Chapter 1

Sets

Notation:

$f : X \to Y \qquad A \subset X \qquad B \subset Y$

$f(A) := \{f(a) \mid a \in A\} \subset Y$

$f^{-1}(B) := \{x \in X \mid f(x) \in B\}$

Note: $f^{-1}(\cap_{\alpha\in I} V_\alpha) = \cap_{\alpha\in I} f^{-1}(V_\alpha)$

$f^{-1}(\cup_{\alpha\in I} V_\alpha) = \cup_{\alpha\in I} f^{-1}(V_\alpha)$

$f(P \cup Q) = f(P) \cup f(Q)$ but in general $f(P \cap Q) \neq f(P) \cap f(Q)$

Theorem 1.0.1 *The following are equivalent (assuming the other standard set theory axioms):*

1. *Axiom of Choice*
2. *Zorn's Lemma*
3. *Zermelo well-ordering principle*

where the definitions are as follows.

Axiom of Choice: Given sets A_α for $\alpha \in I$, $A_\alpha \neq \emptyset \Rightarrow \prod_{\alpha\in I} \neq \emptyset$

(i.e. may choose $a_\alpha \in A_\alpha$ for each $\alpha \in I$ to form an element of the product)

To state (2) and (3):

Definition 1.0.2 *A* partially ordered set *consists of a set X together with a relation $\leq$ s.t.*

1. $x \leq x \qquad \forall x \in X \qquad$ *reflexive*
2. $x \leq y, y \leq z \Rightarrow x \leq z \qquad$ *transitive*
3. $x \leq y, y \leq x \Rightarrow x = y \qquad$ *(anti)symmetric*

Notation: $b \geq a$ means $a \leq b$.
If X is a p.o. set:

Definition 1.0.3 *1. m is* maximal *if $m \leq x \Rightarrow m = x$.*

2. *For $Y \subset X$, an element $b \in X$ is called an* upper bound*for Y if $y \leq b\ \forall y \in Y$. an element $b \in X$ is called an* lower bound*for Y if $y \geq b\ \forall y \in Y$.*

3. *X is called* totally ordered *if $\forall x, y \in X$, either $x \leq y$ or $y \leq x$. A totally ordered subset of a p.o. set is called a* chain*.*

4. *X is called* well ordered *if each $Y \neq \emptyset$ has a least element. i.e. if $\forall Y \neq \emptyset$, $\exists y_0 \in Y$ s.t. $y_0 \leq y\ \forall y \in Y$.*

Remark 1.0.4 *In contrast to "well-ordered", which requires the element y_0 to lie in Y, a "lower bound" is an elt. of X which need not lie in Y.*

Note: well ordered $\Rightarrow$ totally ordered (given x, y apply defn. of well ordered to the subset $\{x, y\}$), but totally ordered $\not\Rightarrow$ well ordered (e.g. $X = \mathbb{Z}$).

Zorn's Lemma: A partially ordered set having the property that each chain has an upper bound (the bound not necessarily lying in the set) must ahve a maximal element.

Zermelo's Well-Ordering Principle: Given a set X, $\exists$ relation $\leq$ on X such that $(X, \leq)$ is well-ordered.
Proof of Theorem: Pf. of Theorem:
2$\Rightarrow$ 3:
Given X, let $\mathcal{S} := \{(A, \leq_A) \mid A \subset X$ and $(A, \leq_A)$ well ordered $\}$
Define order on $\mathcal{S}$ by:

$$(A \leq_A) \leq (B, \leq_B) \text{ if } A \subset B \text{ and } \begin{cases} a \leq_B a' \Leftrightarrow a \leq_A a' & \forall a, a' \in B \\ a \leq b & \forall a \in A, b \in B - A \end{cases}$$

(i) This is a partial order

Trivial. e.g. Symmetry: If $(A, \leq_A) \leq (B, \leq_B) \leq (A, \leq_A)$ then $A \subset B \subset A$ so $A = B$ and defns. imply order is the same.

