9.2 Eilenberg-Steenrod Homology Axioms

Historically:

1. Simplical homology was defined for simplicial complexes.

2. It was proved that the homology groups of a simplicial complex depend only on its geometric realization, not upon the actual triangulation.

3. Various other "homology theories" were defined on various subcategories of topological spaces. (e.g. singular homology, de Rham (co)homology, Čech homology, cellular homology,...) The subcollection of spaces on which each was defined was different, but they had similar properties, were all defined for polyhedra (i.e. realizations of finite simplicial complexes) and furthermore gave the same groups $H_*(X)$ for a polyhedron X.

4. Eilenberg and Steenrod formally defined the concept of a "homology theory" by giving a set of axioms which a homology theory should satisfy. They proved that if X is a polyhedron then any theory satisfying the axioms gives the same groups for $H_*(X)$.

Definition 9.2.1 (Eilenberg-Steenrod) *Let $\mathcal{A}$ be a class of topological pairs such that:*

1) (X, A) *in* $\mathcal{A} \Rightarrow (X, X)$, $(X, \emptyset)$, (A, A), $(A, \emptyset)$, *and* $(X \times I, A \times I)$ *are in* $\mathcal{A}$*;*

2) $(*, \emptyset)$ *is in* $\mathcal{A}$ *(where* $*$ *denotes a space with one point).*

A homology theory *on* $\mathcal{A}$ *consists of:*

E1) an abelian group $H_n(X, A)$ *for each pair* (X, A) *in* $\mathcal{A}$ *and each integer* n*;*

E2) a homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$ *for each map of pairs* $f : (X, A) \to (Y, B)$*;*

E3) a homomorphism $\partial : H_n(X, A) \to H_{n-1}(A)$ *for each integer* n *(where* $H_n(A)$ *is an abbreviation for* $H_n(A, \emptyset)$ *),*

such that:

A1) $1_* = 1$*;*

A2) $(gf)_* = g_* f_*$*;*

A3) ∂ *is natural. That is, given* $f : (X, A) \to (Y, B)$*, the diagram*

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{f_*} & H_n(Y,B) \\ \downarrow \partial & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow{(f|_A)_*} & H_{n-1}(B) \end{array}$$

commutes;

A4) *Exactness:*

$$\ldots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \longrightarrow \ldots$$

is exact for every pair (X, A) *in* $\mathcal{A}$*, where* $H_*(A) \to H_*(X)$ *and* $H_*(X) \to H_*(X, A)$ *are induced by the inclusion maps* $(A, \emptyset) \to (X, \emptyset)$ *and* $(X, \emptyset) \to (X, A)$*;*

A5) *Homotopy:* $f \simeq g \Rightarrow f_* = g_*$.

A6) *Excision: If* (X, A) *is in* $\mathcal{A}$ *and* U *is an open subset of* X *such that* $\overline{U} \subset \mathring{A}$ *and* $(X \smallsetminus U, A \smallsetminus U)$ *is in* $\mathcal{A}$ *then the inclusion map* $(X \smallsetminus U, A \smallsetminus U) \to (X, A)$ *induces an isomorphism* $H_n(X \smallsetminus U, A \smallsetminus U) \xrightarrow{\cong} H_n(X, A)$ *for all* n*;*

A7) *Dimension:* $H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases}$

A8) *For each* $\alpha \in H_n(X, A)$ *there exists a pair of compact subspaces* (X_0, A_0) *in* $\mathcal{A}$ *such that* $\alpha \in \operatorname{Im} j_*$*, where* $j : (X_0, A_0) \to (X, A)$ *is the inclusion map.*

Remark 9.2.2

1. *The inclusion of the* 8th *axiom is not completely standard. Some people would call anything satisfying the* 1st 7 *axioms a homology theory.*

2. A1 *and* A2 *simply say that* $H_n(\ )$ *is a functor for each* n*.*

Remark 9.2.3 *Under the presence of the other axiom, the excision is equivalent to the Mayer-Vietoris property, stated below as Theorem 9.4.15 and to the Suspension property, stated below as Theorem 9.5.6.*

9.3 Singular Homology Theory

Definition 9.3.1 *A set of points* $\{a_0, a_1, \ldots, a_n\} \in \mathbb{R}^N$ *is called* geometrically independent *if the set*

$$\{a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0\}$$

is linearly independent.

Proposition 9.3.2 $a_0, \ldots, a_n$ *geometrically independent if and only if the following statement holds:* $\sum_{t=0}^{n} t_i a_i = 0$ *and* $\sum_{t=0}^{n} t_i = 0$ *implies* $t_i = 0$ *for all* i.

Proof: Exercise □

Definition 9.3.3 *Let* $\{a_0, \ldots, a_n\}$ *be geometrically independent. The* n-simplex σ *spanned by* $\{a_0, \ldots, a_n\}$ *is the convex hull of* $\{a_0, \ldots, a_n\}$. *Explicitly*
$\sigma = \{x \in \mathbb{R}^n \mid x = \sum_{i=0}^{n} t_i a_i$ *where* $t_i \geq 0$ *and* $\sum t_i = 1\}$.

For a given n-simplex σ, each $x \in \sigma$ has a unique expression $x = \sum_{i=0}^{n} t_i a_i$ with $t_i \geq 0$ and $\sum t_i = 1$. The t_i's are called the *barycentric coordinates* of x (with respect to $a_0, \ldots, a_n$). The *barycentre* of the n-simplex is the point all of whose barycentric coordinates are $1/(n+1)$.

$a_0, \ldots, a_n$ are called the *vertices* of σ.

n is called the *dimension* of σ.

Any simplex formed by a subset of $\{a_0, \ldots, a_n\}$ is called a *face* of σ.

Special case:

$a_0 = \epsilon_0 := (0, 0, \ldots, 0)$, $a_1 = \epsilon_1 := (1, 0, \ldots, 0)$, $a_2 = \epsilon_2 := (0, 1, 0, \ldots, 0)$, $a_n = \epsilon_n := (0, 0, \ldots, 0, 1)$ in $\mathbb{R}^n$ gives what is known as the *standard n-simplex*, denoted Δ^n.

Definition 9.3.4 *Suppose* $A \subset \mathbb{R}^m$ *is convex. A function* $f : A \to \mathbb{R}^k$ *is called* affine *if* $f\big(ta + (1-t)b\big) = tf(a) + (1-t)f(b)$ $\forall a, b \in \mathbb{R}^m$ *and* $0 \leq t \leq 1 \in \mathbb{R}$.

Let σ be an n-simplex with vertices $v_0, \ldots, v_n$. Given $(n+1)$ points $p_0, \ldots, p_n$ in $\mathbb{R}^k$, $\exists!$ affine map f taking v_j to p_j.
Note: $p_0, \ldots, p_n$ need not be geometrically independent.

Notation: Given $a_0, \ldots, a_n \in \mathbb{R}^N$, let $l(a_0, \ldots, a_n)$ denote the unique affine map taking e_j to a_j. Explictly, $l(a_0, \ldots, a_n)(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n}(a_i - a_0)x_i$

Note: $l(\epsilon_0, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_n)$ is the inclusion of the (ith face of Δ^n) into Δ^n.

Definition 9.3.5 *Given a topological space* X, *a continuous function* $f : \Delta^p \to X$ *is called a* singular p-simplex *of* X.

