14.1 Cohomology Ring Calculations

S^n

degree	H_*	H^*
n	$\mathbb{Z}$	$\mathbb{Z}$
$n-1$	0	0
	$\vdots$	$\vdots$
1	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$S^1 \times S^1$

degree	H_*	H^*
2	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{C}P^n$

degree	H_*	H^*
$2n$	$\mathbb{Z}$	$\mathbb{Z}$
$2n-1$	0	0
$2n-2$	$\mathbb{Z}$	$\mathbb{Z}$
	$\vdots$	$\vdots$
1	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{R}P^{2n+1}$

degree	H_*	H^*
$2n+1$	$\mathbb{Z}$	$\mathbb{Z}$
$2n$	0	$\mathbb{Z}/(2\mathbb{Z})$
$2n-1$	$\mathbb{Z}/(2\mathbb{Z})$	0
	$\vdots$	$\vdots$
3	$\mathbb{Z}/(2\mathbb{Z})$	0
2	0	$\mathbb{Z}/(2\mathbb{Z})$
1	$\mathbb{Z}/(2\mathbb{Z})$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{R}P^2$ (nonorientable)

degree	H_*	H^*
2	0	$\mathbb{Z}/(2\mathbb{Z})$
1	$\mathbb{Z}/(2\mathbb{Z})$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{R}P^2$

degree	$H_*(\ ;\mathbb{Z}/(2\mathbb{Z}))$	$H^*(\ ;\mathbb{Z}/(2\mathbb{Z}))$
2	$\mathbb{Z}/(2\mathbb{Z})$	$\mathbb{Z}/(2\mathbb{Z})$
1	$\mathbb{Z}/(2\mathbb{Z})$	$\mathbb{Z}/(2\mathbb{Z})$
0	$\mathbb{Z}/(2\mathbb{Z})$	$\mathbb{Z}/(2\mathbb{Z})$

Cup Products:

$H^*(S^n)$:
Group generators: $1 \in H^0(S^n)$, $x \in H^n(S^n)$.
No choices: $1 \cup 1 = 1 \qquad 1 \cup x = x \cup 1 = x \qquad x \cup x = 0$ $\quad\checkmark$

Before proceding to the other spaces we need a lemma.

Let X be a connected compact oriented manifold s.t. all the boundary maps in some cellular chain complex for X are trivial. (e.g. $X = S^n$; $S^1 \times S^1$; $\mathbb{C}P^n$. Also $X = \mathbb{R}P^n$ if we use $\mathbb{Z}/(2\mathbb{Z})$

coefficients.)

$H^n(X) \cong H_0(X) \cong \mathbb{Z}$ (in the cases with $\mathbb{Z}$-coefficients). Let μ be a generator of $H^n(X)$. Replacing μ by $-\mu$ is necessary, we may assume that $\langle \mu, \zeta \rangle = 1$, where $\zeta \in H_n(X)$ the chosen orientation. Let $g \in H^q(X)$ be a basis element. (Note: The boundary maps equal to 0 implies that $H^q(X) \cong \mathrm{Hom}\big(D_q(X), \mathbb{Z}\big)$ is a free abelian group.)

Lemma 14.1.1 *$\exists f \in H^{n-q}(X)$ s.t. $f \cup g = \mu$*

Proof: Being a basis element, g is not divisible by p for any p so neither is $D(g) \in H_{n-q}(X)$ (since D is an isomorphism). Therefore by the hypothesis on the cellular chain complex for X, $\exists f \in H^{n-q}(X)$ s.t. $\langle \mu, \zeta \rangle = 1 = \langle f, D(g) \rangle \langle f, g \cap \zeta \rangle = \langle f \cup g, \zeta \rangle$ Hence $f \cup g$ is a generator of $H^n(X)$ and $f \cup g = \pm \mu$. □

$H^*(S^1 \times S^1)$.

Group generators: $1 \in H^0(\ )$, $y, z \in H^1(\ )$, $\mu \in H^2(\ )$.