(ii) If $\mathcal{C} = \{(A, \leq_A)\}$ is a chain in $\mathcal{S}$ then $(Y := \cup_{A \in \mathcal{C}}, \leq_Y)$ is an upper bound for $\mathcal{C}$ where $\leq_Y$ is defined by:

If $y, y' \in Y$, find $A, A' \in \mathcal{C}$ s.t. $y \in A, y' \in A'$.

$\mathcal{C}$ chain$\Rightarrow$ A, A comparable$\Rightarrow$ larger (say A) contains both y, y'.

So define $y \leq_Y y' \Leftrightarrow y \leq_A y'$.

To qualify as an upper bound for $\mathcal{C}$, must check that $Y \in \mathcal{S}$. i.e. Show Y is well-ordered.

Proof: For $\emptyset \neq W \subset Y$, find $A_0 \in \mathcal{C}$ s.t. $W \cap A_0 \neq \emptyset$.

$A_0 \in \mathcal{S} \Rightarrow A_0$ well-ordered$\Rightarrow W \cap A_0$ has a least elt. w_0.

$\forall w \in W,\ \exists A \in \mathcal{C}$ s.t. $w \in A$.

$\mathcal{C}$ chain$\Rightarrow A_0, A$ comparable in $\mathcal{S}$.

If $A \subset A_0$ then $w \in A_0$ so $w_0 \leq w$ ($w_0 =$ least elt. of A_0).

If $A_0 \subset A$ then $w_0 \leq w$ by defn. of ordering on $\mathcal{S}$.

Therefore $w_0 \leq w\ \forall w \in W$ so every subset of Y has a least elt.

Therefore Y is well-ordered.

Hence $(Y, \leq_Y)$ belongs to $\mathcal{S}$ and forms an upper bound for $\mathcal{C}$.

So Zorn applies to $\mathcal{S}$. Therefore $\mathcal{S}$ has a maximal elt. $(M \leq_M)$.

If $M \neq X$, let $x \in X - M$ and set $M' := M \cup \{x\}$ with $x \geq a\ \forall a \in M$.

Then $(M', \leq) \not\leq (M, \leq)$. $\Rightarrow\Leftarrow$

Therefore $M = X$.

Hence $\leq_M$ is a well-ordering on X.

3$\Rightarrow$ 1:

Well order $\cup_\alpha A_\alpha$. For each α, let $a_\alpha :=$ least elt. of A_α. Then $(a_\alpha)_{\alpha \in A}$ is an elt. of $\prod_\alpha A_\alpha$.

□

Standard consequences of Zorn's Lemma:

1. Every vector space has a basis. (Choose a maximal linearly independent set)
2. Every proper ideal of a ring is contained in a maximal proper ideal
3. There is an injection from $\mathbb{N}$ to every infinite set.

1.0.1 Ordinals

Definition 1.0.5 *If W is well ordered, an* ideal *in W is a subset W' s.t. $a \in W', b \leq a \Rightarrow b \in W'$.*

Note: Ideals are well-ordered.

Lemma 1.0.6 *Let W' be an ideal in W. Then either $W' = W$ or $W' = \{w \in W \mid w < a\}$ for some $a \in W$.*

Proof: If $W' \neq W$, let a be least elt. of $W - W'$. If $x < a$ then $x \in W'$.
Conversely if $x \in W'$:
If $a \leq x$ then $a \in W' \Rightarrow\Leftarrow$
Therefore $x < a$. □

Corollary 1.0.7 *If I, J are ideals of W then either $I \subset J$ or $J \subset I$.*

Notation: $\mathrm{Init}_a := \{w \in W \mid w < a\}$ called an *initial interval*

Proof of Cor. If $I = \mathrm{Init}_a$ and $J = \mathrm{Init}_b$, compare a and b. □

Theorem 1.0.8 *Let X, Y be well ordered. Then*

either a) $Y \cong X$

or b) $Y \cong$ an initial interval of X

or c) $X \cong$ an initial interval of Y

The relevant iso. is always unique.