Let $S_p(X) :=$ free abelian group on {singular p-simplices of X}.
Wish to define a boundary map making $S_p(X)$ into a chain complex.
Given a singular p-simplex T, can define $(p-1)$-simplices by the compositions

$$\Delta^{p-1} \xrightarrow{l(\epsilon_0,\ldots,\hat{\epsilon_i},\ldots,\epsilon_n)} \Delta^p \xrightarrow{T} X.$$

A homomorphism from a free group is uniquely determined by its effect on generators.
Define homomorphism
$\partial : S_p(X) \to S_{p-1}(X)$ by $\partial(T) := \sum_{i=0}^{p} (-1)^i T \circ l(\epsilon_0, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_n)$.

Given $g : X \to Y$, define homomorphism $g_* : S_p(X) \to S_p(Y)$ by defining it on generators by $g_*(T) := g \circ T$. $\quad \Delta^p \xrightarrow{T} X \xrightarrow{g} Y$

Lemma 9.3.6 $g_*\partial = \partial g_*$

(Thus after we show $S_p(X)$, $S_p(Y)$ are chain complexes, we will know that g_* is a chain map.)

Proof: Sufficient to check $g_*\partial(T) = \partial g_*(T)$ $\forall T$. (Exercise: Essentially, left multiplication commutes with right multiplication.) □

Lemma 9.3.7 *$S_*(X)$ is a chain complex. (i.e. $\partial^2 = 0$)*

Proof:
Special Case: $X = \sigma$ spanned by $a_0, \ldots, a_p$ and $T = l(a_0, \ldots, a_p)$.

Then

$$\begin{aligned} \partial T &= \partial l(a_0, \ldots, a_p) \\ &= \textstyle\sum_{j=0}^{p} (-1)^j l(a_0, \ldots, a_p) \circ l(\epsilon_0, \ldots, \hat{\epsilon_j}, \ldots, \epsilon_p) \\ &= \textstyle\sum_{j=0}^{p} (-1)^j l(a_0, \ldots, \hat{a}_j, \ldots, a_p) \end{aligned}$$

Therefore

$$\begin{aligned} \partial^2 T &= \textstyle\sum_{j=0}^{p} (-1)^j \partial l(a_0, \ldots, \hat{a}_j, \ldots a_p) \\ &= \textstyle\sum_{j=0}^{p} (-1)^j \Big(\sum_{i<j} (-1)^i l(a_0, \ldots, \hat{a}_i \ldots, \hat{a}_j, \ldots, a_p) \\ &\qquad + \textstyle\sum_{i>j} (-1)^{i-1} l(a_0, \ldots, \hat{a}_i \ldots, \hat{a}_j, \ldots, a_p) \Big) \\ &\quad \text{(Note: removal of } a_j \text{ moves } a_i \text{ to } (i-1)st \text{ position)} \\ &= 0 \end{aligned}$$

since each term appears twice (once with $i < j$ and once with $j < i$) with opposite signs so they cancel.

General Case: $f : \Delta^p \to X$. Let $I = 1_{\Delta^p} = l(\epsilon_0, \ldots, \epsilon_p) \in S_p(\Delta^p)$. Then $f = f_*(I) \in S_p(X)$. So $\partial^2 f = f_*(\partial^2 I) \overset{\text{(special case)}}{=\!=\!=} f_*(0) = 0$. □

Corollary 9.3.8 (Corollary of previous Lemma)
$g : X \to Y$ implies $g_* : S_*(X) \to S_*(Y)$ is a chain map. □

Definition 9.3.9 *$H_*\big(S_*(X), \partial\big)$ is denoted $H_*(X)$ and called the* singular homology *of the space X.*

Proposition 9.3.10 *Singular homology is a functor from the category of topological spaces to the category of abelian groups.*

Proof: Requirements are $1_* = 1$ and $(gf)_* = g_* f_*$. Both are trivial. □

Corollary 9.3.11 *If $f : X \to Y$ is a homeomorphism then f_* is an isomorphism.* □

Let A be a subspace of X with inclusion map $j : A \hookrightarrow X$. Then $j_* : S_*(A) \to S_*(X)$ is an inclusion ($S_*(X)$ is the free abelian group on a larger set — in general strictly larger since not all functions into X factor through A) so can form the quotient complex $S_*(X)/S_*(A)$ (strictly speaking the denominator is $j_*\big(S_*(A)\big)$).

Definition 9.3.12 *$H_*\big(S_*(X)/S_*(A)\big)$ is written $H_*(X, A)$ and is called the relative homology of the pair (X, A).*

Notice, if $A = \emptyset$ then $S_*(A) = \text{Free-Abelian-Group}(\emptyset) = 0$ so $H_*(X, \emptyset) = H_*(X)$.

9.3.1 Verification that Singular Homology is a Homology Theory

A pair (X, A) gives rise to a short exact sequence of chain complexes:

$$0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) \to 0$$

in such a way that a map of pairs $(X, A) \to (Y, B)$ gives a commuting diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_*(B) & \longrightarrow & S_*(Y) & \longrightarrow & S_*(Y)/S_*(B) & \longrightarrow & 0
\end{array}$$

It follows from the homological algebra section that there are induced long exact homology sequences

$$\begin{array}{ccccccccccccc}
\dots & \xrightarrow{\partial} & H_p(A) & \longrightarrow & H_p(X) & \longrightarrow & H_p(X,A) & \xrightarrow{\partial} & H_{p-1}(A) & \longrightarrow & H_{p-1}(X) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{\partial} & H_p(A) & \longrightarrow & H_p(X) & \longrightarrow & H_p(X,A) & \xrightarrow{\partial} & H_{p-1}(A) & \longrightarrow & H_{p-1}(X) & \longrightarrow & \dots
\end{array}$$

making the squares commute.

This in the definition of a homology theory we immediately have the following: E1, E2, E3, A1, A2, A3, A4.

Proposition 9.3.13 *A7 is satisfied.*

Proof: By definition, if $p \geq 0$,

$S_p(*) = \text{Free-Abelian-Group}\big(\{\text{maps from } \Delta^p \text{ to } *\}\big) = \mathbb{Z}$,

generated by T_p where T_p is the unique continuous map from Δ^p to $*$.

$\partial T_p = \sum_{i=0}^{p}(-1)^i T_p \circ l(\epsilon_0, \dots, \hat{\epsilon_i}, \dots, \epsilon_p)$.

For $p > 0$, $T_p \circ l(\epsilon_0, \dots, \hat{\epsilon_i}, \dots, \epsilon_p) = T_{p-1}\ \forall i$, so $\partial T_p = \begin{cases} T_{p-1} & p \text{ even;} \\ 0 & p \text{ odd.} \end{cases}$ □

Proposition 9.3.14 *A8 is satisfied.*

Proof: Let $\alpha \in H_p(X, A)$. So α is represented by a cycle of $S_p(X)/S_p(A)$ for which we choose a representative $c = \sum_{i-1}^{k} n_i T_i \in S_p(X)$. Thus $\partial c = \sum_{i=1}^{r} m_i V_i \in S_p(A)$.

Let $X_0 = \big(\cup_{i=1}^{k} \operatorname{Im} T_i\big) \cup (\cup_{i=1}^{r} \operatorname{Im} V_i)$ and let $A_0 = (\cup_{i=1}^{r} \operatorname{Im} V_i)$.

Since $T_i : \Delta^p \to X$ and $V_i : \Delta^{p-1} \to A \hookrightarrow X$, each of X_0 and A_0 are a finite union of compact sets and thus compact. It is immediate from the definitions that $\alpha \in \operatorname{Im} j_* : H_*(X_0, A_0) \to H_*(X, A)$ where $j : (X_0, A_0 \hookrightarrow (X, A)$ is the inclusion map, since the chain representing α exists back in $S_*(X_0)/S_*(A_0)$. □

Theorem 9.3.15 $H_0(X) \cong F_{\text{ab}}(\{\text{path components of } X\})$.

Proof: $S_0(X) = F_{\text{ab}}(\{\text{singular 0-simplices of } X\})$.

