9.12 Orientation for Manifolds

Recall

Definition 9.12.1 *A (paracompact) Hausdorff space M is called an n-dimensional* manifold *of for each $x \in M$ $\exists$ open neighbourhood U of x s.t. U is homeomorphic to $\mathbb{R}^n$.*

U is called an open coordinate neighbourhood. (If the neighbourhoods are diffeomorphic to $\mathbb{R}^n$ then M isa called a differentiable manifold. Similarly can define C^∞ manifolds, etc.)

Let M denote an n-dimensional manifold. Given open coordinate neighbourhood V of x, can choose smaller open neighbourhood U of x s.t. the homeomorphism of V to $\mathbb{R}^n$ restricts to a homeomorphism of U with an open ball of radius 1. Thus U is also homeomorphic to $\mathbb{R}^n$. From now on whenever we pick a coordinate neighbourhood U of x we shall always assume that we have chosen one which is contained in a larger coordinate neighbourhood V as above so that $\overline{U} \subset V$ and $V \setminus U \simeq S^{n-1}$.

Proposition 9.12.2 $\forall x \in M$,

$$H_q(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & q = n \\ 0; & q \neq n. \end{cases}$$

Proof: Let U be an open coordinate neighbourhood of x. Then $\overline{M \setminus U} = M \setminus U \subset M - \{x\} = \operatorname{Int}(M \setminus \{x\})$ so

$$\begin{aligned} H_q(M, M \setminus \{x\}) & \overset{\text{(excision)}}{\cong} H_q(U, U \setminus \{x\}) \\ & \cong H_q(\mathbb{R}, \mathbb{R} \setminus \{x\}) \\ & \overset{\text{(long exact sequence)}}{\cong} \tilde{H}_{q-1}(\mathbb{R} \setminus \{x\}) \\ & \cong \tilde{H}_{q-1}(S^{n-1}) \\ & \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases} \end{aligned}$$

□

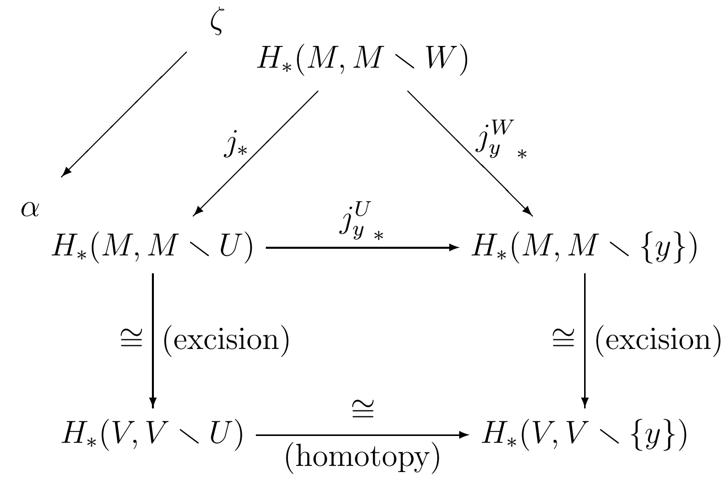
Definition 9.12.3 *A choice of one of the two generators for $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ is called a* local orientation *for M and x.*

Notation: Given $x \in A \subset K \subset M$, let $j_x^A : (M, M \setminus A) \to (M, M - \setminus \{x\})$ denote the map of pairs induced by inclusions. If $A = K = M$, just write j_x for j_x^A.

Lemma 9.12.4 *1. Given open neighbourhood W of x, $\exists$ open neighbourhood U of x s.t. $U \subset W$ and $j^U_{y\,*} : H_*(M, M \smallsetminus U) \to H_*(M, M \smallsetminus \{y\})$ is an isomorphism $\forall y \in U$.*

2. Let $\zeta \in H_n(M, M \smallsetminus W)$. Let U be any open neighbourhood of x satisfying part (1) (i.e. $j^U_{y\,*}$ iso. $\forall y \in U$.) If $\alpha \in H_n(M, M \smallsetminus U)$ s.t. $j^U_{y\,*}(\alpha) = j^W_{y\,*}(\zeta)$ for some $y \in U$ then $j^U_{y\,*}(\alpha) = j^W_{y\,*}(\zeta)$ $\forall y \in U$.

