Chapter 3

Separation axioms

Let X be a topological space.

Definition 3.0.12 *X has the following names if it has the following properties:*

1. X is T_0 if $\forall x \neq y \in X$ either $\exists$ open U s.t. $x \in U, y \notin U$ $x \in U, y \notin U$ or $\exists$ open U s.t. $x \notin U, y \in U$

2. X is T_1 if $\forall x \neq y \in X$ $\exists$ open U s.t. $x \in U$, $y \notin U$ and $\exists$ open V s.t. $y \in V, x \notin V$.

3. X is T_2 or Hausdorff if $\forall x \neq y \in X$ $\exists$ open U, V with $U \cap V = \emptyset$ s.t. $x \in U$ and $y \in V$

4. X is T_3 or regular if X is T_1 and given $x \in X$ and a closed set $F \subset X$ with $x \notin F$, $\exists$ open U and V s.t. $x \in U$, $F \subset V$ and $U \cap V = \emptyset$

5. X is $T_{3\frac{1}{2}}$ or completely regular if X is T_1 and also given $x \in X$ and a closed set $F \subset X$ with $x \notin F$, $\exists f : X \to [0, 1]$ s.t. $f(x) = 0$ and $f(F) = 1$.

6. X is T_4 or normal if X is T_1 and also given closed $F, G \subset X$ s.t. $F \cap G = \emptyset$ $\exists$ open U, V s.t. $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$.

We say U and V *separate* A and B if $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.
Some reformulations:

Proposition 3.0.13

1. *X is T_1 $\Leftrightarrow$ the points of X are closed subsets of X*

2. *X is Hausdorff*

   (a) $\Leftrightarrow \{x\} = \bigcap_{\substack{U \text{open} \\ x \in U}} \bar{U}$

(b) $\Leftrightarrow \Delta(X)$ *is closed in* $X \times X$ *(where* $\Delta(X)$ *means the diagonal subset* $\{(x,x) \mid x \in X\}$ *of* $X \times X$*)*

3. *X is regular* $\Leftrightarrow$ *X is Hausdorff and given* $x \in U$*,* $\exists$ *open V s.t.* $x \in V \subset \bar{V} \subset U$

4. *X is normal*

   (a) $\Leftrightarrow$ *X is Hausdorff and given* $x \in U$ $\exists$ *open V s.t.* $F \subset V \subset \bar{V} \subset U$

   (b) $\Leftrightarrow$ *X Hausdorff and given closed* F, G *with* $F \cap G = \emptyset$ $\exists$ *open* U, V *s.t.* $F \subset U$*,* $G \subset V$ *and* $\bar{U} \cap \bar{V} = \emptyset$*.*

Proof:
1: $(\Rightarrow)$ X T_1. Let $x \in X$. $\forall y \in X$ $\exists$ open V_y s.t. $x \notin V_y$ and $y \in V_y$. Hence $X \setminus \{x\} = \cup_{y \neq x} V_y$ is open so $\{x\}$ is closed.

$(\Leftarrow)$ Suppose points closed. Let $x, y \in X$. $U = X \setminus \{y\}$ is open. $x \in U, y \notin U$. Similarly the reverse.

2a: $(\Rightarrow)$ X is Hausdorff. Let $x \in X$. $\forall y \neq x$ $\exists U_y, V_y$ s.t. $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. $U_y \subset (V_y)^c \Rightarrow \bar{U}_y \subset (V_y)^c \Rightarrow y \notin \bar{U}_y \Rightarrow y \notin \bigcap_{\substack{U \text{open} \\ x \in U}} \bar{U}$.

$(\Leftarrow)$ Let $x \neq y \in X$. $\{x\} = \bigcap_{\substack{U \text{open} \\ x \in U}} \bar{U}$. Find open U s.t. $x \in U$ and $y \notin \bar{U}$. Let $V = \bar{U}^c$, which is open.

2b: $(\Rightarrow)$ Suppose X is Hausdorff.

If $(x,y) \in (\Delta(x))^c$ find U, V s.t. $x \in U, y \in V$, $U \cap V = \emptyset$

Then $(x,y) \in U \times V$ but $U \times V \subset \big(\Delta(X)\big)^c$. Since $U \times V$ is open, $(x,y) \in$ interior of $\big(\Delta(X)\big)^c$. This is true $\forall (x,y) \in \big(\Delta(X)\big)^c$, so $\big(\Delta(X)\big)^c$ is open, and $\big(\Delta(X)\big)$ is closed.

$(\Leftarrow)$ Suppose $\Delta(X)$ is closed.