$S_1(X)$ is generated by maps $f : I = \Delta^1 \to X$.

$\partial f = f(1) - f(0)$. Hence $\operatorname{Im}\partial = \{f(1) - f(0) \mid f : I \to X\}$.
Therefore

$$\begin{aligned} H_0(X) &= \ker\partial_0/\operatorname{Im}\partial_0 = S_0(X)/\operatorname{Im}\partial_1 \\ &= F_{\text{ab}}(\text{points of } X)/\!\sim \qquad \text{where } f(1) - f(0) \sim 0 \ \forall f : I \to X \\ &\cong F_{\text{ab}}(\{\text{path components of } X\}). \end{aligned}$$

□

9.3.2 Reduced Singular Homology

Define the "augmentation map" $\epsilon : S_0(X) \to \mathbb{Z}$ by $\epsilon(\sum_{i \in I} n_i x_i) = \sum_{i \in I} n_i$.

If f is a generator of $S_1(X)$ with $f(0) = x$ and $f(1) = y$ then $\partial f = y - x$ so $\epsilon\partial f = 0$.

$$\begin{array}{ccccccccccccc} \to & S_p(X) & \xrightarrow{\partial} & S_{p-1}(X) & \to & \dots & \to & S_1(X) & \xrightarrow{\partial} & S_0(X) & \to & 0 & \to \\ & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle\epsilon} & & \downarrow & \\ \to & 0 & \to & 0 & \to & \dots & \to & 0 & \to & \mathbb{Z} & \to & 0 & \to \end{array}$$

commutes.

The chain complex formed by taking termwise kernels of this chain map is denoted $\tilde{S}_*(X)$ and its homology, denote $\tilde{H}_*(X)$, is called the *reduced homology* of X.

The short exact sequence of chain complexes defining $\tilde{S}_*(X)$ yields a long exact sequence

$$0 \to \tilde{H}_p(X) \to H_p(X) \to 0 \to \dots \to 0 \to \tilde{H}_1(X) \to H_1(X) \to 0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Therefore $H_n(X) \cong \begin{cases} \tilde{H}_n(X) & n > 0; \\ \tilde{H}_0(X) \oplus \mathbb{Z} & n = 0. \end{cases}$

Consider the special case $X = *$.

$$\begin{array}{lccccccccccc} S_*(*) & \to & \mathbb{Z} & \xrightarrow{\cong} & Z & \to & \dots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle\epsilon} & & \downarrow \\ & \to & 0 & \to & 0 & \to & \dots & \to & 0 & \to & \mathbb{Z} & \to & 0 \end{array}$$

In this case ϵ becomes the identity map so that $\epsilon_* : H_0(*) \to \mathbb{Z}$ is an isomorphism. (We already knew $H_*(X) \cong \mathbb{Z}$; just want to check that ϵ_* gives the isomorphism.)

Theorem 9.3.16 $\tilde{H}_*(X) \cong H_*(X, *)$.

Proof: We have a long exact sequence

$$\begin{array}{ccccccccccc} 0 & & & & & & 0 & & & & 0 \\ \| & & & & & & \| & & & & \| \\ H_n(*) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,*) & \longrightarrow & H_{n-1}(*) & \longrightarrow & \dots & \longrightarrow & H_1(*) \longrightarrow \end{array}$$

$$\longrightarrow H_1(X) \longrightarrow H_1(X,*) \xrightarrow{\partial} H_0(*) \xrightarrow{i} H_0(X) \longrightarrow H_n(X,*) \longrightarrow 0$$

Let $f : X \to *$.

$$\begin{array}{ccccc} H_0(*) & \xrightarrow{i} & H_0(X) & & \\ & \overset{1}{\searrow} & \downarrow f_* & \overset{\epsilon_*}{\searrow} & \\ & & H_0(*) & \xrightarrow{\epsilon_*} & \mathbb{Z} \end{array}$$

Therefore $\epsilon_* i = \epsilon_*$ is an isomorphism so i is an injection. It follows algebraically that $\partial = 0$ and that the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(*) & \longrightarrow & H_0(X) & \longrightarrow & H_0(X,*) & \longrightarrow & 0 \\ & & & \overset{\cong}{\searrow} & \downarrow \epsilon & & & & \\ & & & & \mathbb{Z} & & & & \end{array}$$

splits and $\tilde{H}_0(X) = \ker \epsilon \cong \operatorname{coker} i \cong H_0(X*)$. □

Theorem 9.3.17 *Let $X \subset \mathbb{R}^N$ be convex. Then $\tilde{H}_*(X) = 0$.*

Proof: Let $w \in X$ be any point. Define a homomorphism $S_p(X) \to S_{p-1}(X)$ by defining it on generators as follows.

Let $T : \Delta^p \to X$ be a generator of $S_p(X)$.

To define $\phi(T) \in S_{p+1}(X)$: Let $\phi(T) : \Delta^{p+1} \to X$ be the generator of $S_{p+1}(X)$ defined as follows: Given $y \in \Delta^{p+1}$ we can write $y = t\epsilon_p + (1-t)z$ for some $z \in \Delta^p$, $t \in [0,1]$ (where $\epsilon_p = (0, \dots, 0, 1)$). Let $\phi(T)(y) = tw + (1-t)T(z)$.

Lemma 9.3.18 *Let* $c \in S_p(X)$. *Then* $\partial\big(\phi(c)\big) = \begin{cases} \phi(\partial c) + (-1)^{p+1}c & p > 0 \\ \epsilon(c)T_w - c & p = 0 \end{cases}$
where $T_w : \Delta^0 \to X$ *by* $T_x(*) = w$.

Proof: It suffices to check this when c is a generator. Let $T : \Delta^p \to X$ be a generator of $S_p(X)$.
If $p = 0$:
$\phi(T)$ is a line joining $T(*)$ to w so $\partial\big(\phi(T)\big) = T_w - T = \epsilon(T)T_w - T$ as required.
If $p > 0$:
$\partial\big(\phi(T)\big) = \sum_{i=0}^{p+1}(-1)^i\phi(T) \circ l_i$ where l_i is short for $l(\epsilon_0, \ldots \hat{\epsilon}_i, \ldots, \epsilon_p)$.
If $i = p+1$, l_i is the inclusion of Δ^p into Δ^{p+1} so $\phi(T) \circ l_p = \phi \circ T\big|_{\Delta^p} = T$.
If $i \leq p$, $\phi(T) \circ l_i = \phi\big(T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\big)$, extended by sending the last vertex to w.
Therefore

$$\begin{aligned} \partial\big(\phi(T)\big) &= \textstyle\sum_{i=0}^{p}(-1)^i\phi\big(T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\big) + (-1)^{p+1}T \\ &= \phi\big(\textstyle\sum_{i=0}^{p}(-1)^i T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\big) + (-1)^{p+1}T \\ &= \phi(\partial T) + (-1)^{p+1}T \end{aligned}$$

Proof of Theorem (cont.)
$p = 0$:
Suppose $c \in \tilde{S}_0(X)$. So $\epsilon(c) = 0$.
$\partial\big(\phi(c)\big) = 0 - c$ so $[c] = 0 \in \tilde{H}_0(X)$.
$p > 0$:
Let $c \in Z_p(X)$.
$\partial\big(\phi(c)\big) = \phi(\partial c) + (-1)^{p+1}c = \phi(0) + (-1)^{p+1}c = (-1)^{p+1}c$.
Therefore $[c] = 0$ in $H_p(X) = \tilde{H}_p(X)$. ◻

Corollary 9.3.19 $\tilde{H}_p(\Delta^n) = 0 \ \forall p$. ◻

9.4 Proof that A5 is satisfied: Acyclic Models

Let $f, g : X \to Y$ s.t. $f \overset{H}{\simeq} g$.

$$X \underset{j}{\overset{i}{\rightrightarrows}} X \times I \xrightarrow{H} Y \text{ where } i(x) = (x, 0),\ j(x) = (x, 1).$$

Then $H\circ = f$ and $H \circ j = g$. Therefore $f_* = H_* \circ i_*$ and $g_* = H_* \circ j_*$. Show to show $f_* = g_*$ it suffices to show that $i_* = j_*$.
We show this by showing that at the chain level $i_* \simeq j_* : S_*(X) \to S_*(X \times I)$.

