9.15 Group Structures on Homotopy Classes of Maps

For basepointed spaces X, Y, recall that $[X, Y]$ denotes the based homotopy classes of based maps from X to Y. In general $[X, Y]$ has no canonical group structure, but we define concepts of H-group and co-H-group such that $[X, Y]$ has a natural group structure provided either Y is an H-group or X is a co-H-group.

It is easy to check that if G is a topological group (regarded as a pointed space with the identity as basepoint) then $[X, G]$ has a group structure defined by $[f][g] = [h]$, where $h(x)$ is the product $f(x)g(x)$ in G. But a topological group is more than we need: all we need is a group "up to homotopy". We generalize topological group to H-group as follows:

A pointed space (H, e) is called an *H-space* if $\exists$ a (continuous pointed) map $m : H \times H \to H$ such that

$$\begin{array}{ccc} H & \xrightarrow{\quad i_1 \quad} & H \times H \\ & {\scriptstyle 1_H}\searrow \qquad \swarrow{\scriptstyle m} & \\ & H & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H & \xrightarrow{\quad i_1 \quad} & H \times H \\ & {\scriptstyle 1_H}\searrow \qquad \swarrow{\scriptstyle m} & \\ & H & \end{array}$$

are homotopy commutative, where $i_1(x) := (x, e)$ and $i_2(X) := (e, x)$.

An H-space is called *homotopy associative* if

$$\begin{array}{ccc} H \times H \times H & \xrightarrow{1_H \times m} & H \times 1_H \\ \Big\downarrow {\scriptstyle m \times H} & & \Big\downarrow {\scriptstyle m} \\ H \times H & \xrightarrow{\quad m \quad} & H \end{array}$$

If H is an H-space, a map $c : H \to H$ is called a *homotopy inverse* for H if

$$\begin{array}{ccc} H & \xrightarrow{\quad (1_H, *) \quad} & H \times H \\ & {\scriptstyle *}\searrow \qquad \swarrow{\scriptstyle m} & \\ & H & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H & \xrightarrow{\quad (*, 1_H) \quad} & H \times H \\ & {\scriptstyle m}\searrow \qquad \swarrow{\scriptstyle *} & \\ & H & \end{array}$$

are homotopy commutative.

A homotopy associative H-space with a homotopy inverse is called an H-group.

An H-space is called homotopy abelian if

$$\begin{array}{ccc} H & \xrightarrow{\quad T \quad} & H \times H \\ & {}_{m}\searrow \quad \swarrow_{m} & \\ & H & \end{array}$$

is homotopy commutative, where T is the swap map $T(x, y) = (y, x)$.

Proposition 9.15.1 *Let H be an H-group. Then $\forall X$, $[X, H]$ has a natural group structure given by $[f][g] = [m \circ (f, g)]$. If H is homotopy abelian then the group is abelian.*

Remark 9.15.2 *"Natural" means that any map $q : W \to X$ induces a group homomorphism denoted $q^{\#} : [X, H] \to [W, H]$ defined by $q^{\#}([f]) = [f \circ q]$. The assignment $X \mapsto [X, H]$ is thus a contravariant functor.*

Proof:
Associative:

$m \circ (m \times 1_H) \circ (f, g, h) = m(\circ 1_H) \times m \circ (f, g, h)$ so $([f][g])[h] = [f]([g][h])$.

Identity:

$m \circ (*, f) = m \circ i_1 \circ f = 1_H \circ f = f$ so $[*][f] = [f]$ and similarly $[f][*] = [f]$, and thus $[*]$ forms a 2-sided for $[X, H]$.

Inverse:

Given $[f]$, define $[f]^{-1}$ to be the class represented by $c \circ f$. $m \circ (f, f^{-1}) = m \circ (1_H, c) \circ (f \times f) = 1_H \circ f = (f \times) f \circ * = *$ so $[f][f^{-1}] = [*]$ and similarly $[f^{-1}][f] = [*]$. Thus $[f^{-1}]$ forms a 2-sided inverse for f.

Finally, if H is homotopy abelian then $[f][g] = m \circ (f, g) = m \circ T \circ (f, g) = m \circ (g, f) = [g][f]$, so that $[X, H]$ is abelian. □

Two H-space structures m, m' on X are called *equivalent* if $m \simeq m'$ (rel $*$) as maps from $X \times X$ to X. It is clear that equivalent H-space structures on X result in the same group structure on $[W, X]$.