If $x \neq y$ then $(x,y) \in \big(\Delta(X)\big)^c$. Since $U \times V$ is open, $(x,y) \in$ interior of $\big(\Delta(X)\big)^c$. This is true $\forall (x,y) \in \big(\Delta(X)\big)^c$, so $\big(\Delta(X)\big)^c$ is open, and $\big(\Delta(X)\big)$ is closed.

$(\Leftarrow)$ Suppose $\Delta(X)$ is closed.

If $x \neq y$ then $(x,y) \in (\Delta_c(X))^c$ which is open so there exists a basic open set $U \times V$ s.t. $(x,y) \in U \times V \subset (\Delta(X))^c$. Hence $x \in U, y \in V$, $U \cap V = \emptyset$.

3: $(\Rightarrow)$ Suppose X is regular. Then X is T_1 so points are closed. Hence given $x \neq y \in X$ let $F = \{y\}$ and apply defn. of regular to see that X is Hausdorff. Given $x \in U$, $x \cap U^c = \emptyset$ and U^c is closed so $\exists$ open V, W s.t. $x \in V$, $U^c \subset W$ and $V \cap W = \emptyset$.

$x \in V \subset W^c \subset U$.

Since W^c is closed, $\bar{V} \subset W^c$

$(\Leftarrow$ $)$ Hausdorff $\Rightarrow T_1$.

Let $x \in X$, $F \subset X$ with $x \notin F$.

Then $x \in F^c$, which is open, so $\exists$ open U s.t. $x \in U \subset \bar{U} \subset F^c$. Let $V = (\bar{U})^c$. Then $F \subset V$ and $U \cap V = \emptyset$.

4a: $\Leftrightarrow$ similar to (3.)

4b: ($\Leftarrow$) trivial

($\Rightarrow$) Given closed F, G s.t. $F \cap G = \emptyset$. Then $F \subset G^c$ so $\exists$ open U s.t. $F \subset U \subset \bar{U} \subset G^c$.

$G \subset (\bar{U})^c$ so $\exists$ open V s.t. $G \subset V \subset \bar{V} \subset (\bar{U})^c$.

Hence $\bar{U} \cap \bar{V} = \emptyset$. □

Proposition 3.0.14 *Let $f, g : X \to Y$, with Y Hausdorff. Suppose $A \subset X$ is dense and $f\big|_A = g\big|_A$. Then $f = g$.*

Proof: Define $h : X \to Y \times Y$ by $h(x) = \big(f(x), g(x)\big)$. Then h is continous (since its projections are).

Let $F = \{x \in X \mid f(x) = g(x)\}$.

$F = h^{-1}\big(\Delta(Y)\big)$ which is closed since Y is Hausdorff.

$A \subset F \Rightarrow X = \overline{A} \subset F$

Hence $f(x) = g(x)\ \forall x \in X$. □

Theorem 3.0.15 *metric $\Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_2 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$*

Proof: $T_2 \Rightarrow T_1 \Rightarrow T_0$ is trivial. $T_3 \Rightarrow T_2$ by definition, and part (1) of the previous proposition.

$T_{3\frac{1}{2}} \Rightarrow T_3$: Given x, F let $f : X \to [0,1]$ s.t. $f(X) = 0$, $f(F) = 1$, as in the definition of T_3. Set $U = f^{-1}\big([0, 1/2)\big)$ and $V = f^{-1}\big((1/2, 1]\big)$ which are open in $[0,1]$. Then U, V separate x, F in X.

metric$\Rightarrow T_4$: Let F, G be closed in metric space X s.t. $F \cap G = \emptyset$.

For $x \in F$, let $d_x = \inf_{y \in G}\{d(x, y)\}$.

Claim: $d_x \neq 0$.

Proof:

If $d_x = 0$ then $\forall n\ \exists y_n \in G$ s.t. $d(x, y_n) < 1/n$.

Hence $(y_n) \to x$. Hence $x \in G$.

(Exercise: G closed, $y_n \in G$, $(y_n) \to x \Rightarrow x \in G$)

$\Rightarrow\Leftarrow$

Let $Y = \cup_{x \in F} N_{d_x/2}(x)$ which is open with $F \subset U$.

Claim: $\overline{U} \cap G = \emptyset$

Proof:

Let $y \in \overline{U} \cap G$.

Then $\exists$ sequence $(u_n) \to y$ with $u_n \in U$.

$\forall n$ find $x_n \in F$ s.t. $u_n \in N_{d_{x_n}/2}(x)$

$d_{x_n} \leq d(x_n, y) \leq d(x_n, u_n) + d(u_n, y) < d_{x_n}/2 + d(u_n, y)$.

