Chapter 4

Metric Spaces

4.1 Completeness

Definition 4.1.1 *Let* (x_n) *be a sequence in* (X, d)*. Then* (x_n) *is called a* Cauchy sequence *if* $\forall \epsilon > 0\ \exists N$ *s.t.* $n, m > N \Rightarrow d(x_n, x_m) < \epsilon$*.*

Proposition 4.1.2 $(x_n) \to x \Rightarrow (x_n)$ *Cauchy.*

Proof: Obvious.

Definition 4.1.3 *A* complete metric space *is one in which* $\forall$ *Cauchy sequences* (x_n) $\exists\ x \in X$ *s.t.* $(x_n) \to x$*.*

Definition 4.1.4 *A complete normed vector space is called a* Banach space*.*

Proposition 4.1.5 *Suppose* (X, d) *is complete, and* $Y \subset X$*. Then* Y *is complete* $\Leftrightarrow$ Y *is closed.*

Proof: Exercise.

Theorem 4.1.6 Cantor intersection theorem *Let* (X, d) *be a complete metric space. Let* (F_n) *be a decreasing sequence of nonempty closed subsets of* X *s.t.* $\text{diam}(F_n) \to 0$ *in* $\mathbb{R}$*. Then* $\cap_n F_n$ *contains exactly one point.*

Proof: Let $F = \cap_n F_n$. If F contains two points x and y then we have a contradiction when $\text{diam}(F_n) < d(x, y)$. Hence $|F| \leq 1$.

$\forall n$ choose $x_n \in F_n$. $\text{diam}(F_n) \to 0 \Rightarrow (x_n)$ is Cauchy.

Hence $\exists x \in X$ s.t. $(x_n) \to x$. We show that $x \in F_n\ \forall n$. If $\{x_n\}$ is finite then $x_n = x$ for infinitely many n, so that $x \in F_n$ for infinitely many n. Since $F_{n+1} \subset F_n$ this implies $x \in F_n\ \forall n$. So suppose $\{x_n\}$ is infinite. $\forall m$, $(x_m, x_{m+1}, \ldots, x_{m+k}, \ldots)$ is a sequence in F_m converging to x. Since $\{x_n\}_{n \geq m}$ is infinite, this implies x is a limit point of F_m. But F_m is closed, so $x \in F_m$. $\square$

Theorem 4.1.7 *Let (X, d) be a metric space. Then $\exists!$ metric space $(\tilde{X}, \tilde{d})$ together with an isometry $\imath : X \to \tilde{X}$ s.t.*

1. *$(\tilde{X}, \tilde{d})$ is complete.*
2. *Given any complete (Y, d') and an isometry $j : X \to Y$, $\exists!$ isometry $\tilde{j}$: $\tilde{X} \to Y$ s.t.*

Note: An isometry $f : X \to Y$ is a map s.t. $d\big(f(a), f(b)\big) = d(a, b)$ $\forall a, b \in X$.

Definition 4.1.8 *$\tilde{X}$ is called the completion of X.*

Sketch of Proof:
Let $C = \{$ Cauchy sequences in $X\}$.
Impose an equivalence relation $(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0$ in $\mathbb{R}$.
Let $\tilde{X} = C/\sim$. Define $\tilde{d}\big((x_n), (y_n)\big) = \lim_{n\to\infty} d(x_n, y_n)$.
Define $\imath : X \to \tilde{X}$ by $x \mapsto (x, x, \ldots, x, \ldots)$ Check that it works. (Exercise) □

Proposition 4.1.9 *X is dense in $\tilde{X}$.*

Proof: $\bar{X}$ is closed in $\tilde{X}$, so complete. It also satisfies the universal property of completion so $\bar{X} = \tilde{X}$. □

Definition 4.1.10 *$f : X \to Y$ is called* uniformly continuous *if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d(a, b) < \delta \Rightarrow d\big(f(a), f(b)\big) < \epsilon$.*

Proposition 4.1.11 *$f : X \to Y$ is uniformly continuous, (x_n) is Cauchy in X $\Rightarrow \big(f(x_n)\big)$ is Cauchy in Y.*

Proof: Exercise.