A basepoint-preserving map $f : X \to Y$ between H-spaces is called an *H-map* if

$$\begin{array}{ccc} X \times X & \xrightarrow{f \times f} & Y \times Y \\ \downarrow{\scriptstyle m_X} & & \downarrow{\scriptstyle m_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

homotopy commutes.

An H-map $f : X \to Y$ induces, for any space A, a group homomorphism $f_\# : [A, X] \to [A, Y]$ given by $f_\#([g]) = [f \circ g]$.

Remark 9.15.3 *The collection of H-spaces forms a category with H-maps as morphisms.*

Examples

1. A topological group is clearly an H-group.

2. $\mathbb{R}^8$ has a continous (non-associative) multiplication as the "Cayley Numbers", also called "octonians" $\mathbb{O}$.

3. Loop space on X:

   Given pointed spaces W and X, we define the function space X^W, also denoted $\mathrm{Map}_*(W, X)$. Set $X^W := \{\text{continuous } f : W \to X\}$. Topologize X^W as follows: For each pair (K, U) where $K \subset W$ is compact and $U \subset X$ is open, let $V_{(K,U} = \{f \in X^W \mid f(K) \subset U\}$. Take the set of all such sets $V_{(K,U)}$ as the basis for the topology on X^W.

   X^{S^1} is called the "loop space" of X and denote ΩX. Define a multiplication on ΩX which resembles the multiplication in the group $\pi_1(X)$ by $m(f, g) := f \cdot g$.

   To show that m is continuous:

   Let $V_{(K,U)}$ be a subbasic open set in ΩX. Write $K = K' \cup K''$ where $K' = K \cap [0, 1/2]$ and $K'' = K \cap [1/2, 1]$. Then $m^{-1}(V_{(K,U)}) = V_{(L',U)} \times V_{(L'',U)}$ where L' is the image of K' under the homeomorphism $[0, 1/2] \to [0, 1]$ given by $t \mapsto 2t$ and L'' is the image of K'' under the homeomorphism $[1/2, 1] \to [0, 1]$ given by $t \mapsto 2t - 1$. Therefore m is continuous.

   The facts that ΩX is homotopy associative, that the constant map c_{x_0} is a homotopy identity, and that $f \to f^{-1}$ (where $f^{-1}(t) = f(1 - t)$) is a homotopy inverse follow, immediately from the facts used in the proof that $\pi_1(X, x_0)$ is a group.

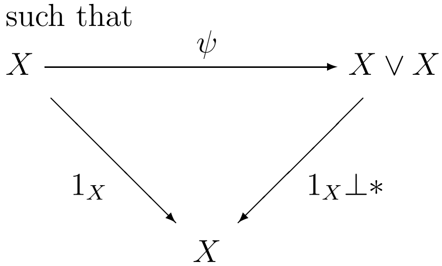
ΩX is an example of an H-group which is not a group. By definition a path from f to g in ΩX is the same as a homotopy $H : f \simeq g \ \mathrm{rel}(0, 1)$. The group $[S^0, \Omega X]$ defined using the H-space structure on ΩX is clearly the same as $\pi_1(X, x_0)$.

Remark 9.15.4 *Given continous $f : X \to Y$, it is easy to see that there is a continous induced map $\Omega f : \Omega X \to \Omega Y$ given by $(\Omega f)(\alpha) := f \circ \alpha$. Thus the correspondence $X \mapsto \Omega X$ defines a functor from the category of topological spaces to the category of H-spaces.*

The preceding can be generalized as follows.

Observe that a pointed map $A \vee B \to Y$ is equivalent to a pair of pointed map $A \to Y$, $B \to Y$. (In other words, $A \vee B$ is the coproduct of A and B in the category of pointed topological spaces.) We write $f \perp g : A \vee B \to Y$ for the map corresponding to f and g.