Lemma 1.0.9 *A, B well-ordered. Suppose $\zeta : A \to B$ is a morphism of p.o. sets mapping A isomorphically to an ideal of B. Let $f : A \to B$ be an injection of p.o. sets. Then $\zeta(a) \leq f(a)$ $\forall a \in A$.*

Proof: If non-empty $\{a \in A \mid \zeta(a) > f(a)\}$ has a least elt. a_0.
$\zeta(a_0) > f(a_0)$.
Since $\mathrm{Im}\,\zeta$ is an ideal, $f(a_0) = \zeta(a)$ for some $a \in A$.
$\zeta(a_0) > \zeta(a) \Rightarrow a_0 > a$ (ζ p.o set injection)
Choice of $a_0 \Rightarrow f(a_0) = \zeta(a) \leq f(a)$
$\Rightarrow a_0 \leq a$ (f p.o. set injection)
$\Rightarrow\Leftarrow$
Therefore $\zeta(a) \leq f(a)$ $\forall a$. □

Proof of Thm. From Lemma, if ζ_1, ζ_2 are both isos. from A onto ideals of B (not necess. the same ideal)
$\forall a$, $\zeta_1(a) \leq \zeta_2(a) \leq \zeta_1(a) \Rightarrow \zeta_1(a) = \zeta_2(a)$.
Therefore $\zeta_1 = \zeta_2$. So uniqueness part of thm. follows.
Claim: $X \ncong$ an initial interval of itself
Proof: If $g : X \cong I$ where $I = \mathrm{Init}_a$,
$I \xrightarrow{j} X$ and $I \xrightarrow{j} X \overset{g}{\cong} I \xrightarrow{j} X$ map I isomorphically onto an ideal of X. (i.e. If $b \leq jgjx = g(jx) \in I$ then $b \in I$ since I ideal $\Rightarrow b = g(y)$ some y. Then $g(y) \leq g(jx) \Rightarrow y \leq jx \Rightarrow y \in I \Rightarrow y = j(y)$, so $b \in \mathrm{Im}\,jgj$.)
Therefore $j = jgj$ (Lemma).
Impossible since $jg(a) \in \mathrm{Im}\,j$ whereas $g(a) \notin \mathrm{Im}\,jgj$ (i.e. $a > jc$ $\forall x \in I \Rightarrow jg(a) > jgj(x) \Rightarrow jg(a) \notin \mathrm{Im}\,jgj$)

Therefore at most one of (a), (b), (c) holds.

Let $\Sigma :=$ set of ideals of X which are isomorphic to some ideal of Y, ordered by inclusion.

($K := \Sigma = \cup_{I \in \Sigma} I$) is an ideal in X

For each $I \in \Sigma$, let $\zeta_I : I \to Y$ be the unique map taking I isomorphically onto an ideal of Y.

Therefore If $J \subset I$, $\zeta_J = \zeta_I|_J$.

So the ζ_I's induce a map $\zeta : K \to Y$ which takes K isomorphically to an ideal of Y. (i.e. If $y < \zeta(K)$, find I s.t. $k \in I$. Then ζ_I iso. to its image $\Rightarrow y = \zeta_I(l)$ for some l. Therefore $\operatorname{Im}\zeta$ is an ideal. And ζ is an injection: Remember, given two elts. $a \in I$, $a' \in I'$ either $a \leq a'$ in which case $a \in I'$ or reverse is true.)

Therefore $K \in \Sigma$.

If (both) $K \neq X$ and $\zeta(K) \neq Y$, let x, y be least elts. of $X - K$, $Y - \zeta(K)$ respectively. Extend ζ by defining $\zeta(x) = y$ to get an iso. from $K \cup \{x\}$ to the ideal $\zeta(K) \cup \{y\}$ of Y. Contradicts defn. of $K \Rightarrow\Leftarrow$

So either $K = X$ or $\zeta(K) = Y$ or both, giving the 3 cases. □

Corollary 1.0.10 *Let $g : X \to Y$ be an injective poset morphism between between well-ordered posets. Then*

either a) $X \cong Y$

or b) $X \cong$ *initial interval of* Y

(i.e. $Y \ncong$ initial interval of X in previous thm.)

