9.13 Poincaré Duality

Let M be an oriented n-dimensional manifold and let $\{\zeta_K\}_{\left(\substack{K\subset M \\ L \text{ compact}}\right)}$ be its chosen orientation, where $\zeta_K \in H_n)M, M \setminus K)$. If M is compact, let $\zeta = \zeta_M$.

(The following also works in M is non-orientable provided $\mathbb{Z}/(2/ZZ)$-coefficients are used.)

Consider first the case where M is compact.

Let $D : H^i(M) \to H_{n-i}(M)$ by $D(z) = z \cap \zeta$.

Theorem 9.13.1 *(Poincaré Duality) $D : H^i(M) \to H_{n-i}(M)$ is an isomorphism $\forall i$.*

In the case where M is not compact:

For each compact $K \subset M$, define $D_K : H^i(M, M \setminus K) \to H_{n-i}(M)$ by $D_K(z) = z \cap \zeta_K$.

If $K \subset L \subset M$, K, L compact, then by theorem 9.12.5 $j^L_{K_*}(\zeta_L) = \zeta_K$ where $j^L_K : (M.M \setminus L) \to (M, M \setminus K)$.

Therefore

$$\begin{array}{ccc} H^i(M, M\setminus K) & & \\ \Big\downarrow {\scriptstyle j^{L^*}_K} & \overset{D_K}{\searrow} & \\ & & H_{n-i}(M) \\ & \underset{D_L}{\nearrow} & \\ H^i(M, M\setminus L) & & \end{array}$$

commutes since $D_K(z) = z \cap \zeta_K = z \cap j^L_{K_*}(\zeta_L) \overset{\text{lemma 9.10.15}}{=} j^{L^*}_K z \cap \zeta_L = D_L j^{L^*}_K(z)$. Thus the various maps D_K induce (by universal property) a unique map

$$D : \varinjlim_{\substack{K\subset M \\ K \text{ compact}}} H^i(M, M\setminus K) \to H_{n-i}(M)$$

where the partial ordering is induced by inclusion.

Notation: Write $H^i_c(M) = \varinjlim_{\substack{K\subset M \\ K \text{ compact}}} H^i(M, M\setminus K)$.

$H_c^*(M)$ is called the cohomology of M with compact support. An element of $H_c^*(M)$ is represented by a singular cochain which vanishes outside of some compact set. Of course, if M is already compact then each element in the direct system maps into $H^i(M)$ so that $H_c^i(M) = H^i(M)$ in this case.

Theorem 9.13.2 *(Poincaré Duality) $D : H_c^i(M) \to H_{n-i}(M)$ is an isomorphism $\forall i$.*

Proof:
Case 1: $M = \mathbb{R}$

Lemma 9.13.3 *Let $B \subset \mathbb{R}^n$ be a closed ball. Then $D_B : H^i(\mathbb{R}, \mathbb{R} \setminus B) \to H_{n-i}(\mathbb{R}^n)$ is an isomorphism $\forall i$.*

Proof: $H_q(\mathbb{R}, \mathbb{R} \setminus B) \cong H_q(\mathbb{R}, \mathbb{R} \setminus \{*\} \cong \tilde{H}_{q-1}(\mathbb{R}^n \setminus \{*\}) \cong \tilde{H}_{q-1}(S^{n-1})$. Similarly $H^q(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong \tilde{H}^{q-1}(S^{n-1})$. Thus if $i \neq n$ the lemma is trivial since both groups are 0.
For i=n:

The groups are isomorphic (both are $\mathbb{Z}$). Must show that D_B is an isomorphism.

ζ_B is a generator of $H_n(\mathbb{R}, \mathbb{R} \setminus B) \cong \mathbb{Z}$. Find generator $f \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ s.t. $\langle f, \zeta_B \rangle = 1$. To see that one of the two generators of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ must have this property, examine the Kronecker pairing of $\tilde{H}_{n-1}(S^{n-1})$ with $\tilde{H}^{n-1}(S^{n-1})$. Using the cellular chain complex $0 \to \mathbb{Z} \to 0 \ldots \to 0$ makes it obvious that the Kronecker pariting gives an ismorphism $\tilde{H}_{n-1}(S^{n-1}) \cong \text{Hom}\big(\tilde{H}_{n-1}(S^{n-1}), \mathbb{Z}\big) \cong \mathbb{Z}$ and that the ring identity $1 \in H^0(\mathbb{R}^n)$ is a generator. Thus

$\langle 1, D_B(f) \rangle = \langle 1, f \cap \zeta_B \rangle = \langle 1 \cup f, \zeta_B \rangle = \langle f, \zeta_B \rangle = 1$

so that $D_B(f)$ must be a generator of $H_0(\mathbb{R}^n)$. Hence D_B is an isomorphism.