A pointed space X is called a *co-H-space* if $\exists$ a (continuous pointed) map $\psi : X \to X \vee X$ such that



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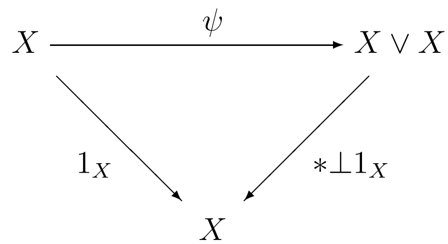
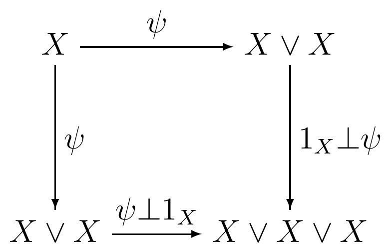
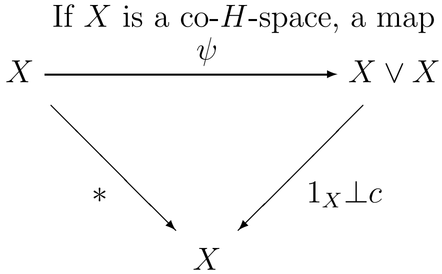


are homotopy commutative.

A co-H-space is called *homotopy coassociative* if



If X is a co-H-space, a map $c : X \to X$ is called a *homotopy inverse* for X if



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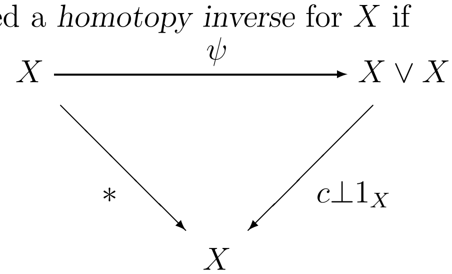
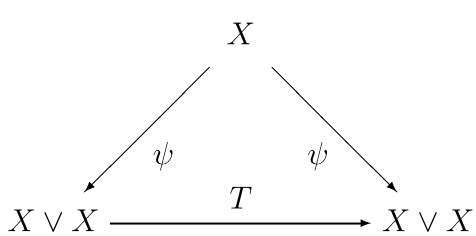


are homotopy commutative.

A homotopy coassociative co-H-space with a homotopy inverse is called a co-H-group.

A co-H-space is called homotopy coabelian if



is homotopy commutative, where T is the swap map.

Proposition 9.15.5 *Let X be a co-H-group. Then for any pointed space Y, $[X,Y]$ has a natural group structure. If X coabelian then $[X,Y]$ is abelian.*

Proof: The group structure is given by $[f][g] = [(f \perp g) \circ \psi]$ where $f, g : X \to Y$. The proof is essentially the same as the dual proof for H-groups with arrows reversed. Further, as before, a map $q : Y \to Z$ induces a group homomorphism $q_\# : [X,Y] \to [X,Z]$ defined by $q_\#([f]) = [q \circ f]$. (The association $Y \mapsto [X,Y]$, $q \mapsto q_\#$ is a functor from topological spaces to groups.) □

Example of a co-H-group:

S^n is a co-H-space for $n \geq 1$. The map $\psi : S^n \to S^n \vee S^n$ is given by "pinching" the equator to a point.

Thus for any pointed space X, $\pi_n(X) := [S^n, X]$ has a natural group structure for $n \geq 1$, called the nth *homotopy group* of X. Looking at the case $n = 1$, the group structure that we get on $[S^1, X]$ is the same as that of the fundamental group.

More generally:

Let X be a topological space. Define a space denoted SX, called the (reduced) *suspension* of X, by $SX := (X \times I)/\big((X \times \{0\}) \cup (X \times \{1\}) \cup (* \times I)\big)$. For any X, SX becomes a co-H-group by pinching the equator, $X \times \{1/2\}$, to a point. That is, $\psi : SX \to SX \vee SX$ by

$$\psi(x,t) = \begin{cases} (x, 2t) \text{ in the first copy of } SX & \text{if } t \leq 1/2; \\ (x, 2t-1) \text{ in the second copy of } SX & \text{if } t \geq 1/2. \end{cases}$$

When $t = 1/2$ the definitions agree since each gives the common point at which the two copies of SX are joined.