Hence $d_{x_n}/2 < d(u_n, y)$.

$(u_n) \to y \Rightarrow d(u_n, y0) \to 0 \Rightarrow d_{x_n}/2 \to 0$.

Hence $d(x_n, y) < d_{x_n}/2 + d(u_n, y) \Rightarrow d(x_n, y) \to 0 \Rightarrow (x_n) \to y$.

So $y \in F \Rightarrow\Leftarrow$.
Hence $\overline{U} \cap G = \emptyset$.

So let $V = (\overline{U})^c \supset G$.

$T_4 \Rightarrow T_{3\frac{1}{2}}$: Corollary of

Theorem 3.0.16 [Urysohn's Lemma] *Suppose X is normal, and F and G are closed subsets of X with $F \cap G = \emptyset$. Then $\exists f : X \to [0,1]$ s.t. $f(F) = 0$ and $f(G) = 1$.*

Proof:
Apply 4(b) of Proposition 3.0.13 to $F \subset G^c$. Then $\exists$ open $U_{1/2}$ s.t. $F \subset U_{1/2} \subset \bar{U_{1/2}} \subset G^c$.

Two more applications of Proposition 3.0.13:

4(b) $\Rightarrow \exists$ open $U_{1/4}, U_{3/4}$ s.t. $F \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_{3/4} \subset \bar{U}_{3/4} \subset G^c$.

Continuing, construct an open set U_t for all t of the form $m/2^n$ for some m and n. For $x \in X$ define

$$f(x) = \begin{cases} 0 & x \in U_t \ \forall t \\ sup(\{t | x \notin U_t\} & \text{otherwise} \end{cases} \tag{3.1}$$

It is clear that $f(F) = 0$ and $f(G) = 1$. We show that f is continuous.

Intervals of the form $[0, a)$ and $(a, 1]$ form a subbasis for $[0, 1]$.

$f(x) < a \Leftrightarrow x \in U_t$ for some $t < a$.

Hence $f^{-1}([0, a)) = \{x | f(x) < a\} = \cup_{t<a} U_t$, which is open.

Similarly $f(x) > a \Leftrightarrow x \notin U_t$ for some $t > a$. which is true iff $x \notin \bar{U}_s$ for some $s > a$.

Hence $f^{-1}((a, 1]) = \cup_{s>a} \left(\bar{U_s}\right)^c$, which is open.

We conclude that f is continuous. □

Lemma 3.0.17 *Suppose X is Hausdorff. Suppose $x \in X$ and $Y \subset X$ is compact s.t. $x \notin Y$. Then $\exists$ open U, V separating x and Y.*

Proof: $\forall y \in Y \; \exists$ open U_y, V_y s.t. $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. $Y = \cup_{y \in Y} V_y$ is a cover of Y by open sets in X so $\exists$ a finite subcover $V_{y_1}, \ldots, V_{y_n}$.

Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$ and $V = V_{y_1} \cup \cdots \cup V_{y_n}$. Then

(i) $x \in U_{y_j} \ \forall j \Rightarrow x \in U$

(ii) $V_{y_1}, \ldots, V_{y_n}$ cover $Y \Leftrightarrow Y \subset V$.

(iii) $U \cap V = \emptyset$.

(Proof: If $z \in U \cap V$ then $z \in V_{y_j}$ for some j and $z \in U_{y_j} \ \forall j$. But $U_{y_j} \cap V_{y_j} = \emptyset$. Contradiction.)

□

Corollary 3.0.18 *A compact subspace of a Hausdorff space is closed.*

Proof: Suppose $A \subset X$ where A is compact and X is Hausdorff. By Lemma, $\forall y \in A^c$ $\exists$ open U_y, V_y separating y and A so $y \in U_y \subset A^c$. Hence y is an interior point of A^c. This is true for all y so A^c is open (equivalently A is closed).

□

Theorem 3.0.19 *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof: Let $f : X \to Y$ where f is compact and Y is Hausdorff. We must show that the inverse to f is continuous, which is equivalent to showing that for any closed set B, $f(B)$ is closed. If $B \subset X$ is closed, then by our earlier Theorem, B is compact, so by another earlier Theorem, $f(B)$ is compact. By a previous Corollary, this implies $f(B)$ is closed. □

Theorem 3.0.20 *A compact Hausdorff space is normal.*

Proof: Suppose X is a compact Hausdorff space. Suppose A and B are closed subsets of X with $A \cap B = \emptyset$. Since A and B are closed and X is compact, we conclude that A and B are also compact.