Proof: If $h : Y \cong$ initial interval of X then

$f : Y \xrightarrow{h}$ initial interval of $X = \operatorname{Init}(a) \hookrightarrow X \xrightarrow{g} Y$ is an injection from Y to Y. Applying earlier Lemma with $\zeta = 1_Y$ gives $g \leq f(y)\ \forall y \in Y$.

But $\operatorname{Im} f \subset \operatorname{Init}\big(g(a)\big)$ so $y < g(a)\ \forall y \in Y$ (i.e. $y \leq f(y) < g(a)$)

$\Rightarrow\Leftarrow$ (letting $y = g(a)$) □

Definition 1.0.11 *An* ordinal *is an isomorphism class of well ordered sets.*

(Generally we refer to an ordinal by giving a representative set.)

Example 1.0.12

1. $\underline{\underline{n}} := \{1, \dots, n\}$ *standard order*

2. $\omega := \mathbb{N}$ *standard order*

3. $\omega + \underline{\underline{n}} := \mathbb{N} \amalg \underline{\underline{n}}$ *with the ordering $x < y$ if $x \in \mathbb{N}$ and $y \in \underline{\underline{n}}$ and standard ordering if both $x, y \in \mathbb{N}$ or both $x, y \in \underline{\underline{n}}$*

   Note: $\amalg :=$ *disjoint union (i.e. union of $\mathbb{N}$ with a set isomorphic to $\underline{\underline{n}}$ containing no elts. of $\mathbb{N}$.)*

4. $2\omega = \mathbb{N} \amalg \mathbb{N}$*)*

Note: For any ordinal *gamma* there is a "next" ordinal $\gamma + 1$, but there is not necessarily an ordinal τ such that $\gamma = \tau + 1$.

Transfinite induction principle: Suppose W is a well ordered set and $\{P(x) \mid x \in W\}$ is a set of propositions such that:
(i) $P(x_0)$ is true where x_0 is the least elt. of W
(ii) $P(y)$ true for $\forall\ y < x \Rightarrow P(x)$ true
Then $P(x)$ is true $\forall\ x$.

1.0.2 Cardinals

Theorem 1.0.13 *(Shroeder-Bernstein). Let X, Y be sets. Then*

1. *Either $\exists$ injection $X \hookrightarrow Y$ or $\exists$ injection $Y \hookrightarrow X$.*
2. *If both injections exist then $X \cong Y$*

Proof: 1. Choose well ordering for X and for Y. Then use iso. of one with other or with ideal of other to define injection.

2. Suppose $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow X$. Choose well ordering for X and Y. If $\exists x \in X$ s.t. X is bijection with $\mathrm{Init}(x)$, let x_0 be least such x. So in this case X is bijective with $\mathrm{Init}(x_0)$ but not with any ideal of $\mathrm{Init}(x_0)$. Replacing X by $\mathrm{Init}(x_0)$ we may assume that X is not bijective with any of its ideals. (And in the case where $\not\exists x \in X$ s.t. X is bijective with $\mathrm{Init}(x)$ then this is clearly also true.) Similarly may assume that Y well ordered such that it is not bijective with any of its ideals. Assuming $X \not\cong Y$, one is iso. to an ideal of the other. Say $Y \cong \mathrm{Init}(x)$. The inclusion $i : X \hookrightarrow Y$ induces a new well-order $(X, \prec)$ on X. from that on Y. By earlier Corollary, either $\exists$ iso. $\zeta : (X, \prec) \to Y$ or $\exists$ iso. $\zeta : (X \prec) \to \mathrm{Init}(y)$ for some $y \in Y$. In the former case we are finished, so suppose the latter. $(X, \prec)) \overset{\zeta}{\cong} \mathrm{Init}(y) \hookrightarrow Y \xrightarrow{\cong} \mathrm{Init}(x)$ gives a bijection from X to an initial interval of X. (Note: Image of init interval under iso. is an init interval, and an init interval within an init interval is an init interval.)