Proof of theorem in case 1: Let $\alpha \in H_c^i(\mathbb{R}^n) = \varinjlim_{\substack{K \subset \mathbb{R}^n \\ K \text{ compact}}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$. Pick a representative $f \in H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ of α for some compact $K \subset \mathbb{R}^n$. Let B be a closed ball containing K. Replacing f by $j_K^{B*}(f)$ gives a new representative for α lying in $H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B)$, and by definition of D, $D(\alpha) - D_B(f)$. Since D_B is an isomorphism by the lemma, if $D(\alpha) = 0$ then $f = 0$ and so $\alpha = 0$. Hence D is $1-1$. Conversely, given $x \in H_{n-i}(\mathbb{R}^n)$, $\exists f \in H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ s.t. $D_B(f) = x$ and so the element α of $H_c^i(\mathbb{R}^n)$ represented by f satisfies $D(\alpha) = D_B(f) = x$. Hence D is onto.

(In effect, there is a cofinal subsystem which has stabilized. Therefore the direct limit map is the same as the map induced by this stabilized subsystem.) $\surd$

Case 2: $M = U \cap V$ where U, V are open subsets of M (thus submanifolds) s.t. the theorem is known for U, V, and $W := U \cap V$

Proof: Let K, L be compact subsets of U, V respectively. Let $A = K \cap L$, $N = K \cup L$. Then have Mayer-Vietoris sequence

$$\begin{array}{ccccccc}
H^q(M, M \setminus A) & \to & H^q(M, M \setminus K) \oplus H^q(M, M \setminus L) & \to & H^q(M, M \setminus N) & \to & H^{q+1}(M, M \setminus A) \\
\cong \big\downarrow \text{(excision)} & & \cong \big\downarrow \text{(excision)} & & & & \cong \big\downarrow \\
H^q(W, W \setminus A) & & H^q(U, U \setminus K) \oplus H^q(V, V \setminus L) & & & &
\end{array}$$

Lemma 9.13.4

$$\begin{array}{ccccccc}
H^{q-1}(M, M \setminus N) & \to & H^q(W, W \setminus A) & \to & H^q(U, U \setminus K) \oplus H^q(V, V \setminus L) & \to & H^{q+1}(M, M \setminus N) \\
D_N \big\downarrow & ① & D_A \big\downarrow & ② & D_K \oplus D_K \big\downarrow & ③ & D_N \big\downarrow \\
H_{n-q+1}(M) & \xrightarrow{\Delta_*} & H_{n-q}(W) & \longrightarrow & H_{n-q}(U) \oplus H_{n-q}(V) & \longrightarrow & H_{n-q}(M)
\end{array}$$

commutes.

Proof:

For square ②: Let $j_W^U : (W, W \setminus A) \to (U, U \setminus A)$ denote the inclusion map of pairs. (It induces an excision isomorphism.)

$$\begin{array}{ccc}
\tilde{f} \in H^q(U, U \setminus A) & \xrightarrow{j_A^{K*}} & H^q(U, U \setminus K) \\
j_W^{U*} \big\downarrow \cong & & \big\| \\
f \in H^q(W, W \setminus A) & \xrightarrow{j_A^{K*}} & H^q(U, U \setminus K) \\
\big\downarrow D_A & ? & \big\downarrow D_K \\
H_{n-q}(W) & \xrightarrow{j_W^{U*}} & H_{n-q}(U)
\end{array}$$

Let $f \in H^q(W, W \setminus A)$.

By the excision isomorphism, $\exists \tilde{f} \in H^q(U, U \setminus A)$ s.t. ${j_W^U}^*(\tilde{f}) = f$.

Let $\zeta_A^U \in H_n(U, U - A)$ be the restriction of ζ_k to A. i.e. $\zeta_A^U := {j_A^K}_* \zeta_K$. By compatibility of orientations, ${j_W^U}_*(\zeta_A) = \zeta_A^U$ (where ζ_A means ζ_A^W).

$$
\begin{aligned}
j^U_{W*} D_A f &= j^U_{W*}(f \cap \zeta_A) \\
&= j^U_{W*}\big(j^{U\,*}_W(\tilde f) \cap \zeta_A\big) \\
&\overset{\text{(lemma 9.10.15)}}{=} \tilde f \cap j^U_{W*}\zeta_A \\
&= \tilde f \cap \zeta^U_A \\
&= \tilde f \cap j^K_{A\,*}\zeta_K \\
&\quad \text{(map of pairs is } (U, U \smallsetminus K) \to (U, U \smallsetminus A) \text{ whose restriction to } U \text{ is 1)} \\
&\overset{\text{(lemma 9.10.15)}}{=} j^{K*}_A \tilde f \cap \zeta_K \\
&= D_K j^{K*}_A \tilde f
\end{aligned}
$$

so the diagram commutes. Get the same diagram with V replacing U, so square ② commutes.

Similarly, doing the same arguments with the pairs (M, U) replacing (U, W) and then (M, V) replacing $U, W)$, we get that the third square commutes.

