Chapter 5

Paracompactness

Let $\{W_\alpha\}_{\alpha\in I}$ be a cover of X. (We do not assume W_α is open.)

Definition 5.0.27 *A cover $\{T_\beta\}_{\beta\in J}$ is called a* refinement *of $\{W_\alpha\}_{\alpha\in I}$ if $\forall\beta \in J$, $\exists\alpha \in I$ s.t. $T_\beta \subset W_\alpha$.*

Definition 5.0.28 *A collection $\{W_\alpha\}_{\alpha\in I}$ of subsets of X is called* locally finite *if each $x \in X$ has an open neighbourhood whose intersection with W_α is non-empty for only finitely many α.*

Proposition 5.0.29 *$\{W_\alpha\}_{\alpha\in I}$ is locally finite $\Rightarrow \cup_\alpha\overline{W}_\alpha = \overline{\cup_\alpha W_\alpha}$*

Proof: $\overline{W_\alpha} \subset \overline{\cup_\alpha W_\alpha} \;\Rightarrow\; \cup_\alpha\overline{W_\alpha} \subset \overline{\cup_\alpha W_\alpha}$

Conversely suppose $y \notin \cup\overline{W}_\alpha$.

Find open U s.t. $y \in U$ and $U \cap W_\alpha = \emptyset$ for $\alpha \neq \alpha_1, \dots \alpha_n$.

$y \notin \overline{W_{\alpha_1}}, \dots, \overline{W_{\alpha_n}}$.

Therefore $y \in V := U \cap (\overline{W_{\alpha_1}})^c \cap \cdots \cap (\overline{W_{\alpha_n}})^c$ open

$V \cap W_\alpha = \emptyset\ \forall\alpha$

Therefore $V^c \subset \overline{\cup_\alpha W_\alpha}$ (since V^c closed)

Therefore $V \cap (\cup_\alpha W_\alpha) = \emptyset$.

Hence $y \notin \overline{\cup W_\alpha}$ □

Definition 5.0.30 *A topological space X is called* paracompact *if every open cover of X has a locally finite refinement.*

Note: Compact $\Rightarrow$ paracompact. (A subcover is also a refinement.)

Proposition 5.0.31 *If A is closed $\subset X$ and X is paracompact, then A is paracompact.*

Proof: Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of A. For all α write $U_\alpha = V_\alpha \cap A$ with V_α open in X. Then $\{V_\alpha\} \cup \{A^c\}$ is an open cover of X so it has a locally finite refinement $\{W_\beta\}_{\beta\in I}$.

Then $\{W_\beta \cap A\}_{\beta\in I}$ is a locally finite refinement of $\{U_\alpha\}_{\alpha\in J}$. □

Proposition 5.0.32 *X paracompact Hausdorff $\Rightarrow X$ normal*

Proof:
First show that X is regular:

Let $a \in X$ and let $B \subset X$ be closed with $a \notin B$.

$\forall b \in B$ $\exists$ open nbhd. U_b s.t. $a \notin \overline{U_b}$ (X Hausdorff)

$\{U_b\}_{b \in B} \cup B^c$ is an open cover of X.

Let $\{W_\alpha\}_{\alpha \in J}$ be a locally finite refinement.

Let $I = \{\alpha \in J \mid W_\alpha \cap B \neq \emptyset\}$ Therefore $\{W_\alpha\}_{\alpha \in I}\}$ covers B.

Set $V := \cup_{\alpha \in I} W_\alpha \supset B$.

$\forall \alpha$ $\exists b \in B$ s.t. $W_\alpha \subset U_b$, and so $\overline{W_\alpha} \subset \overline{U_\beta} \Rightarrow a \notin \overline{W_\alpha}$

Therefore $a \notin \cup_{\alpha \in I} \overline{W_\alpha} = \overline{\cup_{\alpha \in I} W_\alpha} = \overline{V}$.

Therefore X is regular.

Now given closed A, B, s.t. $A \cap B = \emptyset$

$\forall b \in B \exists$ open U_b s.t. $A \cap \overline{U_b} = \emptyset$.

$\{U_b\}_{b \in B} \cup B^c$ covers X.

Let $\{W_\alpha\}_{\alpha \in J}$ be a locally finite refinement.

Let $I = \{\alpha \in J | W_\alpha \cap B \neq \emptyset\}$. Then $\{W_\alpha\}_{\alpha \in I}$ covers B. Set $V = \cup_{\alpha \in I} W_\alpha$.