This generalizes the preceding example since:

Lemma 9.15.6 *SS^n is homeomorphic to S^{n+1}.*

Proof: Intuitively, think of S^{n+1} as the one point compactification of $\mathbb{R}^{n+1}$ and notice that after removal of the point at which the identifications have been made, SS^n opens up to become an open $(n+1)$-disk. For a formal proof, write S^k as $I^k/\partial(I^k)$ and notice that both SS^n and S^{n+1} becomes quotients of I^{n+1} with exactly the same identifications. □

Remark 9.15.7 *As in the case of Ω, given $f : X \to Y$ there is an induced map $Sf : SX \to SY$ defined by $Sf(x,t) := \big(f(x), t\big)$ and so S defines a functor from the category of pointed spaces to itself.*

Theorem 9.15.8 *For each pair of pointed spaces X and Y there is a natural bijection between the sets $\mathrm{Map}_*(SX, Y)$ and $\mathrm{Map}_*(X, \Omega Y)$. This bijection takes homotopic maps to homotopy maps and thus induces a bijection $[SX, Y] \to [X, \Omega Y]$. Furthermore, the group structure on $[SX, Y]$ coming from the co-H-space structure on SX coincides under this bijection with that coming from the H-space structure on ΩY.*

Proof: Define $\phi : \mathrm{Map}_*(SX, Y) \to \mathrm{Map}_*(X, \Omega Y)$ by $\phi(f) = g$ where $g(x)(t) = f(x, t)$. Notice that $g(x)(0) = f(x, 0) = f(*) = y_0$ and $g(x)(1) = f(x, 1) = f(*) = y_0$ since the identified subspace $\big((X \times \{0\}) \cup (X \times \{1\}) \cup (* \times I)\big)$ is used as the basepoint of SX. Thus $g(x)$ is an element of ΩY.

Must show that g is continuous.

Let $q : X \times I \to SX$ denote the quotient map. Let $V_{(K,U)}$ be a subbasic open set in ΩY. Then $g^{-1}(V_{(K,U)}) = \{x \in X \mid f(x, k) \in U \ \forall k \in K\}$. Pick $x \in g^{-1}(V_{(K,U)})$. By continuity of q and f, for each $k \in K$ find basic open set $A_k \times W_k \subset X \times I$ s.t. $x \in A_k$, $k \in W_k$ and $A_k \times W_k \subset (q \circ f)^{-1}(U)$. $\{W_k\}_{k \in K}$ covers I so choose a finite subcover $W_{k_1}, \ldots, W_{k_n}$ and let $A = A_{k_1} \cap \cdots \cap A_{k_n}$. Then $x \in A$ and $A \subset g^{-1}(V_{(K,U)})$ so x is an interior point of $g^{-1}(V_{(K,U)})$ and since this is true for arbitrary x, $g^{-1}(V_{(K,U)})$ is open. Therefore g is continuous.

Show ϕ is $1-1$:

Clearly if $\phi(f) = \phi(f')$ then $f(x, t) = \big(\phi(f)(x)\big)(t) = \big(\phi(f')(x)\big)(t) = f'(x, t)$ for all x, t so $f = f'$.

Show ϕ is onto:

Given $g : X \to \Omega Y$, define $: SX \to Y$ by $f(x, t) = \big(g(x)\big)(t)$. For all x, $f(x, 0) = \big(g(x)\big)(0) = y_0$ and $f(x, 1) = \big(g(x)\big)(1) = y_0$ and for all t, $f(x_0, t) = \big(g(x_0)\big)(t) = c_{y_0}(t) = y_0$ and thus f is well defined.

Must show that f is continuous. Given open $U \subset Y$,

$$f^{-1}(U) = \{(x, t) \in SX \mid \big(g(x)\big)(t) \in U\}.$$

By the universal property of the quotient map, showing that $f^{-1}(U)$ is open is equivalent to showing that $(f \circ q)^{-1}(U)$ is open in $X \times I$.

For a pair $(x, t) \in X \times I$:

Since g is continuous $A := g^{-1}(V_{I,U}) \subset X$ is open. Thus $A \times I$ is an open subset of $X \times I$ which contains (x, t), and if $(a, t') \in A \times I$ then $f \circ q(a, t') = \big(g(a)\big)(t) \in U$ since $g(a)$ takes all of I to U. Thus $A \times I \subset (f \circ q)^{-1}(U)$ and thus (x, t) is an interior point of $A \times I$, and since this is true for arbibrary (x, t), f is continuous. Therefore f lies in $\mathrm{Map}_*(SX, Y)$ and clearly $\phi(f) = g$, so ϕ is onto.

It is easy to see that $f \simeq f' \Leftrightarrow \phi(f) = \phi(f')$. (e.g., if $H : f \simeq f'$ define $\big(g_s(x)\big)(t) := H_s(x, t)$.)

