow X_\alpha$  are continuous.

**Proposition 2.6.7** In  $\prod_\alpha X_\alpha$  sets of the form  $\prod_\alpha U_\alpha$  for which  $U_\alpha = X_\alpha$  for all but finitely many  $\alpha$  form a basis for the topology of  $X$ .

**Proof:** Let  $\mathcal{S} \subset 2^X$  be the collection of sets of the form  $\prod_\alpha U_\alpha$ .

Intersection of two sets in  $\mathcal{S}$  is in  $\mathcal{S}$  so  $\mathcal{S}$  is the basis for some topology  $\mathcal{T}$ .

Claim: In the topology  $\mathcal{T}$  on  $X$ , each  $\pi_\alpha$  is continuous.

Proof: Let  $U \subset X_{\alpha_0}$  be open.

Then  $\pi_{\alpha_0}^{-1}(U) = U \times \prod_{\alpha \neq \alpha_0} X_\alpha \in \mathcal{S} \subset \mathcal{T}$

$\therefore \pi_{\alpha_0}$  is continuous.

Claim: If  $\mathcal{T}'$  is any topology s.t. all  $\pi_\alpha$  are continuous then  $\mathcal{S} \subset \mathcal{T}'$  (and thus  $\mathcal{T} \subset \mathcal{T}'$ )

Proof Let  $V = U_{\alpha_1} \times U_{\alpha_2} \times \dots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \dots, \alpha_n} X_\alpha \in \mathcal{S}$ .

Then  $V = \pi_{\alpha_1}^{-1}U_{\alpha_1} \cap \pi_{\alpha_2}^{-1}U_{\alpha_2} \cap \dots \cap \pi_{\alpha_n}^{-1}U_{\alpha_n}$  which must be in any topology in which all  $\pi_\alpha$  are cont.

$\therefore \mathcal{T} =$  weakest topology on  $X$  s.t. all  $\pi_\alpha$  are cont. □

Note: A set of the form  $\prod_{\alpha} U_{\alpha}$  in which  $U_{\alpha} \neq X_{\alpha}$  for infinitely many  $\alpha$  will not be open.

**Proposition 2.6.8** *Let  $X = \prod_{\alpha \in I} X_{\alpha}$ . Then  $\pi_{\alpha} : X \rightarrow X_{\alpha}$  is an open map  $\forall \alpha$ .*

**Proof:** Let  $U \subset X$  be open, and let  $y \in \pi_{\alpha}(U)$ .

So  $y = \pi_{\alpha}(x)$  for some  $x \in U$ .

Find basic open set  $V = \prod_{\beta} V_{\beta}$  (with  $V_{\beta} = X_{\beta}$  for almost all  $\beta$ ) s.t.  $x \in V \subset U$ .

Then  $y \in V_{\alpha} = \pi_{\alpha}(V) \subset \pi_{\alpha}(U)$ .

$\therefore$  every pt. of  $\pi_{\alpha}(U)$  is interior, so  $\pi_{\alpha}(U)$  is open.

$\therefore \pi_{\alpha}$  is an open map. □

**Proposition 2.6.9** *If  $F_{\alpha}$  is closed in  $X_{\alpha} \forall \alpha$  then  $\prod_{\alpha} F_{\alpha}$  is closed in  $\prod_{\alpha} X_{\alpha}$ .*

$$\prod_{\alpha} F_{\alpha} = \bigcap_{\alpha} \left( F_{\alpha} \times \prod_{\beta \neq \alpha} X_{\beta} \right)$$

$F_{\alpha} \times \prod_{\beta \neq \alpha} X_{\beta}$  is closed (compliment is  $F_{\alpha}^c \times \prod_{\beta \neq \alpha} X_{\beta}$ ).

$\Rightarrow \prod_{\alpha} F_{\alpha}$  is closed □

**Theorem 2.6.10**  $X_1, X_2, \dots, X_k, \dots$  metric  $\Rightarrow X = \prod_{i \in \mathbb{N}} X_i$  metrizable

**Proof:** Let  $x, y \in X$ .

Define  $d(x, y) = \sum_{n=1}^{\infty} d_n(x_n, y_n) / 2^n$ .

Let  $X$  denote  $X$  with the product topology and let  $(X, d)$  denote  $X$  with the metric topology.

Clear that  $\pi_n : (X, d) \rightarrow X_n$  is continuous  $\forall n$ .

$\therefore 1_X : (X, d) \rightarrow X$  is continuous.

Conversely, let  $N_r(x)$  be a basic open set in  $(X, d)$ .

To show  $N_r(x)$  open in  $X$ , let  $y \in N_r(x)$  and show  $y$  interior.