We will show that $i_* \simeq j_*$ by "acyclic models".

Intuitively, acyclic models is a method of inductively constructing chain homotopies which makes use of the fact that in an acyclic space equations of the form $\partial x = y$ can always be "solved" for x provided $\partial y = 0$. (In general there will be many choices for the solution x.) The method does not give an explicit formula for the chain homotopy but merely proves that one exists. In fact, the final result is non-canonical and depends upon the choices of the solutions. In the case of chain homotopy $i_* \simeq j_*$ which we are considering at present, it would be possible to directly write down a chain homotopy and check that it works without using acyclic models. However we will need the method in other places where it would not be so easy to simply write down the formula so we introduce it here.

The acyclic spaces ("models") used in this particular application of the method are the spaces Δ^n. Intuitively we make used of the fact that equations can be solved in Δ^n to solve the same equations in $S_*(X)$ using that elements in $S_*(X)$ are formed from maps $\Delta^n \to X$.

Lemma 9.4.1 *$\exists$ a natural chain homotopy $D_X : i \simeq j : S_*(X) \to S_*(X \times I)$.*

In more detail:

1. *$\forall x$ and $\forall p$, $\exists D_X : S_p(X) \to S_{p+1}(X \times I)$ s.t. $\forall c \in S_p(X)$,*
   $\partial D_X c + D_X \partial c = j_*(c) - i_*(c)$.

2. *$\forall f : X \to Y$,*

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \\ \Big\downarrow f_* & & \Big\downarrow (f \times 1)_* \\ S_p(Y) & \xrightarrow{D_Y} & S_{p+1}(Y \times I) \end{array}$$

   commutes.

Proof: Since $S_p(X)$ is a free abelian group it suffices to define D_X on generators and check its properties on them.

If $p < 0$, $S_p(X) = 0$ so $D_X = 0$-map.

Continue constructing D_X inductively. The induction assumptions are for <u>all</u> spaces. More precisely:

Induction Hypothesis: $\exists$ integer p such that for all $k < p$ and $\forall X$ we have constructed homomorphisms $D_X : S_k(X) \to S_{k+1}(X \times I)$ s.t. $\forall c \in S_k(C)$

1. $\forall x$ and $\forall p$, $\exists D_X : S_p(X) \to S_{p+1}(X \times I)$ s.t. $\forall c \in S_p(X)$,
   $\partial D_X c + D_X \partial c = j_X * (c) - i_X * (c)$.

2. $\forall f : X \to Y$,

$$\begin{array}{ccc} S_k(X) & \xrightarrow{D_X} & S_{k+1}(X \times I) \\ \Big\downarrow f_* & & \Big\downarrow (f \times 1)_* \\ S_k(Y) & \xrightarrow{D_Y} & S_{k+1}(Y \times I) \end{array}$$

   commutes.

(We have this initially for $p = 0$.)

To construct $D_X : S_p(X) \to S_{p+1}(X \times I)$ for any X, consider first the special case ("model case"):

Let $X = \Delta^p$ and let $\iota_p = 1_{\Delta^p} \in S_p(\Delta^p)$.

$i, j : \Delta^p \to \Delta^p \times I$.

Want to define $D_{\Delta^p}(\iota_p)$ so that $\partial D_{\Delta^p}(\iota_p) = j_*(\iota_p) - i_*(\iota_p) - D_{\Delta^p}(\partial \iota_p)$.

That is, solve the equation $\partial x = j_*(\iota_p) - i_*(\iota_p) - D_{\Delta^p}(\partial \iota_p)$ for x and set $D_{\Delta^p(\iota_p)} :=$ solution.

Since $\Delta^p \times I$ is acyclic, solving the equation is equivalent (except when $p = 0$: see below) to checking $\partial(RHS) = 0$.

$$\begin{aligned} \partial(\text{RHS}) &= \partial j_*(\iota_p) - i_*(\iota_p) - D_{\Delta^p}(\partial \iota_p) \\ &\overset{\text{(chain maps)}}{=\!=\!=} j_*(\partial \iota_p) - i_*(\partial \iota_p) - \partial D_{\Delta^p}(\partial \iota_p) \\ &\overset{\text{(induction)}}{=\!=\!=} \partial J_*(\iota_p) - \partial i_*(\iota_p) - (j_* \partial \iota_p - i_* i_* \partial \iota_p - D_{\Delta^p} \partial \partial \iota_p) \\ &= 0. \end{aligned}$$

Hence $\exists$ solution. Choose any solution and define $D_{\Delta^p}(*\iota_p) =$ solution.

Must do the case $p = 0$ separately, since $H_0(\Delta^0 \times I) \neq 0$. For the generator $1_{\Delta^0} : \Delta^0 = * \to *$, set $D_{\Delta^0}(x) := 1_I \in S_1(I = \Delta^0 \times I) = \text{Hom}(\Delta^1, I) = \text{Hom}(I, I)$. Then $\partial D_{\Delta^0}(x) := \partial 1_I = j_*(*) - i_*(*)$ as desired.

Note: We could have avoided doing $p = 0$ separately by writing our argument using reduced homology.

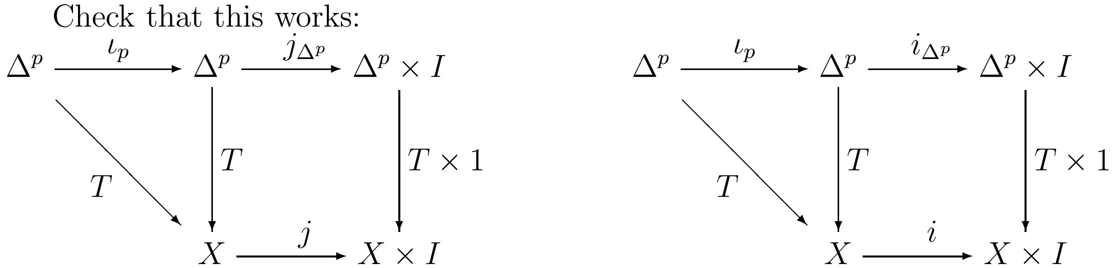
Now to define $S_p(X) \to S_{p+1}(X)$ in general:

Let $T : \Delta^p \to X$ be a generator of $S_p(X)$. Define $D_X(T)$ in the only possible such that (2) is satisfied. That is, want

$$\begin{array}{ccc} S_p(\Delta^p) & \xrightarrow{D_{\Delta^p}} & S_{p+1}(\Delta^p \times I) \\ \downarrow T_* & & \downarrow (T \times 1)_* \\ S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \end{array}$$

Observe that $T = T_*(\iota_p) \in S_p(X)$ so we are forced to define $D_X(T)$ by $D_X(T) := (T \times 1)_* D_{\Delta^p}\iota_0$.