To show that the group structures coincide:

$$(ff')(x,t) = \begin{cases} f(x,2t) & \text{if } t \le 1/2; \\ f'(x,2t-1) & \text{if } t \ge 1/2. \end{cases}$$

so $\big(\phi(ff')\big)(x) = \Big(\big(\phi(f)\big)(x)\cdot\big(\phi(f')\big)(x)\Big)$ by the definition of multiplication of paths. Therefore $\phi(ff') = \phi(f)\phi(f')$. Thus ϕ is an isomorphism, or equivalently, the group structures coincide.

□

Corollary 9.15.9 $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ □

According to the previous theorem, there is a natural bijection between $\mathrm{Map}_*(SX,Y)$ and $\mathrm{Map}_*(X,\Omega Y)$ where natural means that for any map $j : A \to X$.

$$\begin{array}{ccc} \mathrm{Map}_*(SX,Y) & \xrightarrow{\phi} & \mathrm{Map}(X,\Omega Y) \\ \uparrow (Sj)^{\#} & & \uparrow j^{\#} \\ \mathrm{Map}_*(SA,Y) & \xrightarrow{\phi} & \mathrm{Map}(A,\Omega Y) \end{array}$$

commutes, and similarly for any $k : Y \to Z$

$$\begin{array}{ccc} \mathrm{Map}_*(SX,Y) & \xrightarrow{\phi} & \mathrm{Map}(X,\Omega Y) \\ \downarrow (Sk)_{\#} & & \downarrow k_{\#} \\ \mathrm{Map}_*(SX,Z) & \xrightarrow{\phi} & \mathrm{Map}(X,\Omega Z) \end{array}$$

For this reason, S and Ω are called adjoint functors. More generally:

Definition 9.15.10 *Functors $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ are called adjoint functors if there is a natural set bijection $\phi : \mathrm{Hom}_{\underline{\underline{C}}}(FX,Y) \to \mathrm{Hom}_{\underline{\underline{D}}}(X,GY)$ for all X in $\mathrm{Obj}\,\underline{\underline{C}}$ and Y in $\mathrm{Obj}\,\underline{\underline{D}}$. F is called the left adjoint or co-adjoint and G is called the right adjoint or simply adjoint.*

Another example: Let T : Vector Spaces/$\mathbf{k}$ $\to$ Algebras/$\mathbf{k}$ by sending V to the tensor algebra on V, and let J : Algebras/$\mathbf{k}$ $\to$ Vector Spaces/$\mathbf{k}$ be the forgetful functor. Then $\mathrm{Hom}_{\mathrm{Alg}}(TV,W) = \mathrm{Hom}_{VS}(V,JW)$ for any vector space V and algebra W over $\mathbf{k}$.

Let X be an H-space. Then ΩX has a second H-space structure (in addition to the one coming from the loop-space structure) given by $m' : \Omega X \times \Omega X \to \Omega X$ with m' is defined by $m'(\alpha, \beta) = \gamma$ where $\gamma(t) = \alpha(t)\beta(t)$ (where $\alpha(t)\beta(t)$ denotes the product $m_X\big(\alpha(t), \beta(t)\big)$ in the H-space structure on X.

Theorem 9.15.11 *Let X be an H-space. Then the H-space structure on ΩX induced from that on X as above is equivalent to the one coming form the loop-space multiplication. Furthermore, this common H-space structure is homotopy abelian.*

Proof: In one H-space structure $(\alpha\beta)(s) = \alpha(s)\beta(s)$, while in the other the product is

$$(\alpha \cdot \beta)(s) := \begin{cases} \alpha(2s) & \text{if } s \leq 1/2; \\ \beta(2s-1) & \text{if } s \geq 1/2. \end{cases}$$

We construct a homotopy by homotoping α until it becomes $\alpha \cdot c_{x_0}$ and β until it comes $c_{x_0} \cdot \beta$ while at all times "multiplying" the paths in X using the H-space structure on X. Explicitly $H : \Omega X \times \Omega X \times I \to \Omega X$ by

$$H(\alpha, \beta, t)(s) = \begin{cases} \alpha\big(2s/(t+1)\big)\beta(0) & \text{if } 2s \leq 1-t; \\ \alpha\big(2s/(t+1)\big)\beta\big((2s+t-1)/(t+1)\big) & \text{if } 1-t \leq 2s \leq 1+t; \\ \alpha(0)\beta\big((2s+t-1)/(t+1)\big) & \text{if } 2s \geq 1+t; \end{cases}$$

The definitions agree on the overlaps do the function is well defined and is continuous.
Check that H is a homotopy rel $*$:

The basepoint of $\Omega X \times \Omega X$ is (c_{x_0}, c_{x_0}).