Definition 4.1.12 *Let (f_n) be a sequence of functions $f_n : X \to Y$. We say f_n* converges uniformly *to $f : X \to Y$ if $\forall \epsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow d\big(f(x), f(y)\big) < \epsilon$ $\forall x \in X$.*

Proposition 4.1.13 *Suppose f_n converges uniformly to f and f_n is continuous $\forall n$. Then f is continuous.*

Proof: Let $a \in X$. Show f is continuous at a. Given $\epsilon > 0$, choose N_0 s.t. $n \geq N_0$ $\Rightarrow d\big(f(x), f_n(x)\big) < \epsilon/3$ $\forall x \in X$.

Choose δ s.t. $d(x, a) < \delta \Rightarrow d\big(f_{N_0}(x), f_{N_0}(a)\big) < \epsilon/3$. Then $d(x, a) < \delta \Rightarrow d\big(f(x), f(a)\big) \leq d\big(f(x), f_{N_0}(x)\big) + d\big(f_{N_0}(x), f_{N_0}(a)\big) + d\big(f_{N_0}(a), f(a)\big) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. □

Example 4.1.14 *Sequence of continuous functions whose pointwise limit is not continuous:*

$$f_n : [0,1] \to [0,1],\ f_n(x) = x^n.\ f(x) = \begin{cases} 0 & x < 1; \\ 1 & x = 1. \end{cases}$$

Notation: Let X be a topological space (not necessarily metric).

$\mathcal{C}(X,\mathbb{R})$, resp. $\mathcal{C}(X,\mathbb{C})$ are real-valued (resp. complex-valued) bounded continuous functions on X.

Proposition 4.1.15 $\mathcal{C}(X,\mathbb{R})$ *and* $\mathcal{C}(X,\mathbb{C})$ *are Banach spaces.*

Proof: Let $Y = \mathcal{C}(X,\mathbb{R})$, or $\mathcal{C}(X,\mathbb{C})$.

For $f \in Y$, setting $||f|| = \sup_{x\in X} |f(x)|$ makes Y into a normed vector space. Let (f_n) be a Cauchy sequence in Y. Then $\forall x \in X$, $\big(f_n(x)\big)$ is a Cauchy sequence in $\mathbb{R}$ (resp. $\mathbb{C}$) so set $f(x) = \lim_{n\to\infty} f_n(x)$.

Must show f is bounded and continuous, and show $(f_n) \to f$ in Y.

Given $\epsilon > 0$, find N s.t. $n, m > N \Rightarrow ||f_n - f_m|| < \epsilon/2$.

Given $x \in X$ find $n_x > N$ s.t. $|f_{n_x}(x) - f_n(x)| < \epsilon/2$.

Then $n > N \Rightarrow |f(x) - f_n(x)| \le |f(x) - f_{n_x}(x)| + |f_{n_x}(x) - f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon$ Hence (f_n) converges uniformly to f so f is continuous. $||f|| \le ||f - f_N|| + ||f_N|| < ||f_N|| + \epsilon < \infty$ so f is bounded. Therefore $f \in Y$, and $\{f\} \to f$ in Y since $||f - f_N|| \to 0$.

Theorem 4.1.16 [Tietze's extension theorem]

Let X be normal and $A \subset X$ is closed. Let $f : A \to [p,q]$. Then there exists $F : X \to [p,q]$ s.t. $F|_A = f$.

Proof: If $p = q$ then f is constant and the theorem is trivial so suppose $p < q$. Let $c = \max(p,q)$.

Claim: $\exists\ h : X \to [-c/3, c/3]$ s.t. $|h(a) - f(a)| \le 2/3c\ \forall a \in A$.