For all α $\exists b \in B$ s.t. $W_\alpha \subset U_b$ so $\overline{W_\alpha} \subset \bar{U}_b \Rightarrow A \cap \bar{W}_\alpha = \emptyset$. Hence $\emptyset = A \cap (\cup_{\alpha \in I} \overline{W_\alpha}) = A \cap \overline{\cup_{\alpha \in I} W_\alpha} = A \cap \bar{V}$.

Hence X is normal. $\square$

Definition 5.0.33 *Let X be a topological space and let $\{U_j\}_{j \in J}$ be an open cover of X. A* partition of unity *relative to the cover $\{U_j\}_{j \in J}$ consists of a set of functions $f_j : X \to [0,1]$ such that:*

1. *$\overline{f_j^{-1}((0,1])} \subset U_j$ $\forall j \in J$.*
2. *$\overline{f_j^{-1}((0,1])}_{j \in J}$ is locally finite.*
3. *$\sum_{j \in J} f_j(x) = 1$ $\forall x \in X$.*

Note: (2) implies that if $x \in X, f_j(x) = 0$ for all but finitely many j so the sum in (3) makes sense.

$\{f_j\}_{j \in J}$ is a partition of unity implies that $\left\{ f_j^{-1}((0,1])_{j \in J} \right\}$ is a locally finite refinement of $\{U_j\}$.

Hence if every open cover of X has a partition of unity then X is paracompact.

Conversely

Theorem 5.0.34 *If X is paracompact Hausdorff, then for every open cover $\{U_\alpha\}_{\alpha \in J}$ of X there is a partition of unity relative to $\{U_\alpha\}_{\alpha \in J}$.*

Proof: Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of X where X is paracompact Hausdorff.

Let $\{V_\beta\}_{\beta\in I}$ be a locally finite refinement.

Then $\exists\phi : I \to J$ s.t. $V_\beta \subset U_{\phi(\beta)}$ $\forall\beta \in I$.

Given $\alpha \in J$ set $W_\alpha = \cup_{\{\beta|\phi(\beta)=\alpha\}} V_\beta$. Then $W_\alpha \subset U_\alpha$.

Claim: $\{W_\alpha\}$ is locally finite.

Proof of Claim: Let $x \in X$. Then $\exists U_x$ s.t. $U_x \cap V_\beta = \emptyset$ for all but $\beta_1, \ldots, \beta_n$. Hence $U_x \cap W_\alpha = \emptyset$ unless $\phi(\beta_j) = \alpha$ for some $j = 1, \ldots, n$.

Therefore $U_x \cap W_\alpha = \emptyset$ unless $\phi(\beta_j) = \alpha$, some $j = 1, \ldots, n$.

i.e. $U_x \cap W_\alpha = \emptyset$ for all but $\phi(\beta_1), \ldots, \phi(\beta_n)$ which is a finite set (although it might contain duplicate entries).

Therefore $\{W_\alpha\}$ locally finite. $\checkmark$

Proof of Thm. (cont.) Suff. to show $\exists$ partition of unity relative to $\{W_\alpha\}$ since this gives functions $f_\alpha : X \to [0, 1]$ s.t. $\overline{f^{-1}((0, 1])} \subset W_\alpha \subset U_\alpha$.

Lemma 5.0.35 *Let $\{U_\alpha\}_{\alpha\in J}$ be a locally finite open cover of X where X normal. Then $\exists$ locally finite open cover $\{V_\alpha\}_{\alpha\in J}$ s.t. $V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$ $\forall\alpha \in J$.*

Proof of Thm. (concluded; given Lemma):

Apply Lemma to $\{W_\alpha\}_{\alpha\in J}$ to get cover $\{V_\alpha\}_{\alpha\in J}$ s.t. $V_\alpha \subset \overline{V_\alpha} \subset W_\alpha$ $\forall\alpha$.

$\{W_\alpha\}$ locally finite $\Rightarrow$ $\{V_\alpha\}$ locally finite.

Do it again to get locally finite cover $\{T_\alpha\}_{\alpha\in J}$ s.t. $T_\alpha \subset \overline{T_\alpha} \subset V_\alpha \subset \overline{V_\alpha} \subset W_\alpha$ $\forall\alpha$.

X paracompact Hausdorff $\Rightarrow$ X normal $\Rightarrow$ $\exists g_\alpha : X \to [0, 1]$ s.t. $g_\alpha(\overline{T_\alpha}) = 1$, $g_\alpha(V_\alpha^c) = 0$.

$g_\alpha^{-1}(0, 1] \subset V_\alpha \Rightarrow \overline{g_\alpha^{-1}(0, 1]} \subset \overline{V_\alpha} \subset W_\alpha$.