By the Lemma, $\forall a \in A$ $\exists$ open sets U_a, V_a s.t. $a \in U_a$, $b \in V_a$ and $U_a \cap V_a = \emptyset$.

$\cup_a U_a$ is a cover of A by open sets in X so by compactnss there is a finite subcover $U_{a_1}, \ldots, U_{a_n}$. Let $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cup \cdots \cup V_{a_n}$.

Then as in the proof of the Lemma

(i) $A \subset U$

(ii) $B \subset V$

(iii) $U \cap V = \emptyset$

□

Proposition 3.0.21 *Suppose $A \subset X$.*

If X is T_j for $j < 4$ then so is A.

If A is closed and X is T_4 then A is T_4.

Proof:

$j = 0, 1, 2$: Trivial

$j = 3$: Let $a \in A$ and let $F \subset A$ be closed in A with $a \notin F$.

Let $\overline{F}$ denote the closure of F within X.

Then $a \notin \overline{F}$.

(Proof: $\overline{F} = \bigcap_{\substack{G \supset F \\ G \text{ closed in } X}} G$. Therefore

$$(G \cap A) = \bigcap_{\substack{G' \supset F \\ G' \text{ closed in } X}} (G \cap A) = \bigcap_{\substack{G' \supset F \\ G' \text{ closed in } X}} G' = (\text{closure of } F \text{ in } A) = F.$$

Hence $a \in A, \notin F \Rightarrow a \notin \overline{F}$.)

So $\exists$ open U, V in X s.t. $a \in U$, $\overline{F} \subset V$ and $U \cap V = \emptyset$.

But then $U' = U \cap A$ and $V' = V \cap A$ are open in A and satisfy:

(i) $a \in U \cap A$

(ii) $F = \overline{F} \cap A \subset V \cap A = V'$

(iii) $U' \cap V' = \emptyset$

$j = 3\frac{1}{2}$: Let $a \in F$, $F \subset A$ with F closed in A, $a \notin F$.

$F = \overline{F} \cap A$ with $\overline{F}$ as above.

Since, as above, $a \notin \overline{F}$, $\exists f : X \to [0,1]$ s.t. $f(a) = 0$, $f(\overline{F}) = 1$.

The composition $\hat{f} : A \hookrightarrow X \xrightarrow{f} [0,1]$ is continuous and satisfies $\hat{f}(a) = 0$ and $(\hat{F}) = 1$ (since $F \subset \overline{F}$).

$j = 4$: $A \subset X$ closed.

Let F, G be closed in X. As in previous two cases, $F = \overline{F} \cap A$ and $\overline{F} \cap A = \overline{F}$ since A is closed in X. So F is closed in X and similarly G is closed in X.

Therefore $\exists U, V$ open in X separating F, G in X.

So $U \cap A$ and $V \cap A$ separate F, G in A. $\square$

Proposition 3.0.22 *Let $X = \prod_{\alpha \in I} X_\alpha$ with $X_\alpha \neq \emptyset$ $\forall \alpha$.*

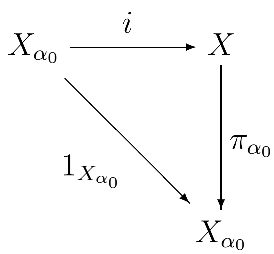
For $j < 4$, X is $T_j \Leftrightarrow X_\alpha$ is T_j $\forall \alpha$. X is $T_4 \Rightarrow X_\alpha$ is $T_j \forall \alpha$.

Proof:

$\Rightarrow$ Suppose X is T_j. Show X_{α_0} is T_j.

For $\alpha \neq \alpha_0$, select $x_\alpha \in X_\alpha$. (Axiom of Choice)

Define $i : X_\alpha \to X$ by $\pi_\alpha\big(i(a)\big) = \begin{cases} a & \alpha = \alpha_0; \\ x_\alpha & \alpha \neq \alpha_0. \end{cases}$



Note: Provided X_α is T_1 for $\alpha \neq \alpha_0$, i(closed) =closed (since a product of closed sets is closed).

If $a \neq b \in X_{\alpha_0}$ then $i(a) \neq i(b)$ in X.

$j = 0$: If $i(a) \in U$, $i(b) \notin U$, find basic open U' s.t. $i(a) \in U' \subset U$. So $i(b) \notin U'$.

But $a = \pi_{\alpha_0} i(a) \in \pi_{\alpha_0}(U')$ (open since projections maps are open maps)

Claim: $\notin \pi_{\alpha_0}(U')$

Proof: Since U' basic, $U' = \prod_\alpha \pi_\alpha(U')$

For $\alpha \neq \alpha_0$, $\pi_\alpha(ib) = x_\alpha = \pi_\alpha(ia) \in \pi_\alpha(U')$.