Proof: Within W find a pair $U \subset V$ of open coordinate neighbourhoods of x (as outlined earlier) s.t. $V \smallsetminus U \simeq S^{n-1}$. Then $\forall y \in U$



Therefore $j^U_{y\,*}$ is an isomorphism as required in (1). If $y_0 \in U$ s.t. $j^U_{y_0}(\alpha) = j^W_{y_0}(\zeta)$ then the diagram with $y = y_0$ shows that $j_*(\zeta) = \alpha$. Hence the diagram with arbitrary $y \in U$ gives $j^U_y(\alpha) = j^W_y(\zeta)$. □

Theorem 9.12.5 *Let K be compact, $K \subset M$. Then*

1. $H_q(M, M \smallsetminus K) = 0 \qquad q > n$

2. For $\zeta \in H_n(M, M \smallsetminus K)$ if $j^K_{x\,(}(\zeta) = 0$, then $\zeta = 0$.

Proof:
Case 1: $M = \mathbb{R}^n$, K compact convex subset.
Then for $x \in K$, $\mathbb{R}^n \smallsetminus K \cong \mathbb{R}^n \smallsetminus \{x\}$, so (1) and (2) are immediate. $\surd$
Case 2: $K = K_1 \cup K_2$ when theorem is known for K_1, K_2, and $K_1 \cap K_2$.
Apply (relative) Mayer-Vietoris to open sets $M \smallsetminus K_1$, $M \smallsetminus K_2$.
$(M \smallsetminus K_1) \cap (M \smallsetminus K_2) = M \smallsetminus (K_1 \cup K_2) = M \smallsetminus K$
$(M \smallsetminus K_1) \cup (M \smallsetminus K_2) = M \smallsetminus (K_1 \cap K_2)$

$$\longrightarrow H_{n+1}\big(M, M \smallsetminus (K_1 \cap K_2)\big) \xrightarrow{\Delta} H_n(M, M \smallsetminus K) \xrightarrow{(j_{K_1 *}, j_{K_2 *})}$$
$$\text{(with } H_{n+1}\big(M, M \smallsetminus (K_1 \cap K_2)\big) = 0\text{)}$$
$$H_n(M, M \smallsetminus K_2) \oplus H_n(M, M \smallsetminus K_2) \longrightarrow H_n\big(M, M \smallsetminus (K_1 \cap K_2)\big)$$

(1) follows immediately. For (2):

$\forall x \in K_1$

$$\begin{array}{ccc} H_n(M, M \smallsetminus K) & \xrightarrow{\quad j_{K_1} \quad} & H_n(M, M \smallsetminus K_1) \\ & {\scriptstyle j^K_{x\,*}} \searrow \qquad \swarrow {\scriptstyle j^{K_1}_{x\,*}} & \\ & H_n(M, M \smallsetminus \{x\}) & \end{array}$$

Hence $j^K_{x\ X}\big(j_{K_1}(\zeta)\big) = j^K_{x\ *}(\zeta) = 0$. So (since true $\forall x \in K_1$, by the theorem applied to K_1 gives $j_{K_1}(\zeta) = 0$. Similarly $j_{K_1}(\zeta) = 0$.

But by exactness, $\ker(j_{K_1}, j_{K_2}) = 0$ so $\zeta = 0$. $\surd$

Case 3: $M = \mathbb{R}^n$, $K = K_1 \cup \ldots \cup K_r$ where K_i is compact and convex.

Follows by induction on r from Cases 1 and 2.

Note: Intersection of convex sets is convex. To prove the theorem for, say, $K_1 \cup K_2 \cup K_3$ will have to know it already for $(K_1 \cup K_2) \cap K_3$. This will be done by a subsidiary induction. It can best be phrased by taking as the induction hypothesis that the theorem holds for *any* union of $r - 1$ compact convex subsets).

$\surd$.