Define $g(x) = \sum_\alpha g_\alpha(x)$ (finite sum since $f_\alpha(x) = 0$ unless $x \in V_\alpha$ and $\{V_\alpha\}$ locally finite so x in only finitely many V_α)

Set $f_\alpha(x) = g_\alpha(x)/g(x)$.

Then $\{f_\alpha\}_{\alpha\in J}$ is the desired partition of unity.

Proof of Lemma: To help prove Lemma:

Lemma 5.0.36 *(Sublemma). Let X be normal. Suppose $X = U \cup V$ U, V open. Then $\exists$ open W s.t. $W \subset \overline{W} \subset U$ and $X = W \cup V$.*

Proof:(Exercise)

Proof of Lemma (cont.): Well order J.

$X = U_{j_0} \cup W_{j_0}$ where j_0 = least elt. of J and $W_{j_0} = \bigcup_{j>j_0} U_j$.

SubLemma $\Rightarrow$ $\exists$ open V_{j_0} s.t. $V_{j_0} \subset \overline{V_{j_0}} \subset U_\alpha$ and $X = V_{j_0} \cup W_{j_0}$.

Suppose that for all $\gamma < \beta$ we have found open V_γ s.t. $V_\gamma \subset \overline{V_\gamma} \subset U_\gamma$ and

$$X = \bigcup_{j\le\gamma} V_j \cup \bigcup_{j>\gamma} U_j.$$

Claim: $X = \bigcup_{j<\beta} V_j \cup \bigcup_{j\geq\beta} U_j$.

Proof of Claim: Let $x \in X$.
If $x \in U_j$ some $j \geq \beta$, then $x \in$ RHS.
Otherwise, let M be max. s.t. $x \in U_M$. ($\{U_j\}$ locally finite $\Rightarrow \exists$ such max.)
Since $M < \beta$, applying induction hypoth. with $\gamma = M$:

$$X = \bigcup_{j\leq M} V_j \cup \bigcup_{j>M} U_j.$$

$x \notin U_j$ any $j > M$ so $x \in V_j$ some $j \leq M$.
i.e. $x \in$ RHS.

Proof of Lemma (cont.): By Claim, $X = U_\beta \cup W_\beta$ where

$$W_\beta = \bigcup_{j\leq\beta} V_j \cup \bigcup_{j>\beta} U_j.$$

SubLemma$\Rightarrow \exists$ open V_β s.t. $V_\beta \subset \overline{V_\beta} \subset U_\beta$ and $X = V_\beta \cup W_\beta$. i.e.

$$X = \bigcup_{j\leq\beta} V_j \cup \bigcup_{j>\beta} U_j.$$

completing induction step.
Therefore $\exists$ open V_j s.t. $V_j \subset \overline{V_j} \subset U_j$ and

$$X = \bigcup_{j\leq\gamma} V_j \cup \bigcup_{j>\gamma} U_j \qquad \forall\gamma.$$

Claim: $X = \cup_j V_j$.
Proof: Given $x \in X$ find max. M s.t. $x \in U_M$.
Apply above with $\gamma = M$ to see that $x \in V_j$ some $j \leq \gamma$.

Proof of Lemma (concluded): $V_j \subset U_j \quad \forall j$, $\{U_j\}$ locally finite $\Rightarrow \{V_j\}$ locally finite.
$\{V_j\}$ is the required cover. □

Theorem 5.0.37 *Let X be regular. Suppose that every open cover of X has a countable refinement. Then X is paracompact.*

Lemma 5.0.38 *Let $\{B_\beta\}_{\beta\in J}$ be a locally finite cover of X by closed sets. Suppose $\{E_\alpha\}_{\alpha\in I}$ is a collection of sets (arbitrary — not necessarily open, closed, ...) s.t. $\forall\beta$, $B_\beta \cap E_\alpha = \emptyset$ for almost all α. Then $\forall\alpha \in I$ we can choose open U_α s.t. $E_\alpha \subset U_\alpha$ and $\{U_\alpha\}$ locally finite.*

Note: $\{E_\alpha\}$ must be locally finite.

i.e. $\forall x \exists Q_x$ s.t. Q_x intersects only finite many B_β and each such B_β intersects only finitely many E_α.

Proof of Lemma: Set $C_\alpha := \bigcup_{B_\beta \cap E_\alpha = \emptyset} B_\beta$.

$\{B_\beta \mid B_\beta \cap E_\alpha = \emptyset\} \subset \{B_\beta\}$ which is locally finite.

Therefore $\overline{C_\alpha} = \bigcup_{B_\beta \cap E_\alpha = \emptyset} \overline{B_\beta} = \bigcup_{B_\beta \cap E_\alpha = \emptyset} B_\beta = C_\alpha$.

Therefore C_α is closed.