$\Rightarrow\Leftarrow$

Therefore X is bijective with Y. □

Definition 1.0.14 *A* cardinal *is an isomorphism class of sets. (In this context "isomorphism" means "bijection".)*
$\mathrm{card}\, X = \mathrm{card}\, Y$ *means $\exists$ bijection from X to Y.*
$\mathrm{card}\, X \leq \mathrm{card}\, Y$ *means $\exists$ injection from X to Y.*

(Thus previous Thm. says: $\mathrm{card}\, X \leq \mathrm{card}\, Y$ and $\mathrm{card}\, Y \leq \mathrm{card}\, X \Rightarrow \mathrm{card}\, X = \mathrm{card}\, Y$)

1.0.3 Countable and Uncountable Sets

Definition 1.0.15 *A set is called* countable *if either finite or numerically equivalent (i.e. $\exists$ a bijection) to the nature numbers $\mathbb{N}$. A set which is not countable is called* uncountable*.*

Example 1.0.16

1. *Even natural numbers*
2. *Integers*
3. *Positive rational numbers $\mathbb{Q}^+$.* **Proof:** *Define an ordering on $\mathbb{Q}^+$ by $a/b \prec c/d$ if $\big(a+b < c+d$ or $(a+b = c+d$ and $a < c)\big)$ where a/b, c/d are written in reduced form.*

   e.g. 1, 1/2, 2, 1/3, 3, 1/4, 2/3, 3/2, 4, 1/5, 5, ...

   For $f \in \mathbb{Q}^+$, let $S_r = x \in \mathbb{Q}^+ \mid x \le r\}$. This set is finite for each r so define $f(r) = \|S_r|$.

Proposition 1.0.17 *A subset of a countable set is countable.*

Proof: Let A be a subset of X and let $f : X \to \mathbb{N}$ be a bijection. Define $g : A \to \mathbb{N}$ by $g(a) := \big|\{b \in A \mid f(b) \le f(a)\}\big|$. □

Proposition 1.0.18 *Let $g : X \to Y$ be onto. If X is countable then Y is.*

Proof: Let $f : X \to \mathbb{N}$ be a bijection. For $y \in Y$, set $h(y) := \min\{f(x) \mid g(x) = y\}$. Then h is a bijection between Y and some subset of $\mathbb{N}$ so apply prev. prop. □

Proposition 1.0.19 *X, Y countable $\Rightarrow X \times Y$ countable.*

Proof: Use diagonal process as in pf. that rationals are countable. (Exercise.) □

Theorem 1.0.20 *(Cantor). $\mathbb{R}$ is uncountable.*

Proof: Suppose $\exists$ bijection $f : \mathbb{R} \to \mathbb{N}$. Let $g : \mathbb{N} \to \mathbb{R}$ be the inverse bijection. For each $n \in \mathbb{N}$ define

$$a_n := \begin{cases} 1 & \text{if } n\text{th interger after decimal pt. in decimal expansion of } g(n) \text{ is not } 1 \\ 2 & \text{if } n\text{th interger after decimal pt. in decimal expansion of } g(n) \text{ is } 1 \end{cases}$$

Therefore $a_n \neq n$th integer after dec. pt. in the dec. expansion of $g(n)$. Let a be the real number represented by the decimal $0.a_1a_2a_3\cdots$. (i.e. a is defined as the limit of the convergent series $a_1/10 + a_2/100 + a_3/1000 + \ldots + a_n/(10^n) + \ldots$.) Let $f(a) = m$ or equivalently $g(m) = a$. Then $a_m = m$th integer after dec. pt. in dec. expansion of $g(m)$, contradicting defn. of a_m.

$\Rightarrow\Leftarrow$

Therefore no such bijection f exists. □