11 Smooth functions and partitions of unity

11.1 Smooth functions

Example 11.1 The function $\theta : \mathbb{R} \to \mathbb{R}$ given by

$$\theta(t) = 0, \, t \leq 0$$

and

$$\theta(t) = e^{-1/t}, \, t > 0$$

is smooth and all its derivatives are 0 at $t = 0$. In particular it is not represented by its Taylor series at 0.

The open cube $C(r)$ is defined as follows.

Definition 11.2

$$C(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | |x_i| < r \forall i\}$$

The closure of $C(r)$ is denoted $\overline{C(r)}$.

Lemma 11.3 There exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ with

1. $0 \leq h(x) \leq 1$
2. $h(x) = 1, \, x \in \overline{C(1)}$
3. $h(x) = 0, \, x \notin C(2)$

Remark 11.4 The function $h$ is called the bump function.

Proof: Define

$$\phi(x) = \frac{\theta(x)}{\theta(x) + \theta(1-x)}.$$

Then $\phi(x) = 1$ for $x > 1$ and $\phi(x) = 0$ for $x \leq 0$. Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \phi(x + 2)\phi(2 - x).$$

Then $\psi(x) = 1, \, |x| \leq 1$ while $\psi(x) = 0, \, |x| \geq 2$. Thus define $h(x_1, \ldots, x_n) = \psi(x_1) \ldots \psi(x_n)$. 43
Definition 11.5  The support of a smooth function $f : M \to \mathbb{R}$ is the closure of the set of points $x \in M$ where $f(x) \neq 0$.

Consequences of existence of the function $h$:

Proposition 11.6  Let $M$ be a smooth manifold and let $(U, \phi)$ be a chart in an atlas for $M$. There exists a smooth function $f : M \to \mathbb{R}$ with $f(M) \subset [0, 1]$ and $\text{Supp}(f) \subset U$, and $f(x) = 1$ on a neighbourhood of $p \in U$.

Proof:  For a point $p \in U$ choose a cubical neighbourhood $B \subset \mathbb{R}^n$ around $\phi(p)$, say

$$\{x : |\phi(p_i) - x_i| < \epsilon\}.$$  

Define $\alpha : B \to C(2)$ by

$$\alpha(x) = \frac{2}{\epsilon} (x - \phi(p))$$

and define

$$f(x) = \{h \circ \alpha \circ \phi(x), x \in U \cap \phi^{-1}(B)\}$$

and 0 otherwise. Then $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = 1$ if $|x_i| \leq 1$ for all $i$, and $h(x) = 0$ if $|x_i| \geq 2$ for some $i$.

Definition 11.7  A partition of unity subordinate to an open cover $\{U_\alpha\}$ of $M$ is a collection of smooth functions $f_\gamma : M \to \mathbb{R}$ such that

1. For all $\gamma$, $\text{Supp}(f_\gamma) \subset U_\alpha$ for some $\alpha$ (here $\text{Supp}f_\gamma$ is the closure of the subset where $f_\gamma(x) \neq 0$)

2. $0 \leq f_\gamma \leq 1$ on $M$

3. $\forall x \in M$ there is an open neighbourhood $V_x$ of $x$ s.t. there exist only finitely many $f_\gamma$ s.t. $\text{Supp}f_\gamma \cap V_x \neq \emptyset$ are nonzero at any points on $V_x$

4. $\sum_\gamma f_\gamma(x) = 1$ (this sum is finite because of (3))

We shall prove existence of a partition of unity. We require some facts from general topology.

Lemma 11.8  Manifolds are regular (in other words if $C \subset X$ is a closed subset, $C \neq X$ and $x \in X \setminus C$ then these can be separated by disjoint open subsets)

Let $M$ be a Hausdorff space.
Definition 11.9 M is paracompact if

1. M is regular
2. every open cover admits a locally finite refinement

Definition 11.10 An open cover \( \{V\} \) is a refinement of the open cover \( \{U\} \) if there exists \( \iota: \mathcal{I}_V \to \mathcal{I}_U \) (where \( \mathcal{I}_V \) is the indexing set of \( \{V\} \) and similarly for \( \{U\} \)) such that \( V_\beta \subset I_{\iota(\beta)} \).