For square ①:

$$
\begin{array}{ccc}
H^{q-1}(M, M \smallsetminus N) & \xrightarrow{\ \Delta^*\ } & H^q(M, M \smallsetminus A) \\
 & & \big\downarrow \cong \\
\big\downarrow D_N & ? & H^q(W, W \smallsetminus A) \\
 & & \big\downarrow D_A \\
H_{n-q+1}(M) & \xrightarrow{\ \Delta_*\ } & H_{n-q}(W)
\end{array}
$$

apply lemma 9.10.16

$$\begin{array}{ccc}
H^{q-1}(X,B) & \xrightarrow{\Delta^*} & H^q(X,Y) \\
 & & \downarrow \cong \text{(excision)} \\
\Big\downarrow \cap[v] & & H^q(A, A\cap Y) \\
 & & \downarrow \cap[v'] \\
H_{n-q+1}(X) & \xrightarrow{\Delta_*} & H_{n-q}(A)
\end{array}$$

in the case:

$X := M$; $X_1 := U$; $X_2 := V$; $Y := M \smallsetminus A$; $Y_1 := M \smallsetminus K$; $[v] := \zeta_N$.

(Thus $A = U \cap V = W$ and $B = Y_1 \cap Y_2 = M \smallsetminus (K \cup L) = M \smallsetminus N$. Note: $X_1 \cap Y_1 = U \cap (M \smallsetminus K) = M$ since $K \subset U$.) $\surd$

Proof of Case 2 (cont.): Passing to the limit gives a commutative diagram with exact rows (recall the homology commutes with direct limits so exactness is preserved)

$$\begin{array}{ccccccccc}
H_c^q(W) & \to & H_c^q(U)\oplus H_c^q(V) & \xrightarrow{\Delta^*} & H_c^{q+1}(M) & \to & H_c^{q+1}(W) & \to & H_c^{q+1}(U)\oplus H_c^{q+1}(V) \\
D\downarrow\cong & & (D\oplus D)\downarrow\cong & & D\downarrow & & D\downarrow\cong & & \downarrow\cong \\
H_{n-q}(W) & \to & H_{n-q}(U)\oplus H_{n-q}(V) & \xrightarrow{\Delta^*} & H_{n-q}(M) & \to & H_{n-q-1}(W) & \to & H_{n-q-1}(U)\oplus H_{n-q-1}(V)
\end{array}$$

so by the 5-lemma, $D : H_c^q(M) \to H_{n-q}(M)$ is an isomorphism. $\surd$

Case 3: M is the union of a nested family of open sets U_α where the duality theorem is known for each U_α.

Since $M = \cup_\alpha U_\alpha$ and U_α is open, $S_*(M) = \cup_\alpha S_*(U_\alpha)$ so $H_*(M) = \varinjlim_\alpha H_*(U_\alpha)$.

Similarly each generator of $S_c^*(M)$ vanishes outside some compact K, where $S_c^*(M) := \varinjlim_{\substack{K\subset M \\ K \text{ compact}}} S^*(M, M\smallsetminus K)$. (Since homology commutes with direct limits, $H_c^*(N) = H\big(S_*^c(M)\big)$.
)

Find U_{α_0} s.t. $K \subset U_{\alpha_0}$ s.t. $K \subset U_{\alpha_0}$. Then $f \in \operatorname{Im} S_c^*(U_{\alpha_0})$. Thus again $S_c^*(M) = \cup_\alpha s_c^*(U_\alpha)$ and so $H_c^*(M) = \varinjlim_\alpha H_c^*(U_\alpha)$. $\surd$

Case 4: M is an open subset of $\mathbb{R}^n$

If V is a convex open subset of M, then the theorem holds for V by Case 1. (i.e. V is homemorphic $\mathbb{R}^n$.)

If V, W are converx open then so is $V \cap W$ so the theorem holds for $V \cup W$ by Case 2.

Hence if $V = V_1 \cup \ldots \cup V_k$ where V_i is convex open, then the theorem holds for V.

Write $M = \cup_{i=1}^\infty V_i$ by letting $\{V_i\}$ be
$\{N_r(x) \mid N_r(x) \subset M, r \text{ rational}, x \text{ has rational coordinates}\}$ (which is countable).

Let $W_l = \cup_{i=1}^k V_i$. Then by the above, the theorem holds for W_k $\forall k$, $\{W_k\}$ are nested, and $M = \cup_{k=1}^\infty W_k$. Therefore the theorem holds for M by Case 3. $\surd$

Case 5: General Case

By Zorn's Lemma $\exists$ a maximal open subset U of M s.t.the theorem holds for U. If $U \neq M$, find $x \in M \setminus U$ and find an open coordinate neighbourhood C of x. Then by Case 4, the theorem holds for V and $U \cap V$ so by Case 2 the theorem holds for $U \cup V$.$\Rightarrow\Leftarrow$.

Therefore $U = M$. $\square$