Proof: Set $A_- = f^{-1}[-c, -c/3]$ and $A_+ = f^{-1}[c/3, c]$. By Urysohn, $\exists g : X \to [0,1]$ s.t. $g(A_-) = 0$ and $g(A_+) = 1$.

Composing with a homeomorphism of $[0,1]$ with $[-c/3, c/3]$ gives a function $h : X \to [-c/3, c/3]$ s.t. $h(A_-) = -c/3$ and $h(A_+) = c/3$. If $a \in A$ then $|h(a) - f(a)| \le 2/3c$.

Apply the Claim to f. This implies $\exists h_1 : X \to [-c/3, c/3]$ s.t. $|f(a) - h_1(a)| \le 2/3c$. Apply the Claim to $f - h_1$. This implies $\exists h_2 : X \to [-2c/3^2, 2c/3^2]$ s.t. $|f(a) - h_1(a) - h_2(a)| \le (2/3)^2 c$. By induction, we apply the Claim to $f - h_1 - \cdots - h_{n-1}$. This implies $\exists h_n : X \to [-2^{n-1}c/3^n, 2^n c/3^n]$ s.t. $|f(a) - h_1(a) - \cdots - h_{n-1}(a)| \le (2/3)^n c$.

Let $G(x) = \sum_{n=1}^{\infty} h_n(x)$.

$\forall x \in X$,

$$|G(x)| \le \sum_{n=1}^{\infty} |\,|h_n(x)|\,| \le \sum_{n=1}^{\infty} |\,|h_n|\,| = c/3(1 + 2/3 + (2/3)^2 + \dots) = c/3(\frac{1}{1-2/3}) = c.$$

The partial sums of G are a Cauchy sequence in $\mathcal{C}(X, \mathbb{R})$.

Hence by completeness of $\mathcal{C}(X, \mathbb{R})$ their pointwise limit $G : X \to [-c, c]$ is continuous.

Define F by

$$F(x) = \begin{cases} G(x) & \text{if } p \leq G(x) \\ p & \text{if} G(x) < p \\ q & \text{if} G(x) > q \end{cases}$$

$F|_A = G|_A$ since $p \leq f(a) \leq q \ \forall a \in A$. □

4.1.1 Compactness in Metric Spaces

Proposition 4.1.17 *A sequentially compact metric space is complete.*

Proof: Suppose X is sequentially compact, and (x_n) is Cauchy in X.

Some convergent subsequence of (x_n) converges to $x \in X$ so since (x_n) is Cauchy, with $(x_n) \to x$. That is, given $\epsilon > 0$, $\exists N$ s.t. $m, n \geq N \Rightarrow d(x_n, x_m) < \epsilon/2$. Therefore since some subsequence of (x_n) converges, $N_{\epsilon/2}(x)$ contains x_m for infinitely many m, so $\exists m > N$ s.t. $x_m \in N_{\epsilon/2}(x)$ and therefore $n \geq N \Rightarrow d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon$.

Definition 4.1.18 *Given $\epsilon > 0$, a finite subset T of X is called an* ϵ-net *if $\{N_\epsilon(t)\}_{t \in T}$ forms an open cover of X.*

X is called totally bounded *if $\forall \epsilon > 0$, $\exists$ an ϵ-net for X.*

Note: X totally bounded $\Rightarrow \text{diam}(X) < \text{diam}(T) + 2\epsilon$ and $\text{diam}(T) < \infty$ since T finite, so totally bounded implies bounded.

Example 4.1.19 *Suppose X is infinite with*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Then X is bounded but $\nexists$ an ϵ -net for any $\epsilon < 1$.

Theorem 4.1.20 *For metric X, the following are equivalent:*

1. *X compact*
2. *X sequentially compact*
3. *X is complete and totally bounded.*

Proof:

(1) $\Rightarrow$ (2)

Already showed: metric $\Rightarrow$ first countable and Hausdorff
and first countable and Hausdorff and compact $\Rightarrow$ sequentially compact.