Case 4: $M = \mathbb{R}^n$, K arbitrary compact set.
(This is the heart of the proof of the theorem.)

$$H_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K) \overset{\text{(exactness)}}{\cong} H_{q-1}(\mathbb{R}^n \smallsetminus K).$$

Given $z \in H_{q-1}(\mathbb{R}^n - K)$, by axiom $A8$, $\exists$ compact set (depending on z) $L_z \overset{j}{\hookrightarrow} \mathbb{R}^n \smallsetminus K$ s.t. $z = \iota_*(z')$ for some $z' \in H_{q-1}(L_z)$.

Given A s.t. $K \subset A \subset (L_z)^c$,

$$\begin{array}{ccccc} & z' & H_{q-1}(L_z) & & \\ \swarrow & & \swarrow \qquad \searrow {\scriptstyle i_*} & & \\ a_z & H_{q-1}(\mathbb{R}^n \smallsetminus A) & \xrightarrow{\quad i'_* \quad} & H_{q-1}(\mathbb{R}^n \smallsetminus K) & \end{array}$$

shows $z = i'_*(a_z)$ for some $a_z \in H_{q-1}(\mathbb{R} - A)$.

Will also use a_z and z to denote their isomorphic images under $H_q(\mathbb{R}^n, \mathbb{R} \setminus A) \cong H_{q-1}(\mathbb{R}^n \setminus A)$, etc.

Wish to select A_z s.t. A_z is a finite union of compact convex sets and $K \subset A_z \subset (L_z)^c$.

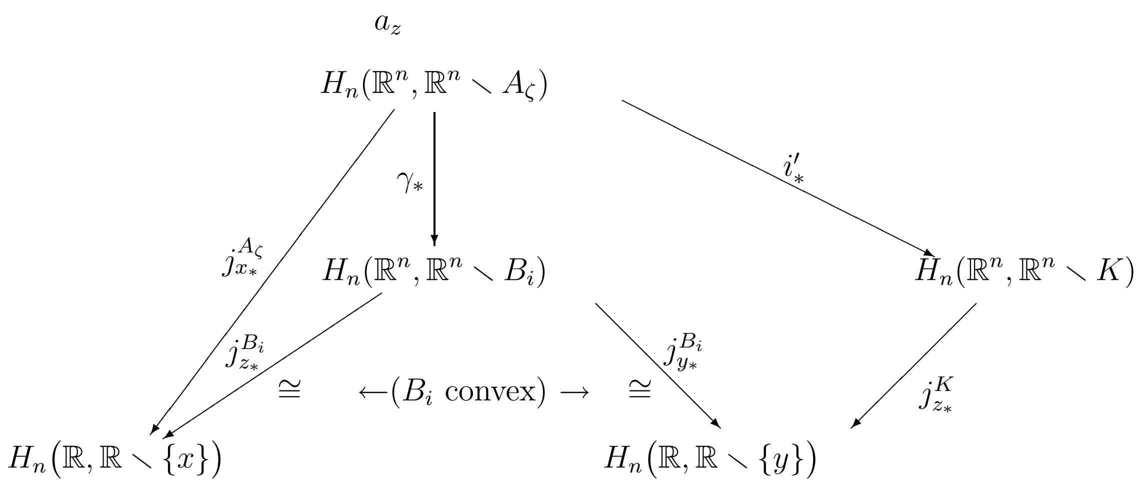
Cover K by open balls whose closures are disjoint from L_z (using normality). By compactness can choose a finite subcover and let A_z be the union of their closures. By Case 3, the theorem holds for A_z.

If $q > n$, by (1) of the theorem applied to A_Z, $A_z = 0$ so $z = 0$. Hence (1) holds for K.
To prove (2):

Suppose $z = \zeta$ where ${j_{x*}}^K(\zeta) = 0\ \forall x \in K$. It suffices to show that ${j_x^{A_\zeta}}_*(a_\zeta) = 0\ \forall x \in A_\zeta$ since we can apply (2) of the theorem for A_ζ to conclude that $a_\zeta = 0$ so that $\zeta = 0$. (It is immediate that ${j_x^{A_\zeta}}_*(a_\zeta) = 0$ if $x \in K \subset A_\zeta$.)

Write $A_\zeta = B_1 \cup \ldots B_r$ where B_i is a closed n-ball s.t. $B_i \cap K \neq \emptyset$ (using defn. of A_ζ).