Theorem 11.11 Manifolds are paracompact.

Proof: There exist compact subsets \( K_1 \subset K_2 \subset \ldots \) of \( M \) such that \( K_r \subset \text{Int}(K_{r+1}) \) and \( M = \bigcup_r \text{Int}(K_r) \). Let \( \{W_i\} \) be a countable base of the topology with each \( W_i \) compact. \( K_1 = W_1, \ldots, K_r \subset \bigcup_{i=1}^\ell W_i \) (let \( \ell \) be the smallest for which this is true) and \( K_{r+1} = \bigcup_{i=1}^\ell W_i \). Let \( \{U_\alpha\} \) be an open cover: to get a locally finite refinement, choose finitely many \( V_i = U_\alpha \) covering \( K_1 \).

Extend this by \( \{U_\alpha\}_{i=\ell_1+1}^{\ell_2} \) to give an open cover of \( K_2 \). \( M \) is Hausdorff so \( K_1 \) is closed, and \( V_i = U_\alpha \setminus K_1 \) is open, \( \ell_1 + 1 \leq \ell_2 \) and \( \{V_i\}_{i=\ell_1+1}^{\ell_2} \) is an open cover of \( K_2 \).

Note that \( V_i \) does not meet \( K_1 \) for \( i > \ell_1 \).

By induction we get \( \{V_i\} \) such that \( K_r \) meets only finitely many elements of \( V \) \( \forall r \geq 1 \).

For any \( x \in M \), \( x \in \text{Int}(K_r) \) for some \( r \), there exists a neighbourhood meeting only finitely many elements of \( V \).

Definition 11.12 A precise refinement of an open cover \( \{U_\alpha\} \) is a locally finite refinement indexed by the same set with \( \bar{V}_\alpha \subset U_\alpha \).

Proposition 11.13 If \( M \) is a paracompact manifold and \( \{U_\alpha\} \) is an open cover of \( M \), then this cover has a precise refinement.

Proof: There exists a refinement \( \{W_k\} \) with \( j: K \to A \) such that \( W_k \subset U_{j(k)} \) (since \( M \) is regular). Passing to a locally finite refinement of \( W \) gives a locally finite refinement \( V' \) of \( U \) with \( \iota: B \to A \) with \( \bar{V}'_\beta \subset U_{\iota(\beta)} \). The \( \bar{V}'_\beta \) are a locally finite family of closed subsets of \( M \). For all \( \alpha \in A \), define \( \beta_\alpha := \iota^{-1}(\alpha) \).

\[
V_\alpha = \bigcup_{\beta_\alpha} V'_\beta
\]

Because \( V' \) is locally finite, \( \bar{V}_\alpha = \bigcup_{\beta_\alpha} \bar{V}'_\beta \subset U_\alpha \).

Definition 11.14 M is normal if whenever \( A \) and \( B \) are disjoint closed subsets of \( M \), there is an open set \( U \) containing \( A \) and disjoint from \( B \) with \( \bar{U} \cap \bar{B} = \emptyset \).

Lemma 11.15 Paracompact spaces are normal.
Proposition 11.16 (Urysohn’s lemma) Suppose $M$ is normal and $A$ and $B$ are closed subsets of $M$. There exists a smooth function $f : M \to [0, 1]$ such that $f |_A = 1$ and $f |_B = 0$.

Theorem 11.17 Suppose $K$ is compact and $K \subset U$ for an open set $U$. Then there exists a smooth function $f : \mathbb{R}^n \to [0, 1]$ with $f |_K = 1$ and $f$ supported in $U$.

Use Lemma 11.3 to show that

Lemma 11.18 If $A = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$ then there is a smooth function $g_A : \mathbb{R}^n \to [0, 1)$ such that $g_A > 0$ for $y \in A$ and $g_A |_{\mathbb{R}^n \setminus A} = 0$.