Find  $\tilde{r}$  such that  $N_{\tilde{r}}(y) \subset N_r(x)$ .

Find  $M$  s.t.  $1/2^{(M-1)} < \tilde{r}$ .

$y \in U := \prod_{k \leq M} N_{1/2^M}(y_k) \times \prod_{k > M} X_k$ , which is open in  $X$

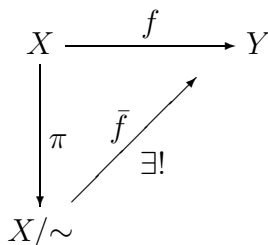
For  $z \in U$ ,

$$d(y, z) \leq \frac{1}{2^M} \left( \frac{1}{2} + \dots + \frac{1}{2^M} \right) + \frac{1}{2^{M+1}} + \frac{1}{2^{M+2}} + \dots < \frac{1}{2^M} + \frac{1}{2^M} = \frac{1}{2^{M-1}} < \tilde{r}.$$

$\therefore U \subset N_{\tilde{r}}(y) \subset N_r(x)$  so  $y$  is interior.

$\therefore N_r(x)$  is open in  $X$ . □

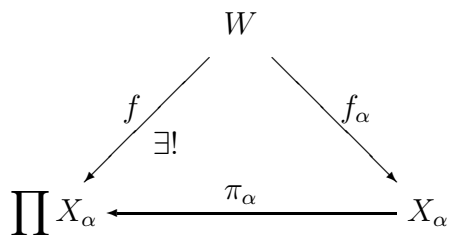
## 2.7 Universal Properties



A set function  $\bar{f}$  making the diagram commute exists iff  $(a \sim b \Rightarrow f(a) = f(b))$

**Proposition 2.7.1**  $\bar{f}$  is cont.  $\Leftrightarrow f$  is cont.

**Proof:** Check definitions. □



A function into a product is determined by its projections onto each component.

**Proposition 2.7.2**  $f$  is continuous  $\Leftrightarrow f_\alpha$  is cont.  $\forall \alpha$

**Proof:**  $\Rightarrow f_\alpha = \pi_\alpha \circ f$  so  $f$  cont.  $\Rightarrow f_\alpha$  cont.  $\Leftarrow$  Suppose  $f_\alpha$  cont.  $\forall \alpha$ .

Let  $V = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \dots, \alpha_n} X_\alpha \in \mathcal{S}$

Then

$$f^{-1}(V) = f^{-1}\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}\pi_{\alpha_n}^{-1}(U_{\alpha_n}) = f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n}) = \text{open}$$

Since  $\mathcal{S}$  is a basis, this implies  $f$  cont. □

## 2.8 Topological Algebraic Structures

**Definition 2.8.1** A topological group consists of a group  $G$  together with a topology on the underlying set  $G$  s.t.

1. multiplication  $G \times G \xrightarrow{\text{mult.}} G$
2. inversion  $G \rightarrow G$

are continuous (using the given topology on the set  $G$  and the product topology on  $G \times G$ )

### Example 2.8.2

1.  $\mathbb{R}^n$  with the standard topology (coming from the standard metric) and  $+$  as the group operation

$(x, y) \mapsto x + y$  is continuous

$x \mapsto -x$  is continuous

2.  $G = S^1 \subset \mathbb{R}^2 = \mathbb{C}$ .

Group operation is multiplication as elements of  $\mathbb{C}$

(a)  $S^1 \times S^1 \rightarrow S^1$

$(e^{it}, e^{iw}) \mapsto e^{i(t+w)}$  is continuous

(b)  $e^{it} \mapsto e^{-it}$  is continuous

Similarly  $G = S^3 \subset \mathbb{R}^4 = \mathbb{H}$

$S^3$  becomes a topological group with multiplication induced from that on quaternions

3.  $G = GL_n(\mathbb{C}) = \{ \text{invertible } n \times n \text{ matrices with entries in } \mathbb{C} \}$

Group operation: matrix multiplication

Topology: subspace topology induced from inclusion into  $\mathbb{C}^{n^2}$  (with standard metric on  $\mathbb{C}^{n^2}$ )

In other words, the topology comes from the metric

$$d(A, B)^2 = \sum_{i,j} |a_{ij} - b_{ij}|^2$$

(a)  $G \times G \xrightarrow{\text{mult.}} G$  is continuous since the entries in the product matrix  $AB$  depend continuously on the entries of  $A$  and  $B$

(b) the inversion map  $G \rightarrow G$  is continuous since there is a formula for the entries of  $A^{-1}$  in terms of entries of  $A$  using only addition, multiplication and division by the determinant.