Given $x \in A_\zeta$, suppose $x \in B_i$ and find $y \in B_i \cap K$.



Since ${j_y^K}_*(\zeta) = 0$ by hypothesis, ${j_y^{B_i}}_*\gamma_*(a_\zeta) = 0$ so $\gamma_*(a_\zeta) = 0$ so that ${j_x^{A_\zeta}}_*(a_\zeta) = {j_x^{B_i}}_*(a_\zeta) = 0$.
Thus ${j_x^{A_\zeta}}_*(a_\zeta) = 0$, as desired. $\checkmark$

Case 5: $K \subset U \subset M$, where U is an open coordinate neighbourhood.

Follows immediate from Case 4 since $H_*(M, M \setminus K) \overset{\text{(excision)}}{\cong} H_*(U, U \setminus K)$. $\checkmark$

Case 6: General Case

By covering K with coordinate neighbourhoods whose closures are contained in larger coordinate neighbourhoods, write $K = K_1 \cup \ldots K_r$ where for each i, $K_i \subset U_i$ with U_i is an open coordinate neighbourhood. Then use Case 5, Case 2, and induction on r. $\square$

Theorem 9.12.6 *For each $x \in M$, let α_x be a generator of $H_n(M, M \smallsetminus \{x\})$. Suppose that these generators are compatible in the sense that $\forall x\ \exists$ open coordinate neighbourhood U_x of x and $\exists \alpha_{U_x} \in H_n(M, M \smallsetminus U_x)$ s.t. $j_y^{U_x} = \alpha_y\ \forall y \in U_x$. Then given $K \subset M$, $\exists! \alpha_K \in H_n(M, M \smallsetminus K)$ s.t. ${j_y^K}_*(\alpha_K) = \alpha_y\ \forall y \in K$.*

Proof: Unique is immediate from the previous theorem. To prove existence:
Case 1: $K \subset U_x$ for some x

Use $\alpha_K = j_*(\alpha_{U_x})$ where $j_* : H_n(M, M \smallsetminus U_x) \to H_n(M.M \smallsetminus K)$.

Case 2: $K = K_1 \cup K_2$ where α_{K_1}, α_{K_2} exist.

$$H_{n+1}\big(M, M \smallsetminus (K_1 \cap K_2)\big) \to H_n(M, M \smallsetminus K) \xrightarrow{(j_{K_1}, j_{K_2})}$$
$$H_n(M, M \smallsetminus K_1) \oplus H_n(M, M \smallsetminus K_2) \xrightarrow{j'_* - j''_*} H_n\big(M, M \smallsetminus (K_1 \cap K_2)\big) \to$$

For any $x \in K_1 \cap K_2$, $j_{x_*}^{K_1 \cap K_2}(j'_* - j''_*)(\alpha_{K_1}, \alpha_{K_2}) = j_{x_*}^{K_1}(\alpha_{K_1} - j_{x_*}^{K_2}(\alpha_{K_2} = \alpha_x - \alpha_x = 0$ Therefore by the previous theorem applied to $K_1 \cup K_2$, $(j'_* - j''_*)(\alpha_{K_1}, \alpha_{K_2}) = 0$ so from the exact sequence $\exists \alpha_K \in H_n(M, M \smallsetminus K)$ s.t. $j_{K_1}(alpha_K) = \alpha_{K_1}$ and $j_{K_2}(alpha_K) = \alpha_{K_2}$. Then α_K satisfies the conditions of the theorem. (To check it from y, find $\epsilon \in K_\epsilon$ and use naturality.)
Case 3: General case

Write $K = K_1 \cup \ldots \cup K_r$ with each $K_i \subset U_x$ for some x by covering K with open sets each having its closure in some U_x. Now use Cases 1, 2 and induction on r. □

Remember: j_x means j_x^M.