Check that this works:



$$\begin{aligned}
\partial D_X T &= \partial (T \times 1)_* D_{\Delta^p}\iota_p \\
&= (T \times 1)_* \partial D_{\Delta^p}\iota_p \\
&= (T \times 1)_*(j_*\iota_p - i_*\iota_p - D_{\Delta^p}\partial\iota_p) \\
&= (T \times 1 \circ j)_*\iota_p - (T \times 1 \circ i)_*\iota_p - (T \times 1)_* D_{\Delta^p}\partial\iota_p \\
&= j_*(T) - i_*(T) - (T \times 1)_* D_{\Delta^p}\partial\iota_p \\
&\overset{\text{((2) of induction hypothesis)}}{=\!=\!=\!=\!=} j_*(T) - i_*(T) - D_X T_*(\partial\iota_p) \\
&\overset{(T_* \text{ is a chain map})}{=\!=\!=\!=} j_*(T) - i_*(T) - D_X \partial T_*\iota_p \\
&= j_*(T) - i_*(T) - D_X\partial T
\end{aligned}$$

Also, if $f : X \to Y$ then

$$(f \times 1)_* D_X(T) \overset{\text{(defn)}}{=} (f \times 1)_*(T \times 1)_* D_{\Delta^p}\iota_p = \big((f \circ T) \times 1\big)_* D_{\Delta^p}\iota_p \overset{\text{(defn)}}{=} D_Y(f \circ T) = D_Y(f_* T).$$

This competes the induction step and proves the lemma. □

Theorem 9.4.2 *Singular homology satisfies A5.*

Proof: Let $f, g : (X, A) \to (Y, B)$ s.t. $f \simeq g$.

Then $\exists F : X \times I \to Y$ s.t. $F : f \simeq g$ and $F\big|_{A\times I} : f\big|_A \to g\big|_A$. That is, $(X, A) \overset{i}{\underset{j}{\rightrightarrows}} (X \times I, A \times I) \xrightarrow{F} (Y, B)$ where $i(x) = (x, 0)$, $j(x) = (x, 1)$, $F \circ i = f$, $F \circ j = g$. Therefore, to show $f_* = g_*$ it suffices to show $i_* = j_*$.

By (2) of the lemma, the restriction of D_X to A equals D_A. (since the diagram commutes and $S_*(A) \to S_*(X)$ is a monomorphism. Thus there is an induced homomorphism on the relative chain groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_p(A) & \longrightarrow & S_p(X) & \longrightarrow & S_p(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle D_A} & & \downarrow{\scriptstyle D_X} & & \downarrow{\scriptstyle D_{X,A}} & & \\
0 & \longrightarrow & S_{p+1}(A) & \longrightarrow & S_{p+1}(X) & \longrightarrow & S_{p+1}(X, A) & \longrightarrow & 0
\end{array}$$

with $D_{X,A}$ a chain homotopy between i_* and j_*. Hence $i_* = j_*$ and so $f_* = g_*$. □

9.4.1 Barycentric Subdivision

(to prepare for excision:)

Definition 9.4.3 *Let σ be a (geometric) p-simplex spanned by $p + 1$ geometrically independent points $v_0, \ldots, v_p$. The* barycenter *of σ, denoted $\hat{\sigma}$ is defined by $\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{p+1} v_i$.*

(This is, the unique point all of whose barycentric coordinates are equal)

$\hat{\sigma}$ = centroid of σ.

Define the *barycentric subdivision* $\operatorname{sd}\sigma$ of a simplex as follows.

Join $\hat{\sigma}$ to the barycenter of each face of σ to get $\operatorname{sd}\sigma$. (This includes joining $\hat{\sigma}$ to each vertex since vertices are faces and are their own barycenters.)

$\operatorname{sd}\sigma$ writes σ as a union of p-simplices.

Can then perform barycentric subdivision on each of these to get $\operatorname{sd}^2\sigma$ and so on.
Notation: $\tau \prec \sigma$ shall mean: τ is a face of σ.

Lemma 9.4.4 *Every p-simplex of $\operatorname{sd}\sigma$ is spanned by vertices $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_p$ where $\sigma_0 \prec \sigma_1 \prec \cdots \sigma_p$.*

Proof: By induction on $\dim\sigma$.

True if $\dim\sigma = 0$.

Observe: $\operatorname{sd}\sigma$ is formed by forming sd(Boundary σ) and then joining $\hat{\sigma}$ to each vertex in sd(Boundary σ). Thus, of the $(p+1)$ vertices spanning a simplex τ in $\operatorname{sd}\sigma$, p of them span a simplex τ' in Boundary σ and the last is $\hat{\sigma}$. By induction, τ' is spanned by $\hat{\sigma_0}$, $\hat{\sigma_1}$, …, $\widehat{\sigma_{p-1}}$ where $\sigma_0 \prec \sigma_1 \prec \cdots \sigma_{p-1}$ and so τ has the desired form with $\sigma_p = \hat{\sigma}$.

Lemma 9.4.5 *Let σ be a p-simplex and let d be any metric on σ which gives it the standard topology. Then $\forall \epsilon > 0$, $\exists N$ s.t. the diameter or each simplex of $\operatorname{sd}^N \sigma$ is less than ϵ.*

Proof:

Step 0: If true for one metric than true for any metric.

Proof:

Let d_1, d_2 be metrics on σ each giving the correct topology. Then $1 : \sigma \to \sigma$ is a homeomorphism so continuous and thus uniformly continuous by compactness of σ. Therefore, given ϵ, $\exists \delta > 0$ s.t. any set with d_1-diameter less than δ has d_2-diameter less then ϵ. Thus if the theorem holds for d_1 then it holds for d_2 also.

For the rest of the proof use the metric on $\mathbb{R}$ given by $d(x, y) = \max_{i=1,\ldots,N} |x_i - y_i|$, which yields the same topology as the standard one. Notice that in this metric:

1. $d(x, y) = d(x - a, y - a)$
2. $d(0, nx) = nd(0, x)$
3. $d(0, x + y) \le d(0, x) + d(x, x + y) = d(0, x) + d(0, y)$
4. For a p-simplex τ spanned by v_0, …, v_p, $\operatorname{diam}(\tau) = \max\{d(v_i, v_j)\}$

Step 1: If $\dim \sigma = p$ then $\forall z \in \sigma$, $d(z, \hat{\sigma}) \le \frac{p}{p+1} \operatorname{diam} \sigma$.

Proof:

First consider the special case $z = v_0$.

$$
\begin{aligned}
d(v_0, \hat{\sigma}) &= d\left(v_0, \sum_{i=0}^{p} \frac{v_i}{p+1}\right) \\
&= d\left(0, \sum_{i=0}^{p} \frac{v_i - v_0}{p+1}\right) \\
&= \frac{1}{p+1}\, d\left(0, \sum_{i=0}^{p}(v_i - v_0)\right) \\
&= \frac{1}{p+1}\, d\left(0, \sum_{i=1}^{p}(v_i - v_0)\right) \\
&\leq \sum_{i=1}^{p} \frac{1}{p+1}\, d(0, v_i - v_0) \\
&= \sum_{i=1}^{p} \frac{1}{p+1}\, d(v_0, v_i) \\
&\leq \sum_{i=1}^{p} \frac{1}{p+1} \operatorname{diam} \sigma \\
&= \frac{p}{p+1} \operatorname{diam} \sigma.
\end{aligned}
$$

Similarly $d(v_j \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$ $\forall$ vertices of σ. Therefore the closed ball $B_{\frac{p}{p+1} \operatorname{diam} \sigma}[\hat{\sigma}]$ contains all vertices of σ so, being convex it contains all of σ. Hence $d(z, \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$ $\forall z \in \sigma$.

Step 2: For any simplex τ of $\operatorname{sd} \sigma$, $\operatorname{diam} \tau \leq \frac{p}{p+1} \operatorname{diam} \sigma$.

Proof: By induction on $p = \dim \sigma$.