Similarly  $SL_n(\mathbb{C})$ ,  $U(n)$ ,  $GL_n(\mathbb{R})$ ,  $SL_n(\mathbb{R})$  and  $O(n)$  are topological groups.

4. Let  $G$  be any group topologized with the discrete topology.

**Lemma 2.8.3** *If  $X$  and  $Y$  have discrete topology then the product topology on  $X \times Y$  is also discrete.*

For  $(x, y) \in X \times Y$  the subset consisting of the single element  $(x, y)$  is open (a (finite) product of open sets).

Every set is a union of such open sets so is open.

Hence multiplication and inversion are continuous. (Any function is continuous if the domain has the discrete topology.)

Similarly one can define topological rings, topological vector spaces and so on.

A *topological ring*  $R$  consists of a ring  $R$  with a topology such that addition, inversion and multiplication are continuous.

A *topological vector space* over  $\mathbb{R}$  consists of a vector space  $V$  with a topology such that the following operations are continuous: addition, multiplication by  $-1$  and

$$\mathbb{R} \times V \rightarrow V$$

$t, v \mapsto tv$  where  $\mathbb{R}$  has its standard topology and  $\mathbb{R} \times V$  the product topology.

Exercise: The standard topology on  $\mathbb{R}^n$  is the only one which gives it the structure of a topological vector space over  $\mathbb{R}$ .

## 2.9 Manifolds

A Hausdorff (see Definition 3.0.12) topological space  $M$  is called an  $n$ -dimensional manifold if  $\exists$  a collection of open sets  $U_\alpha \subset M$  such that  $M = \bigcup_{\alpha \in I} U_\alpha$  with each  $U_\alpha$  homeomorphic to  $\mathbb{R}^n$ .

This is usually known as a “topological” manifold. One can also define differentiable or  $C^\infty$  manifold or complex analytic manifold, by requiring the functions giving the homeomorphisms to be differentiable,  $C^\infty$  or complex analytic respectively. (The last concept only makes sense when  $n$  is even.)

**Example 2.9.1**  $S^n$  is an  $n$ -dimensional manifold.

**Lemma 2.9.2**  $S^n \setminus \{\text{pt}\} \cong \mathbb{R}^n$ .

**Proof:** Stereographic projection:

Place the sphere in  $\mathbb{R}^{n+1}$  so that the south pole is located at the origin. Let the missing point be the north pole (or  $N$ ), located at  $(0, \dots, 0, 2)$ . (Note that we also introduce the notation  $S$  for the south pole.)

Define  $f : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  by joining  $N$  to  $x$  and  $f(x)$  be the point where the line meets  $\mathbb{R}^n$  (the plane where the  $z$  coordinate is 0).

Explicitly  $f(x) = x + \lambda(x - a)$  for the right  $\lambda$ .

$$0 = f(x) \cdot a = x \cdot a + \lambda(x - a) \cdot a$$

so

$$\lambda = -\frac{x \cdot a}{(x - a) \cdot a}.$$

Hence  $f(x) = x - \frac{x \cdot a}{(x - a) \cdot a} \cdot (x - a)$ . This is a continuous bijection.

The inverse map  $g : \mathbb{R}^n \rightarrow S^n \setminus \{N\}$  is given by  $y \mapsto$  the point on the line joining  $y$  to  $N$  which lies on  $S^{n+1}$ .

Explicitly,  $g(y) = ty + (1 - t)a$  where  $t$  is chosen s.t.  $\|g(y)\| = 1$ .

Hence  $(ty + (1 - t)a) \cdot (ty + (1 - t)a) = 1$  so

$$t^2\|y\|^2 + 2t(1 - t)y \cdot a + (1 - t)^2\|a\|^2 = 1$$

The solution for  $t$  depends continuously on  $y$ .

Write  $S^n = (S^n \setminus \{N\}) \cup (S^n \setminus \{S\})$  which is a union of open sets homeomorphic to  $\mathbb{R}^n$ . □

**Lemma 2.9.3**  $\forall r > 0$  and  $\forall x \in \mathbb{R}^n$ ,  $N_r(x)$  is homeomorphic to  $\mathbb{R}^n$ .

**Proof:** It is clear that translation gives a homeomorphism  $N_r(x) \cong N_r(0)$  so we may assume  $x = 0$ .

Define  $f : N_r(0) \rightarrow \mathbb{R}^n$  by  $f(y) = \frac{y}{r-||y||}$  and  $g : \mathbb{R}^n \rightarrow N_r(0)$  by  $g(z) = \frac{r}{1+||z||}z$ . It is clear that  $f$  and  $g$  are inverse homeomorphisms.