$H\big((c_{x_0}, c_{x_0}, t)\big)(s) = x_0 x_0 = x_0 \ \forall s, t$. Hence $H(c_{x_0}, c_{x_0}, t) = c_{x_0} \ \forall t$ so H is a homotopy rel $*$. Note: Although for arbitrary x, $x_0 x$ and $x x_0$ need not equal x, since multiplication by x_0 is only required to be homotopic to the identity rather that equal to the identity, it is nevertheless true that $x_0 x_0 = x_0$ since multiplication is a basepoint-preserving map.

$H(\alpha, \beta, 1)(s) = \alpha(s)\beta(s) \ \forall s$ which is the product of α and β in the H-space structure induced from that on X.

Since $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = x_0$,

$$H(\alpha, \beta, 0)(s) = \begin{cases} \alpha(2s)\beta(0) & \text{if } 2s \leq 1; \\ \alpha(1)\beta(2s-1) & \text{if } 2s \geq 1, \end{cases} = \begin{cases} \alpha(2s)x_0 & \text{if } 2s \leq 1; \\ x_0\beta(2s-1) & \text{if } 2s \geq 1, \end{cases} = \tilde{\alpha} \cdot \tilde{\tilde{\beta}}$$

where $\tilde{\alpha}(s) = \alpha(s)x_0$ and $\tilde{\tilde{\beta}}(s) = x_0\beta(s)$. Since multiplication by x_0 is homotopy to the identity (rel $*$), $\tilde{\alpha} \simeq \alpha$ (rel $*$) and similarly $\tilde{\tilde{\beta}} \simeq \beta$ (rel $*$). Thus the multiplications maps are homotopic and so the two H-space structures are equivalent.

To show that this structure is homotopy abelian, observe there is a homotopy analogous to H given by

$$J(\alpha,\beta,t)(s) = \begin{cases} \alpha(0)\beta\big(2s/(t+1)\big) & \text{if } 2s \le 1-t; \\ \alpha\big((2s+t-1)/(t+1)\big)\beta\big(2s/(t+1)\big) & \text{if } 1-t \le 2s \le 1+t; \\ \alpha\big((2s+t-1)/(t+1)\big)\beta(1) & \text{if } 2s \ge 1+t. \end{cases}$$

As before $J_1(s) = \alpha(s)\beta(s)$ but

$$J_0(s) = \begin{cases} x_0\beta(2s) & \text{if } 2s \le 1; \\ \alpha(2s-1)x_0 & \text{if } 2s \ge 1. \end{cases}$$

Since $J_0 \simeq \beta \cdot \alpha$, we get that the H-space structure is homotopy abelian. □

Corollary 9.15.12 *Suppose Y is an H-space. Then for any space X the group structure on $[SX, Y]$ coming from the co-H-space structure on SX agrees with that coming from the H-space structure on Y. Furthermore this common group structure is abelian.*

Proof: By Theorem 9.15.8 there is a bijection from $[SX, Y] \cong [X, \Omega Y]$ which is a group isomorphism from $[SX, Y]$ with the group structure coming from the suspension structure on $[SX]$, to $[X, \Omega Y]$ with the group structure coming from the loop space H-space structure on ΩY. It is easy to check that the group space structure on $[SX, Y]$ coming from the H-space structure on Y corresponds under this bijection with that on $[X, \Omega Y]$ coming from the H-space structure on Y. Since these H-space structures agree and are homotopy abelian, the result follows. □

Corollary 9.15.13 *If Y is an H-space, $\pi_1(Y)$ is abelian.*

□

Corollary 9.15.14 *For any spaces X and Y, $[S^2X, Y]$ is abelian.*

Proof: $[S^2X, Y] \cong [SX, \Omega Y]$. □

Corollary 9.15.15 *$\pi_n(Y)$ is abelian for all Y when $n \ge 2$.*

□