Therefore $ib \notin U'$ so $b = \pi_{\alpha_0} b \notin \pi_{\alpha 0}(U')$

$j = 1$: Similar

$j = 2$: Begining with open U, V, separating ia, ib, find basic U', V' separating ia, ib.
Claim: $\pi_{\alpha_0}(U')$ and $\pi_{\alpha_0}(V')$ (which are open) separate a and b.
Proof: $\pi_{\alpha_0} \circ i = 1_{X_{\alpha_0}}$ so $a \in \pi_{\alpha_0}(U')$ and $b \in \pi_{\alpha_0}(V')$.

If $c \in \pi_{\alpha_0}(U') \cap \pi_{\alpha_0}(U')$ then $ic \in U' \cap V'$ since U', V' basic and $\pi_\alpha(ic) = x_\alpha \in \pi_\alpha(U') \cap \pi_\alpha(V')$ for $\alpha \neq \alpha_0$.

Contradiction.

$j = 3$: X_{α_0} is T_1 by above.
Let $a \in X_{\alpha_0}$, B closed $\subset X_{\alpha_0}$ with $a \notin B$.
$i(a) \notin i(B)$ (closed because $(x_\alpha)_{\alpha \neq \alpha_0}$ is closed in $\Pi_{\alpha \neq \alpha_0} X_\alpha$ by $j = 1$ case and so $i(B) = B \times \prod_{\alpha \neq \alpha_0} X_\alpha$=closed)
Find U, V separating $i(a)$, $i(B)$ in $X/$
Find basic U' with $i(a) \in U' \subset U$.
$\forall z \in i(B)$, $\exists$ basic open V_z s.t. $z \in V_z \subset V$.
Let $\tilde{V} = \cup_{z \in i(B)} \pi_{\alpha_0}(V_z)$ open in X_α
Therefore $B \subset \tilde{V}$ (i.e. $b \in \pi_{\alpha_0}(V_{i(b)})$)
Claim: $\pi_{\alpha_0}(U')$, $\tilde{V}$ is a separation of a and B.
Proof: $c \in \pi_{\alpha_0}(U') \cap \tilde{V} \Rightarrow \pi_{\alpha_0}(ic) \in \pi_{\alpha_0}(U')$ and $\pi_{\alpha_0}(ic) \in \pi_{\alpha_0}(V_z)$ for some $z \in i(B)$.
For $\alpha \neq \alpha_0$, $\pi_\alpha(ic) = x_\alpha = \pi_\alpha(a) \in \pi_\alpha(U')$ and $\pi_\alpha(ic) = x_\alpha = \pi_\alpha(z) \in \pi_\alpha(V_z)$
That is, $ic \in U' \cap V_z \subset U \cap V$. $\Rightarrow\Leftarrow$.
Therefore case $j = 3$ follows.

$j = 3\frac{1}{2}$: X_{α_0} is T_1 by above.
Let $a \in X_{\alpha_0}$, B closed $\subset X_{\alpha_0}$, $a \notin B$.
$i(a) \notin i(B)$ (which is closed) implies $\exists g : X \to 0, 1$ s.t. $g(ia) = 0$, $g(oB) = 1$.
Let $f = g \circ i$.

$j = 4$: X_{α_0} is T_1 as above. Find separating function as in previous case, using Urysohn.

$\Leftarrow$ Suppose X_α is T_j for all α.
First consider cases $j < 3$.
Let $x, y \in X$ with $x_{\alpha_0} \neq y_{\alpha_0}$ for some α_0.

$j = 0$: If $x_{\alpha_0} \in U_0$, $y_{\alpha_0} \notin U_{\alpha_0}$ then $U = U_0 \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is open in X and $x \in U$, $y \notin U$.

$j = 0$: Similar

$j = 2$: If U_0, V_0 separate x_{α_0}, y_{α_0} in X_{α_0} then $U = U_0 \times \prod_{\alpha \neq \alpha_0} X_\alpha$ and $V = V_0 \times \prod_{\alpha \neq \alpha_0} X_\alpha$ separate x and y in X.

$j = 3$: By above X is T_1.
Let $x \in U$ (open)
Find basic open U' s.t. $U' \subset U$. Write $U' = \prod_\alpha U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \ldots, \alpha_n$.