Definition 9.12.7 *Suppose M is a compact n-dimensional manifold. If $\exists \zeta \in H_n(M)$ s.t. $j_{x_*}(\zeta)$ is a local orientation for M at x for each $x \in M$ then M is called* orientable *and ζ is called a (global)* orientation *for M.*

If M is not compact than such a global orientation class will not exist. (Consider, for example, $M = \mathbb{R}^n$). More gnerally we define:

Definition 9.12.8 *An* orientation *for M consists of a family of elements $\{\zeta_K\}_{K \subset M}$ with $\zeta_K \in H_n(M, M \smallsetminus K)$ such that $J_{x_*}^K(\zeta_K)$ is a local orientation for M at $x\ \forall x \in K$, K compact and furthermore if $x \in K_1 \cap K)2$ then $j_{x_*}^{K_1}(\zeta_{K_1}) = j_{x_*}^{K_2}(\zeta_{K_2})$.*

Of course, this second definition works equally well in the compact case, since a global class can be restricted.

The preceding theorem says that if M has a "compatible" collection of local orientations at each point then M is orientable.

Corollary 9.12.9 *Let M be orientable and connected. Then any two orientations of M which induce the same local orientation at any point are equal.*

Proof: Let $\{\alpha_y\}_{y\in M}$ and $\{\beta_y\}_{y\in M}$ be the sets of local orientations induced by the two orientations $\{\zeta_{K\subset M}$ and $\{\zeta'_{K\subset M}$.

By earlier lemma, if the orientations agree at x then they agree on an open neighbourhood of x ($\exists U$ s.t. $J^U_{y*} : H_*(M.M \smallsetminus U) \to H_*(M, M \smallsetminus \{y\})$ is iso. $\forall y \in U$) so $A = \{x \mid \alpha_x = \beta_x\}$ is open.

On the other hand, if $\alpha_x \neq \beta_x$, then $\alpha_x = -\beta_x$ (there are only 2 generators of $\mathbb{Z}$ and they are related in this way) so by the same lemma $\exists$ open set U containing x s.t. $\alpha_y = -\beta_y$ $\forall y \in U$. Hence $B = \{x \mid \alpha_x \neq \beta_x\}$ is also open.

Since $A \cup B = M$ and $A \cap B = \emptyset$, by connectivity of M one of A, B is $\emptyset$. By hypothesis $A \neq \emptyset$ so $B = \emptyset$ and $A = M$. Hence $\alpha_x = \beta_x$ $\forall x \in M$, which by earlier theorem says that $\zeta_K = \zeta'_K$ $\forall$ K. □

Corollary 9.12.10 *If M is connected and orientable then it has precisely 2 orientations and a choice of orientations at one point uniquely determines one of the orientations.* □

9.12.1 Orientability with Coefficient

Let R be a commutative ring with 1/ We can make the same definitions of orientability using homology with R-coefficients (e.g., a local orientation is a generator of $H_n(M, M \smallsetminus \{x\}) \cong R$) although the theorems might not all work. In practice, besides $\mathbb{Z}$ the only useful coefficient ring for the purpose of orientations is $R = \mathbb{Z}/(2\mathbb{Z})$. In that case there is only one generator so all compatiblity conditions are automatic. This means that every manifold is ($\mathbb{Z}/(2\mathbb{Z})$-orientable. Sometimes theorems which hold (using $\mathbb{Z}$-coefficients) only for orientable manifolds *can* be extended to non-orientable manifolds if ($\mathbb{Z}/(2\mathbb{Z})$-coeffiecients are used.

Example 9.12.11 *Consider $\mathbb{R}P^2$. It is a 2-dimensional manifold.*

$$H_q(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/(2\mathbb{Z}) & q=1 \\ 0 & q=2 \end{cases} \qquad H_q\big(\mathbb{R}P^2;\mathbb{Z}/(2\mathbb{Z})\big) = \begin{cases} \mathbb{Z}/(2\mathbb{Z}) & q=0 \\ \mathbb{Z}/(2\mathbb{Z}) & q=1 \\ \mathbb{Z}/(2\mathbb{Z}) & q=2 \end{cases}$$

Examining the $\mathbb{Z}$-coefficients, since $H_2(\mathbb{R}P^2) = 0$ there can be no global orientation class, so $\mathbb{R}P^2$ is non-orientable. Notice that there *is* a candidate for a global $\mathbb{Z}/(2\mathbb{Z})$-orientation calss, and since every manifold is $\mathbb{Z}/(2\mathbb{Z})$-orientable it must indeed *be* a $\mathbb{Z}/(2\mathbb{Z})$-orientation class.