Trivial if $p = 0$. Suppose true in dimensions less than p.

Write $\tau = \hat{\sigma_0} \ldots \hat{\sigma_p}$ where $\sigma_p = \sigma$.

Then $\operatorname{diam} \tau = \max\{d(\hat{\sigma_i}, \hat{\sigma_j})\}$. Suppose $i < j$.

If $j < p$ then by induction: $d(\hat{\sigma_i}, \hat{\sigma_j}) \leq \frac{j}{j+1} \operatorname{diam} \sigma_j \leq \frac{p}{p+1} \operatorname{diam} \sigma_j \leq \frac{p}{p+1} \operatorname{diam} \sigma$ since $j < p$ and $\sigma_j \subset \sigma$.

If $j = p$ then $d(\hat{\sigma_i}, \hat{\sigma_p}) = d(\hat{\sigma_i}, \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$ by Step 1.

Hence $\operatorname{diam} \tau \leq \frac{p}{p+1} \operatorname{diam} \sigma$. □

Definition 9.4.6 *Let X be a topological space. Define the* barycentric subdivision operator*,* $\operatorname{sd}_X : S_p(X) \to S_p(X)$ *inductively as follows:*

$\operatorname{sd}_X : S_0(X) \to S_0(X)$ *is defined as the identity map.*

Suppose sd_X *defined in degrees less than p for all spaces.*

Recall: Given convex $Y \subset \mathbb{R}^N$ *and* $y \in Y$*, in the proof of Theorem 9.3.17 we defined a homomorphism* $S_q(Y) \to S_{q+1}(Y)$*, which we will denote* $T \mapsto [T, y]$*, by*

$$[T, y](v) := ty + (1-t)T(z)$$

where $v = t\epsilon_{p+1} + (1-t)z$ *with* $z \in \Delta^p$*. Recall that* $\partial[c, y] = \begin{cases} [\partial c, y] + (-1)^{q+1}c & q > 0; \\ \epsilon(c)T_y - c & q = 0, \end{cases}$
where $T_y : \Delta^0 \to Y$ *by* $T_y(*) = y$*.*

We will apply this with $Y = \Delta^p$*,* $y = \hat{\sigma} =$ *barycenter of* Δ^p*.*

To define $S_p(X) \xrightarrow{\mathrm{sd}_X} S_p(X)$*, first consider* $\iota_p :=$ *identity map* $: \Delta^p \to \Delta^p \in S_p(\Delta^p)$*.*

Define $\mathrm{sd}_{\Delta^p}\,\iota_p) := (-1)^p[\mathrm{sd}_{\Delta^p}(\partial\iota_p), \hat{\sigma}] \in S_{p+1}(\Delta^p)$*.*

Then given generator $T : \Delta^p \to X \in S_p(X)$ *for arbitrary* X*, define*
$\mathrm{sd}_X(T) := T_*\big(\mathrm{sd}_{\Delta^p}(\iota_p)\big) = (-1)^p\big[T_*\big(\mathrm{sd}_{\Delta^p}(\partial\iota_p)\big), T(\hat{\sigma})\big]$*.*

Letting SD denote geometric barycentric subdivision, by construction, $\mathrm{sd}_{\Delta^p}(\iota_p) = \sum \pm\sigma_i$ where $\mathrm{SD}(\Delta^p) = \cup_i \tau_i$ and $\sigma \in S_p(\Delta^p)$ is the affine map sending ϵ_j to $\hat{\tau}_j$ where $\hat{\tau}_0, \ldots, \hat{\tau}_p$ are the vertices of $\hat{\tau}_i$.

Lemma 9.4.7 sd_X *is a natural augmentation-preserving chain map.*

Note: Natural means $\begin{array}{ccc} S_p(X) & \xrightarrow{\mathrm{sd}_X} & S_p(Y) \\ \downarrow f_* & & \downarrow f_* \\ S_p(Y) & \xrightarrow{\mathrm{sd}_Y} & S_p(Y) \end{array}$ commutes.

Proof:

Let $\epsilon : S_0(X) \to \mathbb{Z}$ be the augmentation. If $c \in S_0(X)$ then $\operatorname{sd}_X(c) = c$ so $\epsilon\big(\operatorname{sd}(c)\big) = \epsilon(c)$. Hence sd_X is augmentation preserving.

To show naturality:

$f_X \operatorname{sd}_X T = f_* T_* \operatorname{sd}_{\Delta^p} \iota_p = (f \circ T)_* \operatorname{sd}_{\Delta^p} \iota_p = \operatorname{sd}_Y (f \circ T)_* \iota_p = \operatorname{sd}_Y f_* T.$

We show that sd_X is a chain map by induction on p. Suppose we know, for all spaces, that $\partial \operatorname{sd}_X = \operatorname{sd}_X \partial$ in degrees less than p. Then in Δ^p we have

$$\begin{aligned}
\partial \operatorname{sd} \iota_p &= (-1)^p \partial [\operatorname{sd} \partial \iota_p, \hat{\sigma}] \\
&= \begin{cases} (-1)^p [\partial \operatorname{sd} \partial \iota_p, \hat{\sigma}] + (-1)^p (-1)^p \operatorname{sd} \partial \iota_p & p > 1 \\ -\epsilon(\operatorname{sd} \partial \iota_1) T_{\hat{\sigma}} + \operatorname{sd} \partial \iota_1 & p = 1 \end{cases} \\
&= \begin{cases} (-1)^p [\operatorname{sd} \partial \partial \iota_p, \hat{\sigma}] + \operatorname{sd} \partial \iota_p & p > 1 \\ -\epsilon \partial \iota_1 T_{\hat{\sigma}} + \operatorname{sd} \partial \iota_1 & p = 1 \end{cases} \\
&= \begin{cases} 0 + \operatorname{sd} \partial \iota_p & p > 1 \\ 0 + \operatorname{sd} \partial \iota_1 & p = 1 \end{cases} \\
&= \operatorname{sd} \partial \iota_p .
\end{aligned}$$

Now for arbitrary $T \in S_p(X)$,

$\partial \operatorname{sd} T = \partial T_* (\operatorname{sd} \iota_p) = T_* (\partial \operatorname{sd} \iota_p) \overset{\text{(naturality of sd)}}{=} \operatorname{sd} T_* \partial \iota_p = \operatorname{sd} \partial T_* \iota_p = \operatorname{sd} \partial T.$ □

Theorem 9.4.8 *Let $\mathcal{A}$ be a collection of subset of X whose interiors cover X. Let $T : \Delta^p \to X$ be a generator of $S_p(X)$. Then $\exists N$ s.t. $\operatorname{sd}^N T = \sum_i n_i T_i$ with $\operatorname{Im} T_i$ contained in some set in $\mathcal{A}$ for each i. (Need not be the same set of $\mathcal{A}$ for different i.)*

Proof: Since $\{\operatorname{Int} A\}_{A \in \mathcal{A}}$ covers X, $\{T^{-1}(\operatorname{Int} A)\}_{A \in \mathcal{A}}$ covers Δ^p which is compact. Let λ be a Lebesgue number for the covering $\{T^{-1}(\operatorname{Int} A)\}_{A \in \mathcal{A}}$ of Δ^p. Choose N s.t. for each simplex σ of $\operatorname{SD}^N \Delta^p$, $\operatorname{diam} \sigma < \lambda$ (where SD denotes geometric barycentric subdivision).

Thus writing $\operatorname{sd}^N \sigma = \sum n_i \sigma_i$, for each i $\exists A \in \mathcal{A}$ s.t. $\operatorname{Im} \sigma_i \subset T^{-1}(\operatorname{Int} A)$. (Each n_i is ± 1, but we don't need this.)