$S^{\times} S^1 \xrightarrow{\pi_1} (S^1) \qquad \pi_1^*(x) = y$, $\pi_2^*(x) = z$.

Since $x^2 = 0$ in $H^*(S^1)$, $y^2 = (\pi_1^* x)^2 = 0$ (ring homomorphism). Similarly $z^2 = 0$.

By the lemma, $y \cup f = \mu$ for some f so $f = \pm z$.

Reversing the roles of y and z if necessary, $y \cup z = \mu$ and $z \cup y = (-1)^{1 \cdot 1} y \cup z = -\mu$.

Aside from the multiplications by the identity and the multiplications which must be 0 for degree reasons, this describes all of the cup products in $H^*(S^1 \times S^1)$. $\checkmark$

Lemma 14.1.2 *Let $X = Y \vee Z$ so that $\tilde{H}^*(X) \cong \tilde{H}^*(Y) \oplus \tilde{H}^*(Z)$ If $f \in H^p(X)$ and $g \in H^q(Z)$ then $f \cup g = 0$ in $H^{p+q}(X)$.*

Proof: Let $i : Y \to Y \vee Z$ by $y \mapsto (y, *)$ and $j : Z \to Y \vee Z$ by $z \mapsto (*, z)$ denote the injections.

$i^* : \tilde{H}^*(Y) \oplus \tilde{H}^*(Z) \to \tilde{H}^*(Y)$ is the first projection and j^* is the second projection. Thus for $x \in \tilde{H}^*(Y) \oplus \tilde{H}^*(Z)$, $x = 0$ is equivalent to $i^* x = 0$ and $j^* x = 0$.

$i^*(f \cup g) = i^* f \cup i^* g = f \cup 0$ since $g = (0, g) \in H^*(Z)$ has no $H^*(Y)$ component. Thus $i^*(f \cup g) = 0$. Similarly $j^*(f \cup g) = 0$. Thus $f \cup g = 0$. □

Corollary 14.1.3 *$S^1 \times S^1 \not\simeq S^1 \vee S^1 \vee S^2$ (although they have the same homology groups).*

$H^*(\mathbb{C}P^n)$:

Let $x_j \in H^{2j}(\mathbb{C}P^n)$ be a generator, choosing $x_0 = 1$ and x_μ. Set $x := x_1$.

$n = 2$: Basis is 1, $x = x_1$, $\mu = x_2$.

By the lemma, $\exists g$ s.t. $x \cup g = \mu$, and so g must be $\pm x$. Replacing μ by $-\mu$ if necessary, we may assume $x \cup x = \mu$. Aside from the multiplications by the identity and those that must be 0 for degree reasons, this describes all of the multiplications in $H^*(\mathbb{C}P^2)$.

$n = 3$:

Consider $i : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$. It is clear from the cellular chain complex that $i^*(x_j) = x^j$ for $j \leq n-1$ (and $i^* x_n = 0$ for degree reasons). So in $H^*(\mathbb{C}P^3)$, $x \cup x = x_2$ (else applying i^* gives a contradiction to the above calculations in $H^*(\mathbb{C}P^2)$). Now by the lemma, $x \cup (x \cup x)$ must be a generator of $H^6(\mathbb{C}P^3)$, so $x \cup x \cup x = \mu$ (or at least we can choose μ so that this is true). This describe all the non-obvious multiplications in $H^*(\mathbb{C}P^3)$.

For general n: Using induction on n and the same argument as in the previous cases, $x_j = x \cup x \cup \cdots x$ (j times). In other words, as a graded ring $H*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with degree $x = 2$. Passing to the limit gives $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$. $\surd$

If we use $\mathbb{Z}/(2\mathbb{Z})$ coefficients, the same method shows that $H*(\mathbb{R}P^n); \mathbb{Z}/(2\mathbb{Z}) \cong \mathbb{Z}/(2\mathbb{Z})[x]/(x^{n+1})$ with degree $x = 1$ and $H*(\mathbb{R}P^\infty); \mathbb{Z}/(2\mathbb{Z}) \cong \mathbb{Z}/(2\mathbb{Z})[x]$.