(2) $\Rightarrow$ (3):

Suppose X is sequentially compact.

We already showed this implies X is complete.

Given $\epsilon > 0$: Pick $a_1 \in X$.

Having chosen $a_1, \ldots, a_{n-1}$ if $N_\epsilon(a_1) \cup \ldots N_\epsilon(a_{n-1})$ covers X, we are finished.

If not, choose $a_n \in X - \big(N_\epsilon(a_1) \cup \ldots N_\epsilon(a_{n-1})\big)$.

So either we get an ϵ-net $\{a_1, \ldots, a_n\}$ for some n, or we get an infinite sequence $(a_1, a_2, \ldots, a_n, \ldots)$.

If the latter: By construction $d(a_k, a_n) \geq \epsilon \; \forall k, n$ so (a_n) has no convergent subsequence. This is a contradiction. So the former holds. $\square$

(2) $\Rightarrow$ (1):

Definition 4.1.21 *Let $\{G_\alpha\}_{\alpha \in J}$ be an open cover of the metric space X. Then $a > 0$ is called a* Lebesgue number *for the cover if* $\operatorname{diam}(A) < a \Rightarrow A \subset G_\alpha$ *for some* α.

Theorem 4.1.22 Lebesgue's Covering Lemma *If X is sequentially compact, then every open cover has a Lebesgue number.*

Proof: Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover.

Say $A \subset X$ is "big" if A is not contained in any U_α.

If $\nexists$ big subsets then any $a > 0$ is a Lebesgue number, so assume $\exists$ big subsets.

Let $a = \inf\{\operatorname{diam}(A) \mid A \text{ big}\}$

If $a > 0$, a is a Lebesgue number, so we assume $a = 0$.

Hence $\forall n > 0$, $\exists$ a big B_n s.t. $\operatorname{diam}(B_n) < 1/n$.

$\forall n$, pick $x_n \in B_n$. Find x s.t. a subsequence of (x_n) converges to x.

Find α_0 s.t. $x \in U_{\alpha_0}$.

U_{α_0} is open, so $\exists r > 0$ s.t. $N_r(x) \subset U_{\alpha_0}$.

For infinitely many n, $x_n \in N_{r/2}(x)$.

Find N s.t. $N > 2/r$ and $x_N \in N_{r/2}(x)$.

$\operatorname{diam}(B_N) < 1/N < r/2$ and $B_N \cap N_{r/2}(x) \neq \emptyset$ $\Big(\text{since } x \in B_N \cap N_{r/2}(x)\Big)$ so $B_N \subset N_r(x) \subset U_{\alpha_0}$. This is a contradiction, since B_N is big.

Hence $a > 0$ so X has a Lebesgue number.

Proof that (2) $\Rightarrow$ (1):

Given an open cover $\{U_\alpha\}_{\alpha \in J}$, find a Lebesgue number a for $\{U_\alpha\}$.

Let $\epsilon = a/3$ and using (2) $\Rightarrow$ (3) from the above, pick an ϵ-net $T = \{t_1, t_2, \ldots, t_n\}$. For $k = 1, \ldots, n$ $\operatorname{diam} N_\epsilon(t_k) = 2\epsilon < a$ so $N_\epsilon(t_k) \subset U_{\alpha_k}$ for some α_4.

Since $\{N_\epsilon(t_1), N_\epsilon(t_2), \ldots, N_\epsilon(t_n)\}$ covers X (by definition of ϵ-net), so does $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$.

$3 \Rightarrow 2$:

Suppose X is complete and totally bounded.

Let $S^{(0)} = (x_1, x_2, \ldots, x_m, \ldots)$ be a sequence in X.

Since X is complete, to show $S^{(0)}$ has a convergent subsequence, it suffices to show $S^{(0)}$ has a Cauchy subsequence.