For $j = 1, \ldots, n$ find V_{α_j} s.t. $x_{\alpha_j} \in V_{\alpha_j} \subset \overline{V_{\alpha_j}} \subset U_{\alpha_j}$
Let $V = V_{\alpha_1} \times \cdots \times V_{\alpha_r} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha$ closed
Therefore $\overline{V} \subset W$.
Hence $x \in V \subset \overline{V} \subset W \subset U' \subset U$
Therefore X is T_3.

$j = 3\frac{1}{2}$: A corollary of the Stone-Cech Compactification Thm (below) is

Corollary 3.0.23 *X is completely regular $\Leftrightarrow$ X is homeomorphic to a subspace of a compact Hausdorff space.*

Proof of Case $j = 3\frac{1}{2}$ (Given Corollary)

By Corollary, $\forall \alpha$, find compact Hausdorff Y_α s.t. X_α homeomorphic to a subspace of Y_α.

Hence X is homeomorphic to a subspace of $Y := \prod_\alpha Y_\alpha$.

By Tychonoff, Y is compact and by case $j = 2$, Y is Hausdorff. Hence X is homeomorphic a subspace of a compact Hausdorff space so is completely regular by the Corollary.

Proof of Corollary:

$\Leftarrow$: By earlier theorems, a compact Hausdoff space is normal and thus completely regular and a subpace of a completely regular space is completely regular.

$\Rightarrow$ Follows from:

Theorem 3.0.24 [Stone-Cech Compactification] *Let X be completely regular. Then there exists a compact Hausdorff space $\beta(X)$ together with a (continuous) injection $X \hookrightarrow \beta(X)$ s.t.*

1. *$i : X \hookrightarrow \beta(X)$ is a homeomorphism*

2. *X is dense in $\beta(X)$*

3. *Up to homeomorphism $\beta(X)$ is the only space with these properties*

4. *Given a compact Hausdorff space W and $h : X \to W$ there is a unique $\bar{h}$ s.t. $h = \bar{h} \circ i$*

Definition 3.0.25 *$\beta(X)$ is called the* Stone-Cech *compactification of X.*

Example 3.0.26 *Let $X = (0, 1]$. Let $f : X \to [-1, 1]$ by $f(x) = \sin(1/x)$. Then f is a continuous function from X to the compact Hausdorff space $[-1, 1]$, but f does not extend to $[0, 1]$. Thus although $[0, 1]$ is a compact Hausdorff space containing $(0, 1]$ as a dense subspace, it is not the Stone-Cech compactification of $(0, 1]$.*

Proof of Theorem: Let $J = \{f : X \to \mathbb{R} \mid f \text{ bounded and continuous}\}$.

For $f \in J$, let I_f be the smallest closed interval containing $\text{Im}(f)$. As f is bounded, I_f is compact.

Let $Z = \prod_{f \in J} I_f$. It is compact Hausdorff.

Define $i : X \to Z$ by $(ix)_f = f(x)$. Since X is completely regular, $x \neq y \Rightarrow \exists f : X \to [0,1]$ s.t. $f(x) \neq f(y)$. Thus i is injective.

Claim: $i : X \xrightarrow{\cong} i(X)$.

Proof: Use the injection i to define another topology on X – the subspace topology as a subset of Z.

The Claim is equivalent to showing the subspace topology is equals to the original topology.

Since i is continuous (because its projections are), if U is open in the subspace topology then U is open in the original topology.

Conversely suppose U is open in the original topology.

Let $x \in U$. To show x is interior (in the subspace topology):

By definition of the subspace and product topologies, the subspace topology is the weakest topology s.t. $f : X \to \mathbb{R}$ is continuous $\forall f \in J$.

Because X is completely regular, $\exists f : X \to [0,1]$ s.t. $f(x) = 0$, $f(U^c) = 1$

$f \in J \Rightarrow f^{-1}([0,1))$ is open in the subspace topology.

$f^{-1}([0,1)) \subset U$ since $f(x) = 1$ $\forall x$ not in U.

Therefore $x \in \text{Int}(U)$ (in the subspace topology).

True $\forall x \in U$, so U is open in the subspace topology.

Let $\beta(X) = \overline{i(X)}$.

Then $\beta(X)$ is compact Hausdorff, as it is a closed subspace of a compact Hausdorff space and $X \cong i(X)$ is dense in $\beta(X)$ by construction.

To show the extension property and uniqueness of $\beta(X)$ up to homeomorphism,

Lemma 3.0.27

1. *Given $g : X \to Y$, $\exists!$ $\hat{g} : \beta(X) \to \beta(Y)$ s.t.*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ \beta(X) & \xrightarrow{\hat{g}} & \beta(Y) \end{array}$$

2. *If X is compact Hausdorff then $X \to \beta(X)$ is a homeomorphism.*

Proof:

1. *Uniqueness:* Since $\beta(Y)$ is Hausdorff and X is dense in $\beta(X)$ any two maps from $\beta(X)$ agreeing on X are equal. So $\hat{g}$ is unique.