Set

$$U_\alpha := (C_\alpha)^c = \bigcup_{\substack{B_\beta \cap E_\alpha = \emptyset \\ \Leftrightarrow \\ E_\alpha \subset B_\beta^c}} B_\beta^c \supset E_\alpha.$$

Show $\{U_\alpha\}$ locally finite.

Let $x \in X$.

Find open V s.t. $x \in V$ and $V \cap B_\beta = \emptyset$ for $\beta \neq \beta_1, \ldots, \beta_n$.

Therefore $V \subset B_{\beta_1} \cup \ldots \cup B_{\beta_n}$.

$\forall j$, $B_{\beta_j} \cap E_\alpha = \emptyset$ for all but finitely many α.

Let $\{\alpha_1, \ldots, \alpha_k\}$ be the set of all such α for all $j = 1, \ldots, n$.

For $\alpha \neq \alpha_1, \ldots, \alpha_k$:

$V \subset B_{\beta_1} \cup \ldots \cup B_{\beta_n} \subset \bigcup_{B_\beta \cap E_\alpha = \emptyset} B_\beta = C_\alpha$

Therefore $V \cap U_\alpha = \emptyset$ for $\alpha \neq \alpha_1, \ldots, \alpha_k$. □

Proof of Thm. Let $\{U_j\}_{j \in J}$ be an open cover of X.

$\forall x \in X$, $x \in U_{j(x)}$ for some $j(x)$.

X regular $\Rightarrow \exists W_x$ s.t. $x \in W_x \subset \overline{W_x} \subset U_{j(x)}$

$\{W_x\}$ is an open cover refining $\{U_\alpha\}_{\alpha \in J}$

Applying hypothesis to $\{W_x\}$ gives a countable refinement of $\{W_x\}$ (thus a refinement of $\{U_\alpha\}_{\alpha \in J}$) $V_1, V_2, \ldots, V_n, \ldots$, where $\forall j$ $V_j \subset \overline{V_j} \subset U_{\alpha(j)}$ for some $\alpha(j)$

Set

$$\begin{aligned}
E_1 &:= \overline{V_1} \\
E_2 &:= \overline{V_2 - V_1} \\
&\vdots \\
E_n &:= \overline{V_n - \textstyle\bigcup_{j=1}^{n-1} V_j} \subset \overline{V_n} \subset U_{\alpha(n)} \\
&\vdots
\end{aligned}$$

For $x \in X$:

$\exists$ least n s.t. $x \in V_n$.

$x \in E_n$ for this n.

Therefore $\{E_n\}$ covers X.

If $k > n$, $V_n \cap \left(V_k - \bigcup_{j=1}^{k-1} V_{k-1}\right) = \emptyset$

Since E_k is the closure of $V_k - \bigcup_{j=1}^{k-1} V_{k-1} = \emptyset$, V open $\Rightarrow V_n \cap E_k = \emptyset$.

Therefore $\{E_k\}$ locally finite (since each $x \in V_n$ for some n.)
$\{E_k\}$ is a locally finite refinement of $\{U_\alpha\}$.
Repeat procedure on cover $\{V_n\}$ to get a locally finite closed refinement $\{B_\beta\}$ of $\{V_n\}$.
By construction $\forall \beta$, $B_\beta \subset V_n$ for some n so $B_\beta \cap E_k = \emptyset$ for almost all k.
Therefore Lemma $\Rightarrow \forall k \; \exists$ open W_k s.t. $E_k \subset W_k$ and $\{W_k\}$ locally finite.
Set $W'_k := W_k \cap U_{\alpha(k)} \subset U_{\alpha(k)}$ open.
$E_k \subset W_k$ and $E_k \subset U_{\alpha(k)} \Rightarrow E_k \subset W'_k$.
$\{E_k\}$ covers so $\{W'_k\}$ covers.
$W'_k \subset U_{\alpha(k)} \Rightarrow \{W'_k\}$ is a refinement.
$W'_k \subset W_k$, $\{W_k\}$ locally finite $\Rightarrow \{W'_k\}$ locally finite. □

Corollary 5.0.39 *X regular and 2nd countable $\Rightarrow$ X paracompact.*

Proof: Let $\{U_\alpha\}$ be an open cover of X.
Let W_1, W_2, …, W_n, … be a countable basis.
If $x \in X$ then $x \in U_\alpha$ some α so $\exists$ basic open $W_{n(x)}$ s.t. $x \in W_{n(x)} \subset U_\alpha$.
Therefore $\{W_{n(x)}\}$ is a refinement of $\{U_\alpha\}$ which covers X and is countable (subcollections of a countable collection)
Therefore Thm. $\Rightarrow X$ paracompact. □