Proof: (of Theorem) Let $K \subset \mathbb{R}^n$ be compact and $U \subset \mathbb{R}^n$ an open neighbourhood of $K$. For each $x \in K$, let $A_x$ be an open bounded neighbourhood of $x$ of the form

$$A_x = (a_{1,x}, b_{1,x}) \times \ldots \times (a_{n,x}, b_{n,x})$$

with $A_x \subset U$, $x \in A_x$. By Lemma 11.18, there is a smooth function $g_x : \mathbb{R}^n \to [0, 1)$ with $g_x(y) > 0$ for $y \in A_x$ and $g_x(y) = 0$ for $y \not\in A_x$. Since $K$ is compact, it is covered by finitely many $A_{x_1}, \ldots, A_{x_q}$. Define $G = g_{A_{x_1}} + \ldots + g_{A_{x_q}} : \mathbb{R}^n \to \mathbb{R}$. Then $G$ is smooth on $\mathbb{R}^n$, $G(x) > 0$ if $x \in K$ and $\text{supp}(G) = A_{x_1} \cup \ldots \cup A_{x_q} \subset U$. Since $K$ is compact, there exists $\delta > 0$ such that $G(x) \geq \delta$ for $x \in K$. Define our bump function $\ell$ so it is 0 for $t \leq 0$ and 1 for $t \geq \delta$. Define $f = \ell \circ G : \mathbb{R}^n \to [0, 1]$. Then

1. $f$ is smooth
2. $\text{supp}(f) \subset U$
3. $f |_K = 1$

Theorem 11.19 There is a partition of unity subordinate to any open cover $\mathcal{U}$.

Proof: If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then a partition of unity subordinate to $\mathcal{V}$ induces one subordinate to $\mathcal{U}$.

$$\iota : \mathcal{B} \to \mathcal{U}$$

If $\beta \subset U_{\iota^{-1}(\beta)} \{\mu_\beta\}$ subordinate to $\mathcal{U}$

$$\lambda_\alpha = \sum_{\beta \in \iota^{-1}(\alpha)} \mu_\beta.$$  

So since manifolds are locally compact, WLOG each $U_\alpha$ has compact closure in $M$. 

46
A precise refinement has the property that \( \bar{V}_\alpha \subset U_\alpha \) is a compact subset. We may use Urysohn’s lemma to give \( f \). We choose a precise refinement \( \mathcal{V} \) with \( \mathcal{W} \) a precise refinement of \( \mathcal{V} \).

\( \{ \gamma_\alpha \}_{\alpha \in U} \text{ satisfy } \gamma_\alpha|_{W_\alpha} = 1 \text{ and } \text{supp}(\gamma_\alpha) \subset \bar{V}_\alpha \subset U_\alpha. \)

\( \{ \text{supp} \gamma_\alpha \} \text{ is locally finite. } \gamma := \sum_\alpha \gamma_\alpha \text{ is smooth and } \geq 0. \) Define \( v_\alpha = \gamma_\alpha / \gamma \), which is smooth. The \( v_\alpha \) are a partition of unity.

**Applications of partitions of unity:**

The primary application is integration on manifolds. Let us begin with integration of a function on \( \mathbb{R}^n \). Assume \( \{ U_\alpha \} \) is an open cover of \( \mathbb{R}^n \) and consider a partition of unity \( \{ f_\alpha \} \) subordinate to this open cover. Let \( g \) be a smooth function on \( \mathbb{R}^n \). Then

\[
\int_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} (\sum_\alpha f_\alpha) g = \sum_\alpha \int_{U_\alpha} (f_\alpha g).
\]

**Application to Whitney embedding theorem:**

**Proposition 11.20** Let \( X \) be a compact manifold. Then there is an injective immersion from \( X \) into \( \mathbb{R}^M \) for some \( M \).

**Proof:** Construct a covering of \( X \) by charts \( (U_\alpha, \phi_\alpha) \), and take a partition of unity \( \{ f_\alpha \} \) subordinate to the covering \( \{ U_\alpha \} \). Since \( X \) is compact, WLOG we may assume the number of \( U_\alpha \) is a finite number \( M \). Then define \( F : X \to \mathbb{R}^M \) by

\[
F(x) = (f_1(x)\phi_1(x), \ldots, f_M(x)\phi_M(x))
\]