   Existence: Let $\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ is bounded and continuous}\}$, and let $\mathcal{C}(Y) = \{f : Y \to \mathbb{R} \mid f \text{ is bounded and continuous}\}$.

   Let $z \in \beta(X)$.

   To define $\hat{g}(z)$: For $f \in \mathcal{C}_Y$, define $\Pi_f(\hat{g}z) = \Pi_{f\circ g}(z)$ $\forall x \in X$, and $\forall f \in \mathcal{C}_Y$. Each projection is continous so $\hat{g}$ is continuous.

   $\forall x \in X$ and $\forall f \in \mathcal{C}(Y)$:

   $\Pi_f(i_Y gx) = f\big(g(x)\big)$ while $\pi_f(\hat{g} i_X x) = \pi_{f\circ g}(i_X x) = f \circ g(x)$. Therefore $i_Y \circ g = \hat{g} \circ i_X$ which also shows that $\hat{g}\big(\beta(X)\big) \subset \overline{\hat{g}\big(i(X)\big)} \subset \overline{i(Y)} = \beta(Y)$.

   Hence $\hat{g}$ is the desired extension of g.

2. $i : X \hookrightarrow \beta(X)$ is continuous, and X is compact $\Rightarrow i(X)$ is compact $\Rightarrow i(X)$ is closed in $\beta(X)$ since $\beta(X)$ is Hausdorff.

   But $i(X)$ is dense in $\beta(X)$ so $i(X) = \beta(X)$. Hence i is a bijective map from a compact space to a Hausdorff space and is thus a homeomorphism.

Proof of Theorem (continued): Let $h : X \to Y$ where Y is compact Hausdorff. Then

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Y \\ {\scriptstyle i_X}\big\downarrow & & \big\downarrow{\scriptstyle i_Y}\ \cong \\ \beta(X) & \xrightarrow{\;\hat{h}\;} & \beta(Y) \end{array}$$

So $i_Y^{-1} \circ \hat{h}$ is the desired extension of h to $\beta(X)$. If W is another space with these properties then $X \cong W$ by the standard category theory proof. ☐

Definition 3.0.28 *X is called* 2nd countable *if $\exists$ a countable basis for the open sets of X.*

e.g. $X = \mathbb{R}^n$. Basis $= \{N_r(X) \mid r$ rational and all coordinates of X are rational $\}$

Definition 3.0.29 *X is called* 1st countable *if each $x \in X$ has a countable basis for its neighbourhoods.*

e.g. $X =$ *metric.* $\{N_r(X) \mid r$ rational$\}$ is a basis for the neighbourhoods of X.

Definition 3.0.30 *X is called* separable *if it has a countable dense subset*

Proposition 3.0.31 *2nd countable implies 1st countable and separable.*

Proof: 2*nd* countable implies 1*st* countable is trivial.

Let $\{U_j\}$ be a countable basis of (non-empty) open sets. $\forall j$, select $x_j \in U_j$. Let $A = \{x_j\}$. A is countable. Any open set intersects A so A is dense. □

Example 3.0.32 *Compact subspace which is not closed.*

Let $X := \mathbb{R}$ as a set.

Specify the topology on X to be the one coming from the subbasis;

$$\{U \cap \mathbb{Q} \mid U \text{open in standard topology on } \mathbb{R}\} \cup \{V \mid V \text{is the complement of a finite set of rationals}\}$$

Observe: In corresponding basis, any basis set containing an irrational can be obtained only by intersecting the second type of sets, yielding another set of this type. Therefore any open set in X containing an irrational is the complement of a finite set of rationals.

Hence if $S \subset X$ contains an irrational then S is compact because in any open cover of S at least one set contains all but finitely many points of S, so S can be covered by that set together with one set for each of the missing points. In particular, if y is irrational, $\mathbb{Q} \cup \{y\}$ is compact but not closed. (Its complement contains irrationals, so it can't be open since any open set containing an irrational contains all irrationals.)