Choosing an ϵ-net for $\epsilon = 1/2$, cover X with finitely many balls of radius $1/2$. Since $S^{(0)}$ is infinite, some ball contains infinitely many x_m so discard the x_n outside that ball to get a subsequence $S^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}, \ldots)$ with $d(x_m^{(1)}, x_p^{(1)}) < 2\epsilon = 1$ $\forall m, p$. Repeating this procedure with $\epsilon = 1/4, 1/6, \ldots, 1/(2n), \ldots$ gives for each n a subsequence of $S^{(n-1)}$.

$S^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}, \ldots)$ s.t. $d(x_m^{(n)}, x_p^{(n)}) < 1/n$ $\forall m, p$.

Let $S^{(n)} = (x_1^{(1)}, x_2^{(2)}, \ldots, x_n^{(n)}, \ldots)$

If $m, p \geq n$ then since $S^{(m)}$ and $S^{(p)}$ are subsequences of $S^{(n)}$, $d(x_m^{(m)}, x_p^{(p)}) < 1/n$ so S is a Cauchy subsequence of $S^{(0)}$ as desired. □

Theorem 4.1.23 *If X and Y are metric spaces, and $f : X \to Y$ is a continuous function with X compact, then f is uniformly continuous.*

Proof: Given $\epsilon > 0$, $x \in f^{-1}\big(N_{\epsilon/2}(f(x)\big)$, so $\Big\{f^{-1}\big(N_{\epsilon/2}(f(x))\big\}_{x \in X}$ is an open cover of X.

Let δ be a Lebesgue number for this cover.

$\forall a, b \in X$: $d(a,b) < \delta \Rightarrow \operatorname{diam}\{a,b\} < \delta \Rightarrow \{a,b\} \subset f^{-1}\big(N_{\epsilon/2}(f(x)\big)$ for some x. Hence $d\big(f(a), f(b)\big) \leq d\big(f(a), f(x)\big) + d\big(f(x), f(b)\big) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence f is uniformly continuous.

□

Corollary 4.1.24 *A compact metric space is second countable.*

Lemma 4.1.25 *For metric spaces second countable $\Leftrightarrow$ separable.*

Proof: Second countable $\Rightarrow$ separable in general.
$\Longleftarrow$ Suppose X is a separable metric space. Let $\{x_1, \ldots, x_n, \ldots\}$ be a countable dense subset. Then $\{N_r(x_j) \mid$ r rational $\}$ forms a countable basis for X. (That is: Given $N_{r'}(x)$, find x_n s.t. $d(x_n, x) < r'/3$. Choose rational r s.t. $r < r'/3$. Then $N_r(x_n) \subset N_{r'}(x)$.) □

Proof of Corollary: Suppose X is a compact metric space. Show X is separable.

For each $\epsilon = 1/n$, choose an ϵ-net $T_n = \{x_1^{(n)}, \ldots, x_{k_n}^{(n)}\}$. Let $S = \cup_n T_n$. Then S is a countable dense subset of X.

□

Example 4.1.26 *Normal but not metric:*

Let $X = \prod_{t \in R} I_t$ where $I_t = [0,1]$ $\forall t$. X is compact by Tychonoff and is Hausdorff so X is normal.

If X were metric, then being compact, it would be second countable.

Let $\mathcal{S} = \{U_1, \dots, U_n, \dots\}$ will be a countable basis.

Since $\mathbb{R}$ is uncountable, $\exists t_n \in \mathbb{R}$ s.t. $\pi_{t_0}(U_n) = I_{t_0}$ $\forall n$. But then $\mathcal{S}$ is not a basis. (e.g. The set $(1/4, 3/4) \times \prod_{t \neq t_0} I_t$ is not a union of sets in $\mathcal{S}$.

This is a contradiction. So X is not metric.

$\square$