By naturality $\operatorname{sd}^N T = \sum n_i T(\sigma_i)$ and so $\forall i$ $\exists A \in \mathcal{A}$ s.t. $\operatorname{Im} T \sigma_i \subset A$ □

Theorem 9.4.9 *For each m, $\exists$ natural chain homotopy $D_X : 1 \simeq \operatorname{sd}^m : S_*(X) \to S_*(X)$.*

That is,

1. *$\forall p$ $\exists D_X : S_p(X) \to S_{p+1}(X)$ s.t. $\partial D_X c + D_X \partial c = \operatorname{sd}^m c - c$ $\forall c \in S_p(X)$*

2. Given $f : X \to Y$,

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_X} & S_{p+1}(X) \\ \downarrow f_* & & \downarrow f_* \\ S_p(Y) & \xrightarrow{D_Y} & S_{p+1}(Y) \end{array}$$

commutes.

Proof: By "acyclic models". i.e. D_X is defined on all spaces by induction on p.
For $p = 0$, define $D_X = 0 : S_*(X) \to S_1(X)$:

Since for $c \in S_0(X)$, $\operatorname{sd}^m(c) = c$, so $\partial D_X c + D_X \partial c = \partial 0 + D_X 0 = 0 = \operatorname{sd}^m c - c$ is satisfied.

Now suppose by induction that for all $k < p$ and for all spaces X, $D_X : S_k(X) \to S_{k+1}(X)$ has been defined satisfying (1) and (2) above.

Define $D_X T$ first in the special case $X = \Delta^p$, $T = \iota_p : \Delta^p \to \Delta^p \in S_p(\Delta^p)$.

To define $D_X \iota_p$ need to "solve" equation $\partial c = \operatorname{sd}^m \iota_p - \iota_p - D_{\Delta^p}(\partial \iota_p)$ for c and define $D_X \iota_p$ to be a solution.

Since Δ^p is acyclic, it suffices to check that $\partial(RHS) = 0$.

$\partial \operatorname{sd}^m \iota_p - \partial \iota_p - \partial D_{\Delta^p}(\partial \iota_p) = \partial \operatorname{sd}^m \iota_p - \partial \iota_p - \big(\operatorname{sd}^m \partial \iota_p - \partial \iota_p - D_{\Delta^p}(\partial\partial\iota_p)\big) = 0$. Therefore can define $D_X \iota_p$ s.t. (1) is satisfied.

Given $T : \Delta^p \to X \in S_p(X)$, define $D_X T := T_*\big(D_{\partial^p}(\iota_p)\big)$. Then

$$\begin{aligned} \partial D_X T &= \partial T_*(D_{\partial^p} \iota_p) \\ &= T_* \partial (D_{\partial^p} \iota_p) \\ &\overset{\text{(induction)}}{=} \operatorname{sd}^m T_* \iota_p - T_* \iota_p - D_{\Delta^p} T_* \partial \iota_p \\ &= \operatorname{sd}^m T - T - D_{\Delta^p} \partial T \iota_p \\ &= \operatorname{sd}^m T - T - D_{\Delta^p} \partial T \end{aligned}$$

Also $f_X D_X(T) = f_* T_*(D_{\Delta^p} \iota_p) = (f \circ T)_*(D_{\Delta^p} \iota_p) = D_Y(f \circ T) = D_Y f_*(T)$. □

Let A be a subspace of X. Since sd_A is the same as sd_X restricted to A, $\exists$ induced $\operatorname{sd}_{X,A} : S_*(X, A) \to S_*(X, A)$. By property (2) of D_X, restrcion of D_X to A equals D_A so $\exists$ an induced homomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_p(A) & \longrightarrow & S_p(X) & \longrightarrow & S_p(X, A) & \longrightarrow & 0 \\ & & \downarrow D_A & & \downarrow D_X & & \downarrow D_{X,A} & & \\ 0 & \longrightarrow & S_{p+1}(A) & \longrightarrow & S_{p+1}(X) & \longrightarrow & S_{p+1}(X, A) & \longrightarrow & 0 \end{array}$$

with $D_{X,A} : 1 \simeq \mathrm{sd}^m_{X,A} : S_*(X, a) \to S_*(X, A)$.

Notation: Let $\mathcal{A}$ be a collection of sets which cover X.

Set $S^{\mathcal{A}}_p(X) :=$ free abelian group$\{T : \Delta^p \to X \mid \operatorname{Im} T \subset A$ for some $A \in \mathcal{A}\}$.

$S^{\mathcal{A}}_p(X)$ is a subgroup of $S_p(X)$.

Notice that if $\operatorname{Im} T \subset A$ then writing $\partial T = \sum n_i T_i$, for each i $\operatorname{Im} T_i \subset \operatorname{Im} T \subset A$ so $\partial T \in S^{\mathcal{A}}_{p-1}(X)$. Thus the restriction of ∂ to $S^{\mathcal{A}}_p(X)$ turns $S^{\mathcal{A}}_p(X)$ into a chain complex and the inclusion map becomes a chain map.

Notice also that if T is a generator of $S^{\mathcal{A}}_p(X)$ then $D_X T \in S^{\mathcal{A}}_{p+1}(X)$ because:

if $D_{\Delta^p}(\iota_p) = \sum n_i S_i$ then $D_X T = T_*(D_{\Delta^p}\iota_p = \sum n_i T_* S_i = \sum n_i (T \circ S_i)$. But $\operatorname{Im} T \subset A$ for some $A \in \mathcal{A}$ and $\operatorname{Im} T \circ S_i \subset \operatorname{Im} T$.

Theorem 9.4.10 *Let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Then $H_*\big(S^{\mathcal{A}}_*(X), \partial\big) \to H_*\big(S_*(X), \partial\big)$ is an ismorphism.*

Remark 9.4.11 *The even stronger statement $i_* : S^{\mathcal{A}}_*(X) \to S_*(X)$ is a chain homotopy equivalence is true, but we will not show this.*

Proof: The short exact sequence of chain complexes

$0 \to S^{\mathcal{A}}_*(X) \xrightarrow{\ i\ } S_*(X) \xrightarrow{\ q\ } S_*(X)/S^{\mathcal{A}}_*(X) \to 0$

induces a long exact homology sequence. Showing that i_* is an isomorphism on homology for all p is equivalent to showing that $H_p\left(S_*(X)/S^{\mathcal{A}}_*(X)\right) = 0\ \forall p$.

Let $qc \in S_*(X)/S^{\mathcal{A}}_*(X)$ be a cycle representing an element of $H_p\left(S_*(X)/S^{\mathcal{A}}_*(X)\right)$, where $c \in S_p(X)$. That is, $\partial qc = 0$ or equivalently $\partial c \in S^{\mathcal{A}}_{p-1}(X)$.

We wish to show that there exists $d \in S_{p+1}(X)$ s.t. $\partial qd = qc$ or equivalently $c - \partial d \in S^{\mathcal{A}}_p(X)$.

Since c is a finite sum of generators $c = \sum n_j T_j$, find N s.t. we can write $\mathrm{sd}^N T_j = \sum n_{ij} T_{ij}$ where $\forall i, j\ \exists A \in \mathcal{A}$ (depending upon i and j) with $\operatorname{Im} T_{ij} \subset A$. Let D_X be the chain homotopy $D_X : 1 \simeq \mathrm{sd}^N$ for this N. Show $c + \partial D_X c \in S^{\mathcal{A}}_p(X)$ and then let $d = -D_X c$.

$\partial D_X c + D_X \partial c = \mathrm{sd}^N c - c$ so $c + \partial D_X c = \mathrm{sd}^N c - D_X \partial c$.