3.0.1 Convergent Sequences

Definition 3.0.33 *A sequence (x_n) in X* converges *to x, written $(x_n) \to x$, if $\forall$ open U, $\exists N$ s,t, $n \geq N \Rightarrow x_n \in U$.*

Proposition 3.0.34 *X Hausdorff, $(x_n) \to x$, $(x_n) \to y$ implies that $x = y$.*

Proof: If $x \neq y$ separate x, y by open sets and apply definition to give contradiction. □

Proposition 3.0.35 *Suppose $A \subset X$. If $(a_n) \to x$ where $a_n \in A$ $\forall n$ then $x \in \overline{A}$. Conversely, if X is 1st countable and $x \in \overline{A}$ then $\exists$ sequence (a_n) in A s.t. $(a_n) \to x$ in X.*

Proof: Supppose $(a_n) \to x$. Then $\forall$ open U s.t. $x \in U$, $U \cap A \neq \emptyset$ so $x \notin \overline{A}$.

Conversely, suppose X is 1*st* countable and $x \in \overline{A}$.

Then any open neighbourhood of x intersects A.

Let $\{U_1, U_2, \ldots, U_n, \ldots\}$ be a basis for the open neighbourhoods of x.

Select $a_1 \in U$, $a_2 \in U_1 \cap U_2$, …, $a_n \in U_1 \cap U_2 \cdots \cap U_n$, …, with $a_n \in A$ $\forall n$. So $a_n \in U_k$ $\forall n \geq k$.

Given open V s.t. $x \in V$ find basic open U_N s.t. $U_N \subset V$.

Then $\forall n \geq N$, $a_n \in U_N \subset V$ so $(a_n) \to x$. □

Definition 3.0.36 *If $A \subset X$ and $(a_n) \to x$ where $a_n \in A$ then x is called a* limit point *of A.*

Thus previous proposition says that in a 1 countable space, as set is closed if and only if it contains its limit points.

Proposition 3.0.37 *Let $f : X \to Y$ be a (set) function. f is continuous if and only if ($(x_n) \to x \Rightarrow f(x_n) \to f(x)$).*

Proof Suppose f is continous and $(x_n) \to x$.
Given U s.t. $f(x) \in U$ then $x \in f^{-1}(U)$ so $\exists N$ s.t. $nn \geq N \Rightarrow x_n \in f^{-1}(U)$.
Therefore $n \geq N \Rightarrow f(x_n) \in U$ so $f(x_n) \to f(x)$.
Conversely, suppose X $1st$ countable and ($(x_n) \to x \Rightarrow f(x_n) \to f(x)$).
Let $A \subset Y$ be closed. Show $f^{-1}(A)$ is closed.
Let $x \in \overline{f^{-1}(A)}$. Find sequence (x_n) in $f^{-1}(A)$ s.t. $(x_n) \to x$.
Then for all n, $f(x_n) \in A$ and hypothesis implies $\big(f(x_n)\big) \to f(x)$. So A closed implies $f(x) \in A$. Therefore $x \in f^{-1}(A)$.
Thus $\overline{f^{-1}(A)} = f^{-1}(A)$ and hence $f^{-1}(A)$ is closed.
Therefore f is continuous. □

Definition 3.0.38 *X is called* sequencially compact *if every sequence has a convergent subsequence.*

Definition 3.0.39 *Suppose X is Hausdorff and $1st$ countable. Then X compact implies X sequentially compact.*

Proof: Let X be Hausdorff, $1st$ countable and compact.

Let (x_n) be a sequence in X. If any element appeas infinitely many times in (x_n) then (x_n) has a constant (thus convergent) subsequence, so suppose not. Then discarding repeated elements gives us a subsequence so we may assume that (x_n) has no repetitions.

Claim: $\exists x \in X$ s.t. $\forall$ open U containing x, $U \cap \{x_n\}$ is infinite.
Proof: Suppose not. That is, suppose that $\forall x$, $\exists$ open U_x s.t. $x \in U_x$ and $U_x \cap \{x_n\}$ is finite.
Then $\{U_x\}$ is an open cover so ha s a finite subcover $U_x^{(1)}$, $U_x^{(2)}$, …, $U_x^{(k)}$.
Since $\forall j$, $U_x^{(j)} \cap \{x_n\}$ is finite, $\{x_n\}$ is finite.
$\Rightarrow\Leftarrow$.

Choose x as in claim and let $\{V_1, V_2, \ldots, V_k, \ldots\}$ be a basis for the neighbourhoods of x.
Choose $x_{n(1)} \in V_1 \cap \{x_n\}$.
Choose $x_{n(2)} \in V_1 \cap V_2 \cap \{x_n \mid n > n(1)\}$.
$\vdots$
Choose $x_{n(k)} \in V_1 \cap \cdots \cap V_k \cap \{x_n \mid n > n(k-1)\}$.
$\vdots$
Then $(x_{n(1)}, x_{n(2)}, \ldots, x_{n(k)}, \ldots)$ is a subsequence of (x_n) and converges to x. □