**Corollary 2.9.4** *Let  $X$  be a topological space having the property that each point in  $X$  has a neighbourhood which is homeomorphic to an open subset of  $\mathbb{R}^n$ . Then  $X$  is a manifold.*

**Proof:** Let  $x \in X$ .  $\exists U_x$  with  $x \in U_x$  and a homeomorphism  $h_x : U_x \rightarrow V$ .

If  $V$  is open,  $\exists r_x$  s.t.  $N_r(h_x(x)) \subset V$ .

The restriction of  $h_x$  to  $h_x^{-1}(N_r(z))$  gives a homeomorphism  $W_x \rightarrow N_r(z)$ .

(By definition of the subspace topology, the restriction of a homeomorphism to any subset is a homeomorphism.)

Hence  $X = \cup_{x \in X} W_x$  and each  $W_x$  is homeomorphic to  $N_{r_x}(Z)$  for some  $Z$  which is in turn homeomorphic to  $\mathbb{R}^n$ .  $\square$

**Example 2.9.5**  $\mathbb{R}P^n$

Let  $\pi : S^n \rightarrow \mathbb{R}P^n$  be the canonical projection.

Let  $x \in S^n$  represent an element of  $\mathbb{R}P^n$ .

Let  $U = \{y \in \mathbb{R}P^n \mid \pi^{-1}(y) \cap N_r(x) \neq \emptyset\}$

$\pi^{-1}(U) = N_r(x) \cup N_r(-x)$  which is open. Hence  $U$  is open in  $\mathbb{R}P^n$  by definition of the quotient topology.

Because  $r < 1/2$ ,  $N_r(x) \cap N_r(-x) = \emptyset$ .

So  $\forall y \in U$   $\pi^{-1}(y)$  consists of two elements, one in  $N_r(x)$  and the other in  $N_r(-x)$ .

Define  $f_x : U \rightarrow N_r(x)$  by  $y \mapsto$  unique element of  $\pi^{-1}(y) \cap N_r(x)$ .

**Claim:**  $f_x$  is a homeomorphism.

**Proof:** For any open set  $V \subset N_r(x)$   $\pi^{-1}f_x^{-1}(V) = V \cup -V$  which is open in  $S^n$ .

Hence  $f_x^{-1}(V)$  is open in  $\mathbb{R}P^n$  by definition of the quotient topology.

Hence  $f_x$  is continuous.

The restriction of  $\pi$  to  $N_r(x)$  gives a continuous inverse to  $f_x$  so  $f_x$  is a homeomorphism.

Let  $h_x : S^n \setminus \{-x\} \rightarrow \mathbb{R}^n$  be a homeomorphism. So  $h_x(N_r(x))$  is open in  $\mathbb{R}^n$ . So we have homeomorphisms

$$U \xrightarrow{f_x} N_r(x) \xrightarrow{h_x|_{N_r(x)}} h_x(N_r(x))$$

giving a homeomorphism from  $U$  to an open subset of  $\mathbb{R}^n$ .

Since every point of  $\mathbb{R}P^n$  is  $\pi(x)$  for some  $x \in S^n$  we have shown that every point of  $\mathbb{R}P^n$  has a neighbourhood homeomorphic to a neighbourhood of  $\mathbb{R}^n$ . So  $\mathbb{R}P^n$  is a manifold by the previous Corollary.  $\square$

**Definition 2.9.6** *A topological group which is also a manifold is called a Lie group.*

**Examples:**  $\mathbb{R}^n$ ,  $S^1$ ,  $S^3$ ,  $GL_n(\mathbb{R})$ .

To check the last example, we must show  $GL_n(\mathbb{R})$  is a manifold.

Since the topology on  $GL_n(\mathbb{R})$  is that as a subspace of  $\mathbb{R}^{n^2}$ , by Corollary 2.9.4 it suffices to show that  $GL_n(\mathbb{R})$  is an open subset of  $\mathbb{R}^{n^2}$ .

Let  $M_n(\mathbb{R}) = \{n \times n \text{ matrices over } \mathbb{R}\}$  with topology coming from the identification of  $M_n(\mathbb{R})$  with  $\mathbb{R}^{n^2}$ .

So by construction  $M_n(\mathbb{R})$  is homeomorphic to  $\mathbb{R}^{n^2}$ .

$\det : M_n(\mathbb{R}) \rightarrow \mathbb{R}$  is continuous (it is a polynomial in the entries of  $A$ ).

$$\det : A \mapsto \det A$$

$$GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$$

0 is closed in  $\mathbb{R}$  so  $\mathbb{R} \setminus \{0\}$  is open. Hence  $GL(n, \mathbb{R})$  is open in  $\mathbb{R}^{n^2}$ . □