By definition of N, $\mathrm{sd}^N c \in S^{\mathcal{A}}_p(X)$. Also $\partial c \in S^{\mathcal{A}}_{p-1}(X)$ as noted earlier and so $D_X \partial x \in S^{\mathcal{A}}_p(X)$. Thus the requred d exists. Hence ∂c represents the zero homology class in $H_p\left(S_*(X)/S^{\mathcal{A}}_*(X)\right)$.

□

Let X, $\mathcal{A}$ be as in the preceding theorem, and let B be a subspace of X. Let $\mathcal{A} \cap B$ denote the covering of B obtained by intersecting the sets in$\mathcal{A}$ with B. Write $S^{\mathcal{A}}_*(X, B)$ for $S^{\mathcal{A}}_*(X)/S^{\mathcal{A}\cap B}_*(B)$.

Corollary 9.4.12 *$S^{\mathcal{A}}_*(X, B)$ to $S_*(X, B)$ induces an isomorphism on homology.*

Proof:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_*^{\mathcal{A}\cap B}(B) & \longrightarrow & S_*^{\mathcal{A}}(X) & \longrightarrow & S_*^{\mathcal{A}}(X,B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_*(B) & \longrightarrow & S_*^{(}X) & \longrightarrow & S_*(X,B) & \longrightarrow & 0
\end{array}$$

induces

$$\begin{array}{ccccccccccccc}
\to H_{p+1}^{\mathcal{A}}(X,B) & \longrightarrow & H_p^{\mathcal{A}\cap B}(B) & \longrightarrow & H_p^{\mathcal{A}}(X) & \longrightarrow & H_p^{\mathcal{A}}(X,B) & \longrightarrow & H_{p-1}^{\mathcal{A}\cap B}(B) & \longrightarrow & H_{p-1}^{\mathcal{A}}(X) & \to \\
\downarrow & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & \\
\to H_{p+1}(X,B) & \longrightarrow & H_p(B) & \longrightarrow & H_p(X) & \longrightarrow & H_p(X,B) & \longrightarrow & H_{p-1}(B) & \longrightarrow & H_{p-1}(X) & \to
\end{array}$$

Since the marked maps are isomorphisms from the theorem, the remaining vertical maps are also, by the 5-lemma. □

Theorem 9.4.13 *(Excision)*
Let A be a subspace of X and suppose that U is a subspace of A s.t. $\overline{U} \subset \operatorname{Int} A$. Then $j : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism on singular homology.

Remark 9.4.14 *Note that this is slightly stronger than axiom $A5$ which requires that U be open in X.*

Proof: Let $\mathcal{A}$ denote the collection $\{X - U, A\}$ in 2^X.

$\operatorname{Int}(X \setminus U) = X \setminus \overline{U}$. Since $\overline{U} \subset \operatorname{Int} A$, the interiors of $X - U$ and A cover X. Hence $S_*^{\mathcal{A}}(X, A) \to S_*(X, A)$ induces an isomorphism on homology. To conclude the proof we show that $S_*(X \setminus U, A \setminus U) \cong S_*^{\mathcal{A}}(X, A)$ as chain complexes.

Define $\phi : S_p(X \setminus U) \to S_p^{\mathcal{A}}(X)/S_p^{\mathcal{A}\cap A}(A)$ by $T \mapsto [T]$, which makes sense since $\operatorname{Im} T \subset X - U$ which belongs to $\mathcal{A}$.

Every element of $S_p^{\mathcal{A}}(X)$ can be written $c = \sum m_i S_i + \sum n_j T_j$ where
$\operatorname{Im} S_i \subset A\ \forall i$ and $\operatorname{Im} T_j \subset X \setminus U\ \forall j$. Since $\sum m_i S_i \in S_p^{\mathcal{A}\cap A}(A)$, in $S_p^{\mathcal{A}}(A)/S_p^{\mathcal{A}\cap A}(A)$, $[c] = \left[\sum n_j T_j\right] = \phi(\sum_j T_j)$. Therefore ϕ is onto.

$\ker \phi = S_p(X - U) \cap S_p^{\mathcal{A}\cap A}(A)$.

Notice that $\mathcal{A} \cap A = \{(X \setminus U) \cap A, A \cap A\} = \{A - U, A\}$ and since this colleciton includes A itself, $S_p^{\mathcal{A}\cap A}(A) = S_p(A)$.

In general $S_p(A) \cap S_p(B) = S_p(A \cap B)$ since a simplex has image in A and B if and only if its image lies in $A\cap$. Hence $\ker \phi = S_p(X \setminus U) \cap S_p^{\mathcal{A}\cap A}(A) = S_p\big((X \setminus U) \cap A\big) = S_p(A \setminus U)$.

Thus $S_p(X \setminus U, A \setminus U) \cong S_p(X - U)/S_p(A \setminus U) \overset{\phi}{\cong} S_p^{\mathcal{A}}(X)/S + p^{\mathcal{A}\cap A}(A) = S_p^{\mathcal{A}}(X, A)$. □

Let X_1, X_2 be subspaces of Y, let $A = X_1 \cap X_2$ and let $X = X_1 \cup X_2$. Notice that $X_2 \setminus A = X \setminus X_1$. Call this U. Thus $X_2 \setminus U = A$; $X \setminus U = X_1$.

Theorem 9.4.15 *(Mayer-Vietoris): Suppose that* $(X_1, A) \xrightarrow{j} (X, X_2)$ *induces an isomorphism on homology. (e.g. if* $\overline{U} \subset \operatorname{Int} X_2$. *) Then there is a long exact homology sequence*

$$\ldots \to H_{n+1}(X) \xrightarrow{\Delta} H_n(A) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \to \ldots$$

Remark 9.4.16 *The hypothesis is satisfied of* X_1 *and* X_2 *are open since that* $\overline{U} = U$ *and* $\operatorname{Int} X_2 = X_2$.

Proof: Follows by algebraic Mayer-Vietoris from:

$$\begin{array}{ccccccccccc}
\longrightarrow & H_{n+1}(X_1, A) & \longrightarrow & H_n(A) & \longrightarrow & H_n(X_1) & \longrightarrow & H_n(X_1, A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow \\
 & \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow & \\
\longrightarrow & H_{n+1}(X, X_2) & \longrightarrow & H_n(X_2) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, X_2) & \xrightarrow{\partial} & H_{n-1}(X_2) & \longrightarrow
\end{array}$$

□

9.4.2 Exact Sequences for Triples

Suppose $A \hookrightarrow B \hookrightarrow C$.

$0 \to S_*(B)/S_*(A) \to S_*X/S_*(A) \to S_*(X)/S_*(B) \to 0$ is a short exact sequence of chain complexes. Therefore we have a long exact sequence

$$\ldots \to H_{n+1}(X, B) \xrightarrow{\partial} H_n(B, A) \to H_n(X, A) \to H_n(X, B) \xrightarrow{\partial} H_{n-1}(X, A) \to \ldots$$

called the long exact homology sequence of the triple. From

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_*(B) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S_*(B)/S_*(A) & \longrightarrow & S_*(X)/S_*(A) & \longrightarrow & S) * (X)/S_*(B) & \longrightarrow & 0
\end{array}$$

we get

$$\begin{array}{ccccccccc}
\longrightarrow & H_n(X) & \longrightarrow & H_n(X,B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow \\
 & \downarrow & & \| & & \downarrow{\scriptstyle j} & \\
\longrightarrow & H_n(A) & \longrightarrow & H_n(X,B) & \xrightarrow{\tilde{\partial}} & H_{n-1}(B,A) & \longrightarrow
\end{array}$$

so $\tilde{\partial} = j\partial$ which relates the boundary homomorphism of the triple to ones we have seen before.